Quantum chromatic number

Johanna SEIF

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Contents

Introduction

This report gather some notable results found in [SS12], giving also details on some foundation results presented in [CMN$^+$07]. The results covered in Section 2 are answers given in [SS12] to questions of the article [CMN$^+$07].

Graph parameters are studied a lot as they are a way to highlight a general structure from a graph. In that sense, the chromatic number is studied a lot as it shows the multipartitions. The chromatic number can be seen as the smallest number needed to color a graph, with respecting the condition that two adjacent vertices must not have the same color. But it can also be studied with the following protocol. We consider an integer c and a graph G. We denote $[c] = \{1, \dots, n\}$. A referee gives to Alice and Bob two vertices that are either the same one, or belong to one edge. Alice and Bob answer two colors in $[c]$ that should be the same if they were given the same vertex, and different in the other case. The chromatic number coincide with the smallest c for which Alice and Bob have a probability 1 winning strategy.

This game gives a way to define the quantum chromatic number. We consider the same game but this time Alice and Bob share an entangled state. The smallest c for which they win with probability 1 is the quantum chromatic number. This parameter is a way to see how different the quantum and the classical world are. Indeed it have been shown that for a class of graphs, called the Hadamard graphs, the quantum chromatic number is strictly smaller than the chromatic number: the article [DHKS06] gives a protocol that provides a winning strategy for some Hadamard graphs (defined in 2). This kind of game in which we have a separation between the quantum and the classical worlds are called pseudo-telepathy games because the reader choose either to believe in entanglement or in a form of telepathy between Alice and Bob. Therefore pseudo-telepathy games highlight the entanglement phenomenom (see [GWT02] for more details).

Another interesting thing about the quantum chromatic number is that it is linked to other graph parameters, as we will see in 2. Also, this graph parameter computation is hard. Therefore this new parameter is another way to obtain information on NP-hard problems.

1 Model and first properties

In this section, we present the formalization of the problem and give properties that help to understand the quantum chromatic number.

1.1 The coloring game

In the coloring game, Alice and Bob play against a referee. The entry is a graph $G = (V, E)$ and Alice and Bob can agree on a strategy before playing. The referee provides two vertices u and v to respectively Alice and Bob, with either $u = v$ or $uv \in E$. Alice and Bob have to answer respectively α and β in $[c]$. The winning conditions are the following: if $u = v$ they win iff $\alpha = \beta$; if $uv \in E$ they win if and only if $\alpha \neq \beta$. The classical chromatic number coincides with the smallest c for which Alice and Bob can win with probability 1. Indeed, if a graph is c-colorable, a coloration of the graph is a strategy to win with c colors. And any strategy with strictly less than $\chi(G)$ colors cannot win with probability 1 since 2 vertices belonging to the same edge will have the same color.

The quantum strategy uses an entangled state $|\psi\rangle$ of local dimension d ($|\psi\rangle \in \mathbb{C}^{d\times d}$). Alice and Bob both have, for each $v \in V$, a family of POVMs $\{E_{v\alpha}\}_{\alpha=1,\ldots,c}$ and $\{F_{v\beta}\}_{\beta=1,\ldots,c}$ respectively. The idea is that, when $u = v$, if Alice measures α on her part, then the probability that Bob measures $\beta \neq \alpha$ is 0, and vice versa. When $uv \in E$, if Alice measures α on her part, then the probability that Bob measures α is 0. This can be mathematically expressed with the following *consistency conditions*:

$$\forall v \in V, \forall \alpha \neq \beta, \langle\psi|E_{v\alpha} \otimes F_{v\beta}|\psi\rangle = 0 \tag{1}$$

$$\forall (u, v) \in E, \forall \alpha, \langle\psi|E_{u\alpha} \otimes F_{v\alpha}|\psi\rangle = 0 \tag{2}$$

If this conditions are fulfilled we have a quantum c-coloring of G.

Definition 1. For all graphs G, the quantum chromatic number $\chi_q(G)$ is the minimum number c such that there exists a quantum c-coloring coloring of G.

1.2 A more convenient model

This model can be narrowed by choosing certain types of POVMs and entangled states that will only make the chromatic number smaller. That is the aim of following proposition, first stated in [CMN$^+$07].

Proposition 1. *Without loss of generality the state $|\psi\rangle$ can be considered maximally entangled in a winning strategy. And there exists an optimal solution where the POVM elements are all projectors of the same rank r and for all $v \in V$, $\alpha \in [c]$, $E_{v,\alpha} = \overline{F_{v,\alpha}}$. And this choice can be made with $|\psi\rangle$ of dimension rc.*
This being given, the consistency conditions become:

$$\forall uv \in E, \forall \alpha \in [c],\ \langle E_{u\alpha}, F_{v\alpha}\rangle = 0 \tag{3}$$

Proof idea. Here we give mostly the qualitative ideas behind the proof and some step that make it easier to read.

- If $|\psi\rangle$ is not full Schmidt, we restrict the POVMs to the support of $|\psi\rangle$. We still have the consistency condition, and the entangled state has now full Schmidt rank.
- The main computation here lies on (which is a more general property on traces):

$$\begin{aligned} \mathrm{Tr}(E_{v\alpha}\,\mathrm{Tr}_B((I\otimes F_{v\beta})|\psi\rangle\langle\psi|)) &= \mathrm{Tr}((E_{v\alpha}\otimes I)(I\otimes F_{v\beta})|\psi\rangle\langle\psi|) \\ &= \mathrm{Tr}((E_{v\alpha}\otimes F_{v\beta})|\psi\rangle\langle\psi|) \\ &= \langle\psi|E_{v\alpha}\otimes F_{v\beta}|\psi\rangle \\ &= 0 \end{aligned}$$

  This gives that for all α, v and $\beta \neq \alpha$ $\langle E_{v\alpha}, \mathrm{Tr}_B(I \otimes F_{v\beta}|\psi\rangle\langle\psi|) = 0$. In other words, once one of the two players has made a measurement, it lowers the degrees of freedom of the other. And this is why $E_{v\alpha}$ can be seen as a projector on all the allowed values: this can only make the probability of winning higher. And this leads to $E_{v,\alpha} = \overline{F_{v,\alpha}}$: the winning strategy does not depend on the Schmidt coefficients of $|\psi\rangle$. Therefore we can assume that $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle|i\rangle$.

- To make all the projectors of the same rank, we consider $|\psi\rangle$ to be maximally entangled of dimension dc. Then the $E'_{v\alpha} = \sum_{i=0}^{c-1} E_{v,(\alpha+i) \mod c} \otimes |i\rangle\langle i|$ all have the same rank $r = d$ and still verify the conditions.
- Condition (3) follows from a direct computation after considering the previous points.

□

This justifies to consider the quantum chromatic number obtained when the rank of the POVMs is given.

Definition 2. The rank-r quantum chromatic number $\chi_q^{(r)}(G)$ of G is the minimum number of colors c such that G has a quantum c-coloring with projectors of rank r and a maximally entangled state of local dimension rc.

The bigger the entangled state, the more room for strategy Alice and Bob have. Therefore when $r \geqslant s$, $\chi_q^{(r)}(G) \leqslant \chi_q^{(s)}(G)$. Moreover we have:

$$\chi_q(G) = \inf_r \{\chi_q^{(r)}(G)\}$$

Starting from now we mainly focus on the rank-1 chromatic number. We will see that it is enough to observe a gap between the chromatic number and the quantum chromatic number. As we will see, the rank-1 chromatic number is closely linked to another graph parameter: the orthogonal representation parameter. Moreover it is quite powerful since in many cases the rank-1 chromatic number is equal to the quantum chromatic number, and studying it was enough to show the gap that can be obtained between classical and quantum parameter (see [CMN$^+$07]).

In this case the maximally entangled state has local dimension c and that the rank-1 projectors for each vertex v can be seen as the outer product $|a_{v\alpha}\rangle\langle a_{v\alpha}|_{\alpha\in[c]}$ of an orthonormal basis $(|a_{v\alpha}\rangle)_{\alpha\in[c]}$. Now consider U_v a matrix with column vectors $a_{v\alpha}$. Alice measures v by performing $U_v^\dagger$, and Bob measures w by performing $U_w^\dagger$. But for the rank-1 case, the consistency equation (3) becomes: $\forall uv \in E, \forall \alpha \in [c], \langle a_{u\alpha}, a_{v,\alpha}\rangle = 0$ i.e. the diagonal of $U_v^\dagger U_w$ is null.

This leads to the definition of a matrix representation ([CMN$^+$07]) of a graph which is a map $\Phi : V \to C^{c\times c}$ such that if $uv \in E$ the diagonal of $\Phi(u)^\dagger\Phi(v)$ is null.

1.3 Properties on the quantum chromatic number

A first question that arise is: does the quantum chromatic number behaves like a chromatic number in a graph?

Definition 3. Let G and H be two graphs.
A homomorphism φ from G to H is a vertex-mapping function that preserves edges i.e. if $uv \in V(G)$ then $\varphi(u)\varphi(v) \in V(H)$. We write $G \to H$.

Proposition 2. *If $G \to H$ then $\chi_q^{(r)}(G) \leqslant \chi_q^{(r)}(H)$ and therefore $\chi_q(G) \leqslant \chi_q(H)$.*

This is just because the strategy on G for a vertex u is to apply the strategy on H and answer the color of $\varphi(u)$.

Proposition 3. *Let G and H be two graphs.*
For all r, s we have $\chi_q^{(rs)}(G \cup H) \leqslant \chi_q^{(r)}(G)\chi_q^{(s)}(H)$.

These two results show similarities in the behavior of the quantum chromatic number and the chromatic number. But this does not mean that these two parameters are close in value. And in fact the gap can be huge.

The following proposition is an interesting special case that is used in 2 to find examples with strict inequalities between the different parameters.

Proposition 4. *[CMN+ 07] Let G be a graph.*
Then $\chi_q^{(1)}(G) = 3$ if and only if $\chi(G) = 3$.

2 Quantum chromatic number and orthogonal representation

2.1 The orthogonal representaion

The orthogonal representation provides a parameter that is closer to the quantum chromatic number that the classical chromatic number is.

Definition 4. Let $G = (V, E)$ be a graph.
A c-dimensional orthogonal representation of G is a map $\varphi : V \to \mathbb{C}^c$ such that for all $(v, w) \in E, \langle \varphi(v), \varphi(w) \rangle = 0$. The orthogonal rank of a graph G, denoted by $\xi(G)$, is defined as the minimum c such that there exists an orthogonal representation of G in $\mathbb{C}^c$.

The definition of the orthogonal representation echoes with the matrix representation, and we will see that they are somehow linked in the following part.

One can reciprocally define the orthogonal graph G_S associated to a set $S \subseteq \mathbb{C}^c$ as follows: the vertices are indexed by the vectors and two vertices are linked if their associated vectors are orthogonal.

Theorem 5. *[CMN+ 07] $\omega(G) \leqslant \xi(G) \leqslant \chi_q^{(1)}(G) \leqslant \chi(G)$.*

Proof. Let's give some details of the proof of this theorem.

- The inequality $\xi(G) \leqslant \chi_q^{(1)}(G)$ comes from the fact that adjacent vectors have orthogonal POVMs, so mapping a vertex to the first measurement element gives a valid orthogonal representation.
- A clique of size k in a graph can only be associated to k pairwise orthogonal vectors so dimension k is needed: $\omega(G) \leqslant \xi(G)$.

□

The results of [SS12], presented in the next session, give a broader view on these inequalities by answering if some are strict, and give conditions of equality for one of them. It focuses on $\xi(G) \leqslant \chi_q^{(1)}(G) \leqslant \chi(G)$. However the whole property is interesting since it shows the link between the quantum chromatic number and classical graph parameters. And this gives a hint on the fact that studying quantum graph parameters can be a manner to obtain results for classical parameters.

2.2 Expressing rank-1 chromatic number in term of orthogonal representation

Definition 5. Let G and H be two graph. We define the Cartesian product $G\square H$ as a graph with the following sets: $V(G\square H) = V(G) \times V(H)$; there is an edge between (u, i) and (v, j) if, either $u = v$ and $ij \in E(H)$, or $i = j$ and $uv \in E(G)$.

The fact that the POVMs can be taken as projectors tends to show that there is a link between the quantum chromatic number and the orthogonal representation.

Lemma 6. *For all graphs G, $\chi_q^{(1)}(G) = \min\{c : \xi(G\square K_c) = c\}$.*

Proof. The rank-1 chromatic associates to each vertex a family of pairwise orthogonal vectors. But in the orthogonal representation, we want the vectors of two vertices forming an edge to be different. That is why it seems natural to take the Cartesian product with K_c: it is as splitting each vertex into a clique in which one color is associated to one vertex. With this we can show that we can map a matrix representation of G in $\mathbb{C}^{c\times c}$ to an orthogonal representation of $\xi(G\square K_c)$ in $\mathbb{C}^c$. The rest of the proof follows from direct computation, using the mapping we have. □

Theorem 7. *For all graphs G,*

$$\chi_q^{(1)}(G) = \xi(G) \Leftrightarrow \xi(G\square K_{\xi(G)}) = \xi(G)$$

Proof. We have $\chi_q^{(1)}(G) = \xi(G) \Leftrightarrow \min\{c : \xi(G\square K_c) = c\} = \xi(G)$.

If $\min\{c : \xi(G\square K_c) = c\} = \xi(G)$ then $\xi(G\square K_{\xi(G)}) = \xi(G)$.

Now assume $\xi(G\square K_{\xi(G)}) = \xi(G)$: then $\xi(G) \geqslant \min\{c : \xi(G\square K_c) = c\}$.

We have $\xi(G\square K_c) \geqslant \max(\xi(G), \xi(K_c)) = \max(\xi(G), c)$ since $\xi(K_c) = c$.

So $\xi(G\square K_c) \geqslant \xi(G)$, for all c. Therefore $\min\{c : \xi(G\square K_c) = c\} \geqslant \xi(G)$.

We obtain $\min\{c : \xi(G\square K_c) = c\} = \xi(G)$.

□

2.3 Kochen-Specker sets

In quantum theory, hidden variables are a way to say that the randomness observed in quantum system can be explained deterministically, but is not because the models miss some variables. In [KS75a], Kochen and Specker proove the Kochen-Specker theorem using the effect of measurement on quantum variables. It states that there cannot exist non-contextual hidden variable model for quantum theory (non-contexual meaning that an observable is independent of the measurement context).

Mathematically, the Kochen-Specker theorem is linked to the notion of Kochen-Specker set.

Definition 6. A Kochen-Specker (KS) set S in $\mathbb{C}^n$ is a set such that there exists no function $f : \mathbb{C}^n \to \{0, 1\}$ that satisfies for all orthonormal bases $b \subseteq S, \sum_{u\in b} f(u) = 1$.

Lemma 8. *[KS75b, Per91] There exists KS sets in $\mathbb{C}^3$.*

Theorem 9. *There exists a graph such that $\xi(G) < \chi_q^{(1)}(G)$.*

Proof. To show this a good example suffice. The idea here is that visualizing a quantum chromatic number is harder than a chromatic number. So we want the computation of the chromatic number of the graph to give enough information to state the strictness of the inequality. To do so, the main results we need are Proposition 4 and Lemma 8.

We know that there exists a KS set S in $\mathbb{C}^3$. The orthogonal graph G_S is such that $\xi(G_S) = 3$. Also, $\xi(G) \leqslant \chi_q^{(1)}(G)$, so to show that the inequality is strict it is enough to show that $\chi_q^{(1)}(G_S) \neq 3$ i.e. $\chi(G_S) \neq 3$. In G_S an orthonormal basis forms a clique since vector are pairwise orthogonal. Therefore any color appears only one time in the subgraph induction by the orthonormal basis. So if the graph is three colorable, given a color 0, the function f that has value 1 on v of color 0 and 0 otherwise contradicts the fact that S is a KS set. $\square$

Definition 7. A weak Kochen-Specker set S in $\mathbb{C}^n$ is a set such that if there exists a function $f : \mathbb{C}^n \to \{0, 1\}$ that satisfies for all orthonormal bases $b \subseteq S, \sum_{u\in b} f(u) = 1$, then there exists an orthogonal unit vector $u, v \in S$ such that $f(u) = f(v) = 1$.

Theorem 10. *Let G be a graph. Then $\chi_q^{(1)}(G) < \chi(G)$ if and only if for all optimal strategy for the quantum coloring game $\{|a_{v\alpha}\rangle : v \in V, \alpha \in [c]\}$ is a weak KS set.*

Proof. The weak KS sets are a way to define a dependency between two different orthonormal basis (since at most one element in a base value 1). In term of coloring this dependency means that there exists two basis with u and v in each base that are neighbors (orthogonal unit vectors) but have the same color ($f(u) = f(v) = 1$).

This gives ($\Rightarrow$) as if $\chi_q^{(1)}(G) = c$ and $S = \{|a_{v\alpha}\rangle : v \in V, \alpha \in [c]\}$ is not a KS set so there exists a function $f : \mathbb{C}^n \to \{0, 1\}$ that satisfies, for all orthonormal bases $b \subseteq S, \sum_{u\in b} f(u) = 1$, and for all orthogonal unit vectors $u, v \in S$, $f(u) \neq f(v)$. By coloring v in color $\alpha iff f(a_{v\alpha})$ we obtain a valid c-coloring (since $|a_{v\alpha}\rangle$ and $|a_{v\beta}\rangle$ are orthogonal).

For ($\Leftarrow$), in the same way, a classic c-coloration defines a family projectors for each vertex v: $(|\alpha + i\rangle\langle\alpha + i|)_{i\in[c]}$ where α is the color of v. Then a function defined as in the proof of Theorem 9 shows that our set is not weak KS.

$\square$

Let's show that this property allows us to define KS sets, thanks to Hadamard graphs. A Hadamard graph G_N is a graph whose set of vertices is $\{0, 1\}^N$ and set of edges is the pair of vertices with Hamming distance less that $N/2$. Article [DHKS06] gives a proof that for $m \geqslant 3$, $\chi_q^{(1)}(G_{4m}) < \xi(G_{4m})$. So the strategy induces a weak KS set. But in [RW04], it is shown that a weak KS set of dimension d and size n induces a KS set of same dimension and size $O(n^2 d)$. So for $m \geqslant 3$, G_{4m} induces a KS set.

Conclusion

This article studied the relation of the quantum chromatic number with the chromatic number and the minimum dimension of an orthogonal representation. It gave a necessary and sufficient condition to have $\chi_q^{(1)}(G) < \chi(G)$ that stands on the KS sets. And thanks to the result on Hadamard graphs, this leads to the construction of a KS set.

This is interesting to so a link between the chromatic number and KS sets, that are used in a theorem about the physical possibilities of quantum computer science.

Some interesting questions arise from this article.

- The first thing is that we were able to express $\chi_q^{(1)}(G)$ as a Cartesian product. Now another question we can ask is if we can do the same for other rank chromatic numbers, and if this brings more information.
- Another question asked in the article is to find a way to build a graph for which $\chi_q^{(1)}(G) < \chi(G)$ from any KS set. This would be an inverse result to the one provide in the article.

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