# On the expressive power of planar perfect matching and permanents of bounded treewidth matrices

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# The permanent and hamiltonian polynomials

- The permanent : $\mathrm{per}(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma(i)}$.
  $S_n$ : all permutations.
- $\mathrm{ham}(X) = \sum_{\sigma \in HC_n} \prod_{i=1}^{n} X_{i\sigma(i)}$.
  $HC_n$ : permutations made of a single cycle.

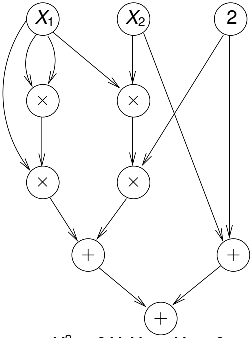
These polynomials are hard to evaluate :

- $\sharp\mathrm{P}$-complete in the boolean model ;
- VNP-complete in the real number model.

# The real-number model

Complexity of a polynomial $f$ measured by number $L(f)$ of arithmetic operations (+,-,$\times$) needed to evaluate $f$ :

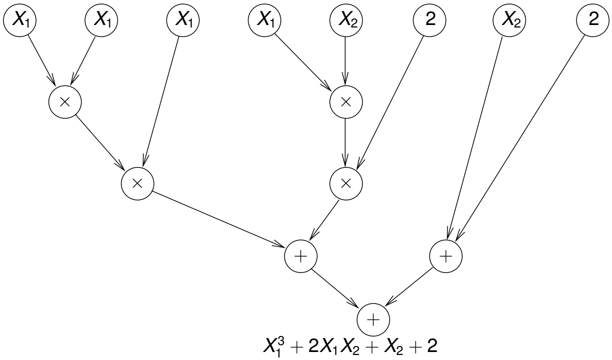
| L(f) = size of smallest arithmetic circuit computing $f$. |
|---|




$X_1^3 + 2X_1X_2 + X_2 + 2$

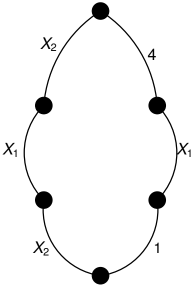
# Arithmetic formulas

A formula is a restricted kind of circuit :
only one outgoing edge for each gate
(intermediate results cannot be reused).



# Some easy special cases

- per, ham for matrices of bounded rank [Barvinok96].
- per, ham (and many other polynomials) for matrices of bounded treewidth [CourcelleMakowskyRotics01].
- sum of weights of perfect matchings for planar graphs [Kasteleyn67].
  **Remark :** for $G$ bipartite, $SPM(G) = \mathrm{per}(A_G)$ where $A_G$ is the adjacency matrix of $G$.



# Can these 3 results be compared ?

Bounded treewidth and bounded matrix rank are incomparable [CMR01].

We take a different point of view :
we will compare the *expressive power* of these methods.

# A recipe for polynomial evaluation

To evaluate $P$, write $P = per(A)$,
where $A$ is a matrix of bounded treewidth [CMR01].
The entries of $A$ are variables of $P$, or constants.

- What polynomials can be evaluated in this way ?
- What polynomials can be evaluated *efficiently* in this way ?

Same questions for the other "easy special cases" :
hamiltonian of bounded treewidth,
Kasteleyn's or Barvinok's algorithms...

# Possibly inefficient evaluation

Barvinok's method is universal for univariate polynomials.

**Theorem**

*Let $K$ be algebraically closed, and $P \in K[X]$ of degree $n$.*
*There is a matrix $A$ of rank 2 and size $2n$ such that $P = \mathrm{per}(A)$.*
*The entries of $A$ are in $K \cup \{X\}$.*

Result due to Saurabh Agrawal
(undergraduate student from IIT Kanpur visiting ENS Lyon).

The other methods (treewidth, planar perfect matching)
are universal for multivariate polynomials.

# Efficient evaluation for bounded treewidth

## Theorem (FKL07)

*Let $(f_n)$ be a family of polynomials with coefficients in a field $K$. The two following properties are equivalent :*

- *$(f_n)$ can be represented by a family of arithmetic formulas of polynomial size.*
- *There exists a family $(M_n)$ of matrices of polynomial size and bounded treewidth such that :*
    1. *the entries of $M_n$ are constants of $K$ or variables of $f_n$ ;*
    2. *$f_n = \mathrm{per}(M_n)$.*

Same result applies to ham polynomial.
Morally : treewidth-based methods $\Leftrightarrow$ arithmetic formulas.

# Efficient evaluation for planar perfect matchings

## Theorem (FKL07)

*Let $(f_n)$ be a family of polynomials with coefficients in a field $K$. The two following properties are equivalent.*

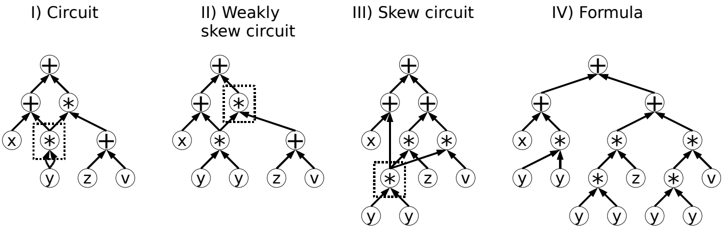
- $(f_n)$ *can be computed by a family of polynomial size* **weakly skew circuits.**
- *There exists a family $(G_n)$ of polynomial size planar graphs such that :*
  1. *The edges of $G_n$ are weighted by variables of $f_n$, or constants from $K$ ;*
  2. $f_n = SPM(G_n)$.

Morally : Kasteleyn's method $\Leftrightarrow$ weakly skew circuits.

# Weakly Skew Circuits

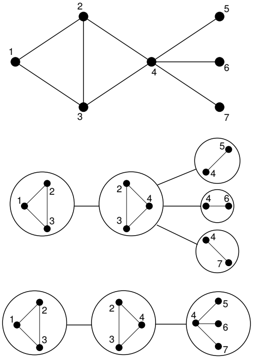
- Formulas, Skew Circuits $\subseteq$ WS Circuits.
  WS circuits are conjectured to be more powerful than formulas (example of the determinant).
  Therefore, Kasteleyn's method $>$ treewidth methods.
- Skew circuits are polynomially equivalent to WS circuits [Toda92,MalodPortier06].
  Circuit size increases by a constant factor only [KalftofenKoiran08].

The polynomial $x + y^2 + y^2 \cdot z + y^2 \cdot v$ represented in four ways :



# Tree decompositions and treewidth

Two tree decompositions for the same graph :



# Definition of treewidth [RobertsonSeymour84]

A *tree decomposition* of a graph $G = (V, E)$ is a tree $T$ where each vertex $i$ is labeled by $X_i \subseteq V$.
The following conditions must be satisfied :

- $\bigcup_{i \in I} X_i = V$ ;
- $\forall xy \in E, \exists i \in I$ such that $x, y \in X_i$ ;
- $\forall x \in V$ :
  $\{i \in I;\ x \in X_i\}$ is connected (i.e., it is a subtree of $T$).

The width of a decomposition is $\max_{i \in I} |X_i| - 1$.
The *treewidth* $tw(G)$ is the minimum width over all tree decompositions.

# From matrices to graphs

- To $n \times n$ matrix $M$ associate directed graph $G_M = ([n], E)$ where $(i, j) \in E$ if $M_{ij} \neq 0$. Weight on this edge is $M_{ij}$.
- Treewidth of $M$ is obtained from $G_M$
  by forgetting orientations (and self-loops).
- Graph-theoretic interpretations of $\mathrm{per}(M) = \sum_\sigma M_{i\sigma(i)}$ :
  1. undirected graphs :
     sum of weights of perfect matchings in bipartite graph.
  2. directed graphs :
     sum of weights of cycle covers of $G_M$
     (use cycle decomposition of $\sigma$).

# Evaluation of permanents of bounded treewidth

## Proposition (FKL07)

*Permanents of matrices of bounded treewidth can be expressed as formulas of polynomial size.*

**From [CMR01] : "The algorithms can also be parallelized so as to be in NC, but we shall not pursue this further."**

- We show that $\mathrm{per}(M)$ (or $\mathrm{ham}(M)$) can be evaluated by an arithmetic circuit of depth $O(\log n)$.
  Is this general ?
- For such a circuit there is an equivalent arithmetic formula of size $\mathrm{poly}(n)$.

# The parallel algorithm

**Theorem (Bodlaender 1988)**

*Let $G$ be a graph with $n$ vertices and treewidth at most $k$. There exists a binary tree decomposition of $G$ of width $3k+2$ and depth at most $2\lceil log_{\frac{5}{4}}(2n)\rceil$.*

We construct from this decomposition a log depth circuit using dynamic programming.
(Constant number of arithmetic operations at each node of the decomposition.)

# More details for the proof of a simpler result...

### Exercise (FlumGrohe, *Parameterized complexity theory*)

*Let $G$ be a graph with $n$ vertices and treewidth $k$.*
*Is there a Hamiltonian cycle in $G$ ?*
*This can be decided in time $2^{k^{O(1)}} \cdot n$.*

**Observation :** removing some vertices from Ham. cycle yields disjoint paths.
**Algorithm :** At each non-root node $t$ of the tree decomposition, compute all sets $\{(v_1, w_1), \ldots, (v_l, w_l)\}$ such that :

1. $v_i, w_i \in V(G)$ lie in $t$.
2. There exists disjoint paths $P_1, \ldots, P_l$ which use only nodes that lie in $t$, or below $t$.
3. All such nodes occur on one of the $P_i$.

At root, try to combine paths to form a Hamiltonian cycle.

# From arithmetic formulas to permanents of bounded treewidth.

## Proposition

*An arithmetic formula of size $n$ can be expressed as the permanent of a matrix of treewidth at most 2 and size at most $n+1$.*
*All entries of the matrix are 0, 1, variables or constants of the formula.*

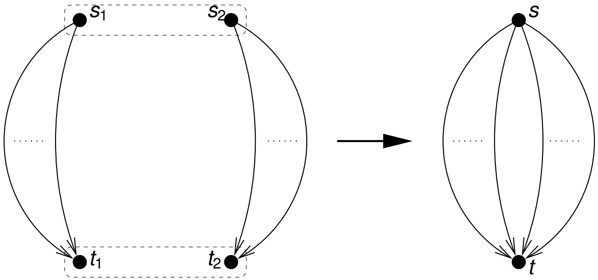
- Construction goes back to [Valiant79] (without treewidth bound).
- We check that Valiant's matrix has treewidth 2.
- Similar construction for the hamiltonian.

# Valiant's construction

1. Express the formula as the **sum of weights of paths** from $s$ to $t$ in a series-parallel graph $G_0$.
   - Addition gate simulation : parallel composition.
   - Multiplication gate simulation : serie composition.
2. To construct final graph $G$, add backward edge of weight 1 from $t$ to $s$ and loops on every vertex distinct from $s$ and $t$ : 1-1 correspondance between $st$-paths in $G_0$ and cycle covers in $G$.

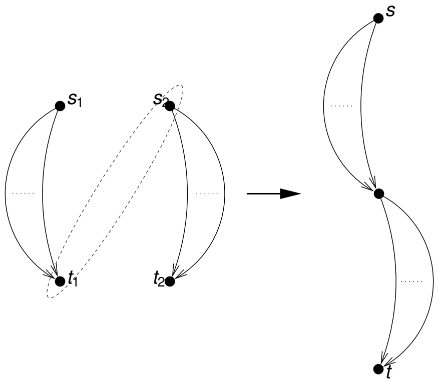
# Addition gate simulation : Parallel composition.

$$\phi = \phi_1 + \phi_2$$



# Multiplication gate simulation : Series composition.

$$\phi = \phi_1 \times \phi_2$$



# Treewidth and graph grammars

Graph grammars offer a bottom-up (constructive) definition of treewidth.

## Theorem (Courcelle)

*$G$ has treewidth $\leq k$ iff there is a set $\mathcal{S}$ of $\leq k+1$ vertex labels such that $G$ can be constructed from the following operations :*

(i) $\mathrm{ver}_a$, $\mathrm{edge}_{ab}$, $a, b \in \mathcal{S}$ *: basic constructs.*

(ii) $\mathrm{ren}_{a \leftrightarrow b}(H)$, $a, b \in \mathcal{S}$ *: rename operation.*

(iii) $\mathrm{forg}_a(H)$, $a \in \mathcal{S}$ *: forget all a labels.*

(iv) $H \mathbin{//} H'$ *: graph composition (any two vertices with same label are identified).*

# Further Work...

- Expressive power of permanents of bounded *pathwidth :* Flarup-Lyaudet, CSR 08.
- More on treewidth/cliquewidth in Klaus Meer's talk.
- Evaluation by arithmetic formulas :
  general result along the lines of [CMR01]
  (evaluation problems definable in monadic SO logic) ?
- Expressive power of Barvinok's method ?