HW 5: Mixed states  (due Nov 21st, before class)

1. Let \{ (p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle) \} and \{ (q_1, |\phi_1\rangle), (q_2, |\phi_2\rangle) \} be two distributions on qubit states such that 
\[ p_1 |\psi_1\rangle \langle \psi_1 | + p_2 |\psi_2\rangle \langle \psi_2 | = q_1 |\phi_1\rangle \langle \phi_1 | + q_2 |\phi_2\rangle \langle \phi_2 |. \]
Show that for any choice of basis \{ |b_1\rangle, |b_2\rangle \}, the statistics of the measurement outcomes are the same for the two distributions.

2. Compute the eigenvalues and eigenvectors of the following matrices:
\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

3. (Purification) Show that for any density operator \( \rho_X \in D(X) \), there is a pure state \( |\psi\rangle \in X \otimes Y \) such that
\[ \text{tr}_Y (|\psi\rangle \langle \psi |) = \rho_X. \]

4. (Distinguishing between states) Suppose I have a system which is either in the state \( \rho_0 \) or in the state \( \rho_1 \), each with a priori probability \( \frac{1}{2} \). We would like to determine which is the case using a measurement. Show that the maximum probability of successfully guessing which one is given by
\[ \frac{1}{2} + \frac{1}{4} \| \rho_0 - \rho_1 \|_{tr}, \]
where \( \| A \|_{tr} = \sum_i |\lambda_i| \) if \( A \) is a Hermitian operator with eigenvalues \( \{ \lambda_i \} \). You might want to start with the classical case where \( \rho_0 \) and \( \rho_1 \) are probability distributions.

5. Given the definition of the cone \( \mathcal{CS}^n_+ \) of \( n \times n \) completely positive semidefinite matrices. What is the relation to the cone \( \mathcal{S}^n_+ \) of positive semidefinite matrices?