Quantum State Tomography via Compressed Sensing

Laureline Pinault

January 5, 2017

Abstract

This report aims at presenting the article *Quantum State Tomography via Compressed Sensing* by David Gross, Yi-Kai Liu, Steven T. Flammia, Stephen Becker and Jens Eisert [GLF+10]. As the article is a short letter – only four pages – it could not spend a lot of time on bibliographical studies or physical interpretations. Hence I focused on those two aspects to try and explain the article's main ideas.

I first present shortly what is quantum state tomography. Then I explain the main result of the article and where lies the novelty of their work. I briefly expose the other results and I end up summarizing and suggesting further research directions.

1 Context and motivations

Quantum state tomography is the process of reconstructing the quantum state produced by a quantum source by some measurements on systems coming from the source. The idea is that by repeated measurements of the quantum system on a "good basis" one can approximate the density matrix describing the system with small error.

Quantum state tomography has applications in quantum physics in order to understand some quantum systems – for instance describing the entangled state of trapped ions [HHR+05], as well as in quantum computation in order to determine the exact state of qubits – for instance assuring an optical system performs a CNOT gate [OPW+03].

One of the main stake is to make this process tractable.

2 Main result

If we consider a n-qubit system it is represented by a density matrix of size $d \times d$ where $d = 2^n$. Then to recover perfectly this density matrix we would need to know at least d^2 pieces of information, which can in practice be really slow – for instance it took hundreds of thousands of measurements and weeks of postprocessing to get a maximum likelihood estimate of a quantum state of 8 ions [HHR+05]. But if we suppose we have a matrix with small rank r, most of those pieces of information are redundant. Since this is a very common case because pure states subject to some noise are well approximated by low-rank density matrices, the article aims at improving the known methods in this particular case.

The method presented by the article is inspirated from compressed sensing but for recovering low-rank matrices rather than sparse vectors. It is called matrix completion and is detailed in its more general setting in the article [Gro11]. The idea behind matrix completion is to use some pieces of information coming from random measurements in a well-suited basis *i.e.* a basis where coefficients countain non-trivial information, an idea formalized by the notion of "incoherence to a basis". Then we want to find the minimum rank matrix satisfying the measurements. In order to do this efficiently, the minimization over the rank is replaced by a minimization over the trace norm[1], yielding a convex optimization problem. Justifications of this transformation can be found in [MP97] and [FHB01].

The framework of the article [GLF+10] consists in recovering the state of a spin based qubit, which seems to be among the most popular implementations of qubits [KGS+07]. It is then natural to take as a basis the one formed by the Pauli matrices, with the hope that it would yield good results in term of incoherence. They indeed allow to access the expectation of the observables "spin" along the different axes[2].

More formally speaking, the problem is the following. Name $\{W_a, a \in [\![1, d^2]\!]\}$ the set of Pauli matrices for n-particles. Draw m integers a_1, a_2, ... a_m at random in $[\![1, d^2]\!]$. Measure the expectations of your unknown system ρ for the corresponding observables and solve the following convex optimization problem:

$$\text{Minimize } \|\sigma\|_{tr} \text{ subject to } \begin{cases} tr(\sigma) = 1 \\ tr(W_{a_i}\sigma) = tr(W_{a_i}\rho) \quad \text{for } i = 1..m \end{cases}$$

where $\|\cdot\|_{tr}$ is the trace norm.

Then if we note r the rank of ρ and take $m = cdrlog^2 d$ with c a parameter, ρ can be uniquely determined with probability of failure exponentially small in c. The main stake of the proof is to show that the unknown density matrix ρ is the global minimum of the problem, *i.e.* that for any deviation Δ, either $\rho + \Delta$ does not satisfy the constraint or its trace norm is greater than ρ's one. It does so by using a classical method in convex optimization problems: it constructs a strict subgradient which will serve as a dual certificate. Then the bound on the failure of the completion matrix algoritm is computed from the probability of failing to construct such a strict subgradient.

The complexity of the postprocessing is in $O(d^2)$ for classical implemetations.

[1]The trace norm or Schatten 1-norm or nuclear norm is the sum of the singular values

[2]To measure the expectation of an observable represented by a hermitian matrix A for a system represented by its density matrix ρ is getting the value $\langle A \rangle = tr(A\rho)$

3 Other results

Robustness to noise. There might be two sources of imprecision. The first one is that due to some interferences with the exterior the unknown density matrix is not low rank but only well-approximated by a low rank density matrix. It is a physical phenomenon known as decoherence [Sch14]. The second is that measures are also approximations. To cope with it, they relax their constraints and introduce a parameter error ε depending on estimated errors from both sources. Then the same method allows to approximate the state of the unknown system with an error in $O(\varepsilon\sqrt{rd})$. The same failure rate holds up to adding a term depending on the error parameter.

Coping with unknown rank. So far we supposed we had a quantum system well approximated by an unknown density matrix but of known low rank. Yet in realistic experiments the rank may be unknown or we may want to certify that the rank is the one assumed. The authors distinguish two cases. Either the quantum system is nearly pure, *i.e.* well approximated by a rank 1 density matrix. Then they can find a certificate for this assumption and reconstruct the approching pure state without increasing the asymptotic number of measures needed or the failure rate. Or the quantum state has approximatively low-rank but different from 1. Then the authors advise to perform tomography with different values for m. If m is larger than necessary the method recover the correct density matrix, if it is smaller the method does not converge.

Hybrid approach. The authors also describe a variant of their method they call "the hybrid approach". The idea is to take the Pauli matrices according to a pattern instead of completely random. It allows to solve the convex optimisation problem more efficiently – $O(d)$ instead of $O(d^2)$ – but the theoritical guarantees are weaker. However this approach is claimed to yield accurate results in practice.

Experiments. For their experiments the authors solve a slightly different optimzation problem thanks to the singular value thresholding algorithm which is described in the article [CCS08]. This modified problem introcuce a parameter τ and when τ is large enough the two problems are equivalent. The figure 1 show the average fidelity and the trace distance drawn in fonction of the number of measures performed for a noiy system of 8 quantum particles approximated by a rank 3 matrix.The results are quite good – the methods recover in reasonnable time the matrix with around 95% of fidelity even though only 10% of the expansion coefficients of the matrix are sampled. The hybrid approach yields less precise results but do so faster.

4 Conclusion and further work

The article shows a refinement of quantum state tomography in the case where the unknown state is fairly pure. The presented method is able to recover an unknown density matrix of dimension d and rank r in $O(rdlog^2d)$ measurements rather than $O(d^2)$. The postprocessing is then in $O(d^2)$. Moreover the method is stable against the noise either in the state itself or in the measurements. The acquired data also enable to certify the unknown state is indeed close to pure. Besides this method, the article presents an alternative approach that has a lesser postprocessing time and yields good experimental results even if the theoretical results do not hold for it.

I think this article is relevant in its objective even if there are only few justifications about the fact that states represented by low rank density matrices naturally arise in quantum problems where tomography is needed and not only nearly pure states. The article presents a new proof that is inspired from previous work but is more general, more compact and yields tigher bounds. However the compactness of the proof makes it hard to follow and the article does not give any physical interpretation which makes the underlying choices difficult to understand without a lot of research. Some of them are still not very clear for me. For instance the constraints of the optimization problem enforce that the trace of the solution is one but does not require the other properties of density matrices to be fulfilled.

The authors do not present further work. Below is a list of some possible extensions I thought of:

- This work is limited to spin-based quantum system. One could investigate the other possible implementations of qubits such as polarized photons to try to achieve similar or better results. This work would consist in looking how well the matrix completion algorithm works wfor some basis relevant to the given implementation.
- The method lies on the fact that the minimization of the trace norm is a good proxy for the minimization of the rank. It can be linked with the fact that the trace norm is a measure of the coherence of a state, see for instance [SXFL15]. It could be interesting to investigate other coherence measures such as the ones presented in [BCP14].
- When coping with noise, the relaxation of the constraints lies on a error parameter which depends on the bounds of approximations the noise introduces. It is then necessary to have some ways to evaluate those bounds.
- When relaxing the constraints for the case when we only have a state well approximated by a low rank matrix, the norm two over matrices is used. One could wonder what would be the results if we were using another norm.

References

[BCP14] T. Baumgratz, M. Cramer, and M. B. Plenio. Quantifying coherence. *Phys. Rev. Lett.*, 113:140401, Sep 2014.

[CCS08] J.-F. Cai, E. J. Candes, and Z. Shen. A Singular Value Thresholding Algorithm for Matrix Completion. *ArXiv e-prints*, October 2008.

[FHB01] Maryam Fazel, Haitham Hindi, and Stephen P Boyd. A rank minimization heuristic with application to minimum order system approximation. In *American Control Conference, 2001. Proceedings of the 2001*, volume 6, pages 4734–4739. IEEE, 2001.

[GLF+10] David Gross, Yi-Kai Liu, Steven T. Flammia, Stephen Becker, and Jens Eisert. Quantum state tomography via compressed sensing. *Phys. Rev. Lett.*, 105:150401, Oct 2010.

[Gro11] D. Gross. Recovering low-rank matrices from few coefficients in any basis. *IEEE Transactions on Information Theory*, 57(3):1548–1566, March 2011.

[HHR+05] H. Haffner, W. Hansel, C. F. Roos, J. Benhelm, D. Chek-al kar, M. Chwalla, T. Korber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Guhne, W. Dur, and R. Blatt. Scalable multiparticle entanglement of trapped ions. *Nature*, 438, 2005.

[KGS+07] Marek Korkusinski, Louis Gaudreau, S STUDENIKIN, AS Sachrajda, and Pawel Hawrylak. Electron spin based qubits. *Physics in Canada*, 63(4):229, 2007.

[MP97] Mehran Mesbahi and George P Papavassilopoulos. On the rank minimization problem over a positive semidefinite linear matrix inequality. *IEEE Transactions on Automatic Control*, 42(2):239–243, 1997.

[OPW+03] J. L. O'Brien, G. J. Pryde, A. G. White, T. C. Ralph, and D. Branning. Demonstration of an all-optical quantum controlled-not gate. *Nature*, 426, 2003.

[Sch14] M. Schlosshauer. The quantum-to-classical transition and decoherence. *ArXiv e-prints*, April 2014.

[SXFL15] Lian-He Shao, Zhengjun Xi, Heng Fan, and Yongming Li. Fidelity and trace-norm distances for quantifying coherence. *Phys. Rev. A*, 91:042120, Apr 2015.

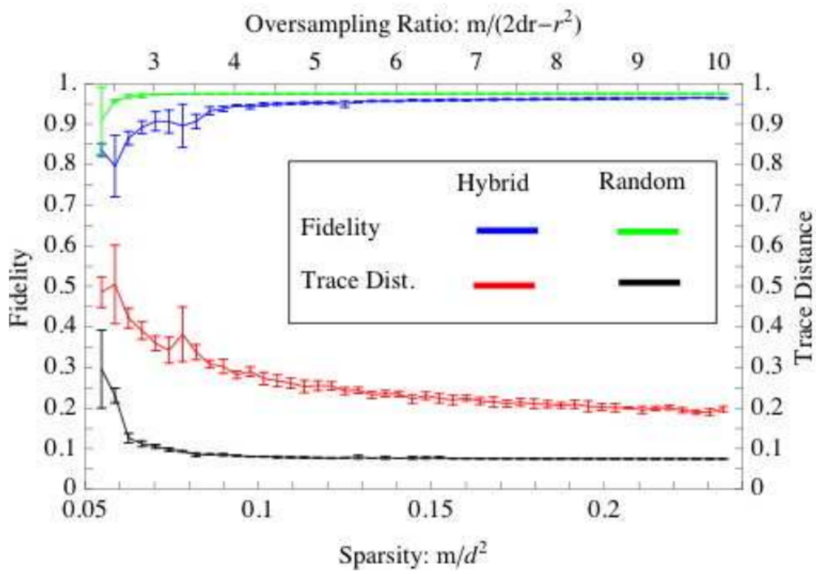

Figure 1: Fidelity and trace distance in fonction of the number of measurements – Image taken from the article [GLF+10]