# Report of *On the reality of the quantum state* article

Deák Norbert

December 6, 2016

## 1 Introduction

Quantum states are the fundamental elements in quantum theory. However, there are a lot of discussions and debates about what a quantum state actually represents. Is it corresponding directly to reality, or is it only a description of what an experimenter knows of some aspect of reality, a physical system? There is a terminology to distinct the two theories: *ontic state* (state of reality) or *epistemic state* (state of knowledge). In [1], this is referred as *the* $\psi$*-ontic/epistemic distinction.*

The arguments started with the beginnings of quantum theory, with the debates between Bohr-Einstein, concerning the foundations of quantum mechanics. In 1935, Einstein, Podolsky and Rosen published a paper stating that quantum mechanics is incomplete. As a repsonse, Bohr also published a paper, in which he defended the so called *Copenhagen interpretation*, a theory strongly supported by himself, Heisenberg and Pauli, that views the quantum state as epistemic [2].

One of the scientific works stating that the quantum state is less real is Spekken's toy theory [3]. It introduces a model, in which a toy bit as a system that can be in one of the four states: (-,-), (-,+), (+,-) and (+,+), representing them on the $xy$ plane, each of them being in a different quadrant. In this representation, the quantum states are epistemic, they are represented by probability distributions. Using this toy theory, Spekkens offers natural explanations for some features of quantum theory, like the non-distinguishibility of the non-orthogonal states and the no-cloning theorem.

There are many others suggesting that the quantum state is merely knowledge, one common argument is the collapse of the state on measurement. If the quantum state is direct representation of reality, the collapse is a mysterious process, which is not well-defined. But if it represents only knowledge, the measurement is simply an update for the probability distribution, after obtaining some new information.

Another argument in favor of the $\psi$-epistemic interpretation is that a single qubit may store an infinite amount of information, since there is an infinite number of qubit states, however only a single bit of classical information can be extracted. This could lead to the statement, that the qubit contains only finite amount of information, representing only the knowledge about the real physical state.

There are also many arguments supporting the theory, that a quantum state represents directly reality. One of the biggest argument is coming from interference phenomena, the famous wave-particle duality problem associated with the double-slit experiment. The experiment was first performed by Thomas Young, demonstrating the interference of waves of light. Later, the experiment was also performed with electrons. The outcome of this experiment is a theory that both light and matter behave like both particle and wave [4]. This became the foundation of quantum mechanics. Many scientists argue that as a consequence of this theory, a real wave function must exist, which describes a quantum state of a system, thus the quantum state is ontic.

There are many publications, that aim to prove that the quantum state represents directly the reality, but none of them managed to provide a complete and accepted proof.

This paper from Pusey, Barrett and Rudolph [5] is the first in a long time that provides a complete proof about the $\psi$-ontic/deterministic distinction, stating that the quantum states represent directly the reality. However, there are two assumptions in their theory, and the question is still not closed, because despite that the authors provided a proof, it isn't accepted by the whole scientific community. There are many discussions addressing this paper, some of them agreeing with it, others contradicting it.

Mixed states are sometimes, without doubt, representing knowledge about which pure states were prepared, so they are sometimes epistemic. For this reason, scientists are only concerned about the nature of pure quantum states [1].

# 2 The Pusey-Barrett-Rudolph (PBR) Theorem

To introduce the main contribution of this paper, first the distinction between ontic and epistemic states must be formalized. Harrigan and Spekkens [6] were the first ones in giving such a model. They stated that if a quantum state is only a representation of knowledge about a real physical system, it is a subjective knowledge, since it also depends on the information available to the experimenter. It follows, that distinct quantum states, which are not orthogonal, may represent the same physical state. This idea is described more formally in [7], and it is presented next.

Let $\lambda \in \Lambda$ be a complete specification of the physical state of a system. Let $|\psi\rangle$ be a quantum state of that system. If $|\psi\rangle$ also corresponds to a single $\lambda$, then the quantum state is also a complete specification of the physical state. But it could also correspond to a probability distribution $\mu(\lambda)$ over the values $\lambda$.

Now let's consider two possible quantum states of the system $|\psi_1\rangle$ and $|\psi_2\rangle$, and their probability distributions $\mu_1(\lambda)$ and $\mu_2(\lambda)$. If they are orthogonal, their probability distributions can't overlap in order to respect one prediction of quantum theory, that "a measurement of the projection operator (...) on a system prepared in the orthogonal state (...) yields 0" [7]. Formally, the non-overlapping can be written as $\mu_1(\lambda)\mu_2(\lambda) = 0, \forall\lambda \in \Lambda$. However, if $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, their probability distributions could overlap: $\mu_1(\lambda)\mu_2(\lambda) \neq 0$, so both quantum states could represent the same physical state $\lambda$. On the other hand, if their probability distribution don't overlap $\mu_1(\lambda)\mu_2(\lambda) = 0$, the quantum states represent directly physical reality.

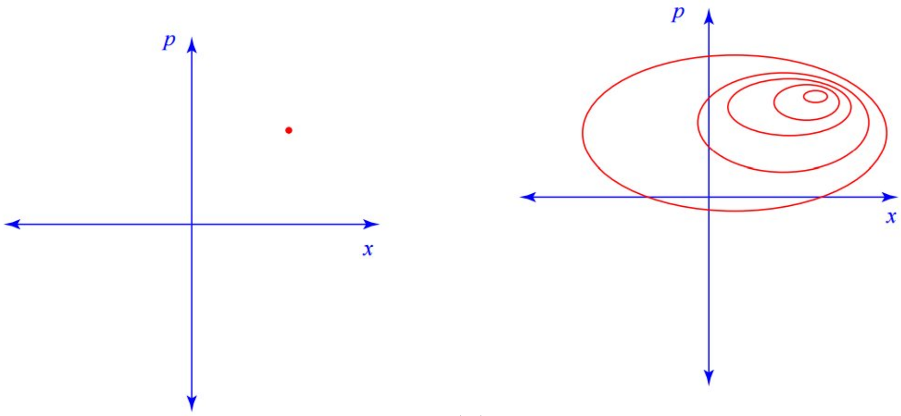
In this paper [5], and also in [1], the notion of two probability measures overlapping is described formally using the variational distance.

**Definition 1** *The variational distance between two probability measures $\mu$ and $\nu$ on the measurable space $(\Lambda, \Sigma)$ is*

$$D(\mu, \nu) = \sup_{\Omega \in \Sigma} |\mu(\Omega) - \nu(\Omega)|$$

If $\mu_1$ and $\mu_2$ are two probability distributions, then $D(\mu_1, \mu_2) = 1$, if they are completely disjoint, i.e. they don't overlap ($\mu_1(\lambda)\mu_2(\lambda) = 0$).

In order to understand better the difference between ontic and epistemic states, which was presented formally above, an example from classical mechanics will be shown, using the same approach as the authors from this paper. Considering a point particle moving in one dimension, it can be




(a) An ontic state is a point in phase space.

(b) An epistemic state is a probability density on phase space. Contours indicate lines of equal probability density.

Figure 1: Figures from [1]

described by its position $x$ on the line and its momentum $p$. All other properties, e.g. energy, can be expresed as function of $x$ and $p$. In the paper, the *physical property* notion is introduceed, meaning some function of the physical state. This way, the physical state of the particle corresponds to a point $(x, p)$ in a two-dimensional phase space, shown in 1a. However, if its position and momentum is not known, the particle's state is represented with a probability distribution $\mu(x, p)$. Since the distribution doesn't represent directly the reality, only some knowledge, it is an epistemic state. Considering a physical property described before, different values of the property correspond to different disjoint regions of the space phase. This is illustrated in 1b. However, if two probability distributions $\mu_1(x, p)$ and $\mu_2(x, p)$ overlap, it cannot refer to the physiscal property of the system.

To illustrate the same in terms of quantum theory, for two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, with probability distributions $\mu_1(\lambda)$ and $\mu_2(\lambda)$, if the distributions don't overlap, the quantum states refer to a physical property of a system, otherwise, if they overlap even for one pair of quantum states, they are epistemic states, representing only information.

After these definitions, the main theorem of the paper can be stated: The quantum state must represent directly the reality, if two assumptions are true. The first assumption is that isolated systems have a real physical state, while the other assumption is that the systems that are prepared independently, have independent physical states. However, the theorem is expressed in a different way, based on contradiction. It states, that if a quantum state represents only information about the real physical system, then the distributions of two distinct quantum states may overlap, thus resulting a contradiction with the predictions of quantum theory.

The theorem, in general case, is reformulated more mathematically.

**Assumption 1** *There exists a measurable set* $\Lambda$ *containing possible physical states* $\lambda$*. Preparation of a quantum system in a state* $|\psi\rangle$ *means, that the system is in one of the physical state* $\lambda$*, resulting from the sample from a probability distribution* $\mu(\lambda)$ *over* $\Lambda$*.*

**Assumption 2** *It is possible to prepare* $n$ *systems independently in states* $|\psi_1\rangle, ..., |\psi_n\rangle$*.*

**Theorem 1** *Consider a preparation device which can produce a quantum system in one of the states* $|\psi_1\rangle, ..., |\psi_n\rangle$*, each state associated with a probability distribution* $\mu_i(\lambda)$ *to result a* $\lambda \in \Lambda$*. Prepare* $n$ *systems independently in states* $|\psi_{x_1}\rangle, ..., |\psi_{x_n}\rangle$*, where* $x_i \in \{1, ..., n\}$*. Then, for a small* $\epsilon > 0$*, the probability distributions* $\mu_1, ..., \mu_n$ *are completely disjoint, thus the quantum states represent directly the physical state of the system.*

# 3 Proof of the PBR Theorem

The authors from the paper divided the proof in three parts: first a simple case is presented, with two distinct qubit states, $|0\rangle$ and $|+\rangle$, then it is extended to arbitrary states, $|\psi_0\rangle$ and $|\psi_1\rangle$. Finally, it is proven for the general case, in formal setting, allowing also experimental errors and noise.

## 3.1 $|0\rangle$ and $|+\rangle$

Consider two preparation methods of a quatum system, which will end up in either state $|\psi_0\rangle = |0\rangle$ or $|\psi_1\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, with probability distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$. Assume, that the distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ overlap. Then there exist a probability $q > 0$ that the physical state associated to the system is from the overlapping region.

If there are two, independently prepared systems, with the same properties (two copies), either one of them could be in state $|\psi_0\rangle$ or $|\psi_1\rangle$. Now, there is a probability of $q^2$ that the systems will be in physical states $\lambda_1$ and $\lambda_2$, both from the overlapping regions. It follows that the physical state of the 2-qubit system is compatible with the four quantum states: $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |+\rangle$, $|+\rangle \otimes |0\rangle$ and $|+\rangle \otimes |+\rangle$.

Consider performing a measurement of the two systems brought together with entangled eigenstates:

$$
\begin{aligned}
|\epsilon_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \\
|\epsilon_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle), \\
|\epsilon_3\rangle &= \frac{1}{\sqrt{2}}(|+\rangle \otimes |1\rangle + |-\rangle \otimes |0\rangle), \\
|\epsilon_4\rangle &= \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle),
\end{aligned} \tag{1}
$$

where $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. It can be easily proven that $|\epsilon_1\rangle$ is orthogonal to $|0\rangle \otimes |0\rangle$, $|\epsilon_2\rangle$ is orthogonal to $|0\rangle \otimes |+\rangle$, $|\epsilon_3\rangle$ is orthogonal to $|+\rangle \otimes |0\rangle$ and $|\epsilon_4\rangle$ is orthogonal to $|+\rangle \otimes |+\rangle$. According to the quantum theory, all outcomes have probability zero, because they are orthogonal, it leads to the contradiction. This way, the assumption, that the two distributions overlap is false, hence the quantum state is a physical property of the system.

## 3.2 Arbitrary $|\psi_0\rangle$ and $|\psi_1\rangle$

Similar to the previous case, consider a preparation device that can produce a system in state $|\psi_0\rangle$ or $|\psi_1\rangle$, but in this case, they can be arbitrary states, distinct and non-orthogonal. Also, assume that there exists a probability $q > 0$ that the resulting physical state $\lambda$ of the system after preparation is from the overlapping region. Consider $n$ systems prepared in this way ($n$ copies of this device), each of them could be in $|\psi_0\rangle$ or $|\psi_1\rangle$ state, so there will be $2^n$ different joint states of the $n$ systems.

For a given large $n$, there exist a joint measurement on the $n$ systems brought together, such that every outcome has 0 probability given one of the $2^n$ possible preparations, which infers the contradiction, i.e. there can be no overlapping, proving the ontic nature of the quantum states.

First, the quantum states are rewritten with a chosen basis of Hilbert space:

$$
\begin{aligned}
|\psi_0\rangle &= cos(\theta/2)\,|0\rangle + sin(\theta/2)\,|1\rangle \\
|\psi_1\rangle &= cos(\theta/2)\,|0\rangle - sin(\theta/2)\,|1\rangle
\end{aligned} \tag{2}
$$

Then, the authors propose a quantum circuit that produces this kind of measurement. For the $n$ input states of the circuit, the following operator will be applied:

$$
U_{\alpha,\beta} = H^{\otimes n} R_\alpha Z_\beta^{\otimes n}, \tag{3}
$$

where $H = |+\rangle \langle 0| + |-\rangle \langle 1|$ is the Hadamard gate, $R_\alpha |00...0\rangle = e^{i\alpha} |00...0\rangle$ is a rotation gate applied to only one state, and $Z_\beta = |0\rangle \langle 0| + e^{i\beta} |1\rangle \langle 1|$. At the end of the circuit, each qubit is measured.

In the paper it is proven that for any $n$ chosen, such that $2^{1/n} - 1 \leq tan(\theta/2)$, for any $0 < \theta < \pi/2$, there exists $\alpha$ and $\beta$ so that the measurement has the desired feature described before.

### 3.3 Formal proof with experimental errors and noise

In this case, consider a more general type of device that can produce a system in a quantum state $|\psi_i\rangle$, which results in a physical state $\lambda$ from the set of possible physical states $\Lambda$, sampled from probability distribution $\mu_i(\lambda)$ over $\Lambda$. By using $n$ copies of this device, $n$ independent system can be prepared with quantum states $|\psi_{x_1}\rangle, ..., |\psi_{x_n}\rangle$, resulting in physical states $\lambda_1, ..., \lambda_n$ distributed according to the physical distribution $\mu_{x_1}(\lambda_1)\mu_{x_2}(\lambda_2)...\mu_{x_n}(\lambda_n)$.

The authors prove that if an experiment is performed using this setting, the probability for each measurement outcome will be within a small $\epsilon > 0$. To show that the probability distributions are distinct, the *total variation distance* is used, which will be close to 1.

## 4 Strengths and weaknesses

This paper can be considered as a milestone in quantum mechanics. It answers the rather old, but much discussed question of whether a quantum state is ontic or epistemic, and it proves that actually it is ontic, given some assumptions. Many papers addressed the result of this paper, agreeing with it, or critisising it.

For instance, Antony Valentini, a theoretical physicist, told the Nature, that this theorem is "the most important general theorem relating to the foundations of quantum mechanics since Bell's theorem" [8]. But others, e.g. on a blog [9], this theorem is called garbage, and they use pretty bad words for those who believe it.

However, there are also some reasonable arguments to criticise this paper. In a paper by Fine and Schlosshauer [10] it is claimed that ontic and epistemic models can be converted into to each other, thus the distinction between these two is just conventional. However, they describe only the procedure, and not the definition of the equivalence after conversion, i.e. how the converted ontic state is actually epistemic.

In my opinion, this paper provides an acceptable proof for the nature of the quantum state, it gives an answer to a very important question, which is the fundamental of quantum mechanics, and which has been in the air for a long time, discussed even by Einstein, Bohr and Schrodinger.

However, there are some apsect that can be considered as weaknesses of the paper. First, the whole theorem is based on some assumptions. There are works that try to weaken these assumptions, for instance the work of Mansfield [11], but the theorem still can't hold without some assumptions. As future improvement the eliminations of these assumptions would be the most important one, because it would answer all those critisisms based on this argument, and it may get closer to officially answer the big question from so many years.

As an interesting project related to this paper is to find all the "gaps" in quantum theory using this theorem and close them, which could lead to a better understanding of the quantum field, and may open many new ways.

# References


[1] Matthew Leifer. Is the quantum state real? an extended review of $\psi$-ontology theorems. *Quanta*, 3(1):67–155, 2014.

[2] The einstein-podolsky-rosen argument in quantum theory. https://plato.stanford.edu/entries/qt-epr/. Accessed: 2016-12-06.

[3] Robert W. Spekkens. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A*, 75:032110, Mar 2007.

[4] Noson S. Yanofsky and Mirco A. Mannucci. *Quantum Computing for Computer Scientists*. Cambridge University Press, New York, NY, USA, 1 edition, 2008.

[5] Jonathan Barrett Matthew F. Pusey and Terry Rudolph. On the reality of the quantum state. *Nature Physics*, 8:475–478, 2012.

[6] Nicholas Harrigan and Robert W. Spekkens. Einstein, incompleteness, and the epistemic view of quantum states. *Foundations of Physics*, 40(2):125–157, 2010.

[7] Daniel Rohrlich Guy Hetzroni. Protective measurements and the pbr theorem. 2014.

[8] Eugenie Samuel Reich. Quantum theorem shakes foundations. 2011.

[9] Nature hypes anti-qm crackpot paper by pusey et al. http://go.nature.com/3vZqkM. Accessed: 2016-12-04.

[10] Fine A Schlosshauer M. Implications of the pusey–barrett–rudolph quantum no-go theorem. *Physical Review Letters*, 2012.

[11] Shane Mansfield. Reality of the quantum state: Towards a stronger $\psi$-ontology theorem. *Phys. Rev. A*, 94:042124, Oct 2016.