# Multivariate multifractal analysis

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**Abstract**

We show how a joint multifractal analysis of a collection of signals unravels correlations between the locations of their pointwise singularities. The multivariate multifractal formalism, reformulated in the general setting supplied by multiresolution quantities, provides a framework which allows to estimate joint multifractal spectra. General results on joint multifractal spectra are derived, and illustrated by the theoretical derivation and practical estimation of the joint multifractal spectra of simple mathematical models, including correlated binomial cascades.

*Keywords:* multifractal analysis, wavelets, spatial regularity correlations, Hausdorff dimension.

## 1. Introduction

### 1.1. Pointwise regularity exponents

The main purpose of multifractal analysis is to make explicit the properties of the sets of points where a function $f$ has a given pointwise regularity, quantified by a regularity exponent $h_f(x_0)$, $x_0 \in \mathbb{R}^d$. The term *multifractal* refers to the fact that the sets $E_f(H)$ of points with same regularity,

$$\forall H \in \mathbb{R}^+ \cup \{+\infty\}, \qquad E_f(H) = \{x \in \mathbb{R}^d : \quad h_f(x) = H\}, \tag{1}$$

often are fractal. A relevant information on these sets is supplied by the *multifractal spectrum*

$$\mathcal{D}_f(H) = \dim(E_f(H)), \tag{2}$$

where dim denotes the Hausdorff dimension and, by convention, $\dim(\emptyset) = -\infty$. The *support* of the multifractal spectrum is $\{H : \ E_f(H) \neq \emptyset\} = \{H : \ \dim(E_f(H)) \neq -\infty\}$.

Different pointwise exponents $h_f(x_0)$ can be used: The most widespread is the Hölder exponent, cf., e.g., [1] and references therein; one recently studied notion for regularity are the $p$-exponents $h^p_f(x_0)$, where $p > 0$ is a parameter, which allow to measure the regularity of non-locally bounded functions, see [2, 3]. Two pointwise exponents fitted to a probability measure $\mu$ on $\mathbb{R}^d$ are the *lower and upper local dimensions*, defined as

$$\underline{H_\mu}(x) = \liminf_{\rho\to 0} \log \mu(B(x,\rho))/\log \rho \quad \text{ and } \quad \overline{H_\mu}(x) = \limsup_{\rho\to 0} \log \mu(B(x,\rho))/\log \rho, \tag{3}$$

respectively. Another category of pointwise exponents, termed *second generation exponents*, measures how regularity exponents change when a regularity parameter varies; they allow to characterize in a sharp way the behavior of the function near its singularities. One example is supplied by the *lacunarity exponent* $\mathcal{L}_f(x_0) = \frac{\partial}{\partial q}\left(h^{1/q}_f(x_0)\right)_{q=0}$, cf., [4]; see [5] for other examples.

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### 1.2. Multivariate multifractal analysis

The above framework permits the multifractal analysis for one single function and regularity exponent. *Multivariate multifractal analysis* deals with the simultaneous multifractal analysis of several pointwise exponents derived from one or several functions (or measures). The relevant information is then given by the sets of points where each of these exponents takes on a given value: If $h_{1,f_1}(x), \dots, h_{m,f_m}(x)$ are $m$ pointwise exponents, one considers the sets

$$E_{f_1,\dots,f_m}(H_1,\dots,H_m) = \{x: \quad h_{1,f_1}(x) = H_1, \dots, h_{m,f_m}(x) = H_m\} \tag{4}$$

and their *joint multifractal spectrum* is

$$D_{f_1,\dots,f_m}(H_1,\dots,H_m) = \dim(E_{f_1,\dots,f_m}(H_1,\dots,H_m)). \tag{5}$$

These notions were introduced by C. Meneveau *et al.* in the seminal paper [6] for the joint analysis of the dissipation rate of kinetic energy and passive scalar fluctuations for fully developed turbulence. A general abstract setting was proposed by J. Peyrière in [7]. Particular situations have also be explored, see, e.g., [8, 9] for a joint analysis of invariant measures of dynamical systems.

This work reformulates the multivariate multifractal formalism in the general setting supplied by multiresolution quantities and describes a framework which allows to estimate joint multifractal spectra (cf., Section 2). Indeed, on one hand, the wavelet characterizations of all previously mentioned exponents imply that they fit into this setting and, on other hand, this reformulation allows to extend all the recently introduced statistical methods of multifractal spectra estimation in this context, see, e.g., [10]. We derive general properties of joint multifractal spectra associated with several exponents; additionally, we work out examples illustrating how correlations between mathematical models are reflected in their joint multifractal spectra. In particular, we study pairs of binomial cascades where multifractal correlations can be "tuned", cf., Section 3.

## 2. Multivariate multifractal formalism

### 2.1. Multiresolution quantities

Numerically feasible ways to estimate multifractal spectra are based on the initial formulation of the multifractal formalism proposed by U. Frisch and G. Parisi in [11]. A key assumption is that the considered exponents can be derived from *multiresolution quantities*. Let $j \in \mathbb{Z}$ and $k = (k_1, \cdots k_d) \in \mathbb{Z}^d$, denote by $\lambda$ $(= \lambda(j,k))$ the dyadic cube $\left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right)$ and by $3\lambda$ the cube of same center and three times wider. We denote by $\Lambda_j$ the collection of dyadic cubes of width $2^{-j}$ and by $\Lambda$ the collection of all dyadic cubes. Further, for $x \in \mathbb{R}^d$, $\lambda_j(x)$ denotes the dyadic cube of width $2^{-j}$ which contains $x$. A multiresolution quantity is a nonnegative sequence $(d_\lambda)_{\lambda\in\Lambda}$; it is hierachical if it satisfies $\lambda' \subset \lambda \Longrightarrow d_{\lambda'} \le d_\lambda$. A pointwise exponent $h(x)$ is *admissible* if it can be recovered from a multiresolution quantity by

$$\forall x \in \mathbb{R}^d, \qquad h(x) = \liminf_{j\to+\infty} \ \log\left(d_{3\lambda_j(x)}\right)/\log(2^{-j}). \tag{6}$$

A simple example of this situation is supplied by probability measures for which $d_\lambda = \mu(3\lambda)$ clearly is a multiresolution quantity associated with $H_\mu$ defined by (3). Multiresolution quantities that yield the Hölder and $p$-exponents of a function $f$ can be derived from the wavelet decomposition of $f$, see [1, 2, 3]. Recall that an orthonormal wavelet basis on $\mathbb{R}^d$ is generated by a function $\varphi$ and $2^d - 1$ functions $\psi^{(i)}$, which are either in the Schwartz class, or compactly supported and smooth enough, and are such that $\varphi(x-k)$ (for $k \in \mathbb{Z}$) together with $2^{dj/2}\psi^{(i)}(2^j x - k)$, (for $j \ge 0$, and $k \in \mathbb{Z}^d$) form an orthonormal basis of $L^2(\mathbb{R}^d)$; the wavelet coefficients of $f$ are defined as $c_\lambda^{(i)} = 2^{dj}\int \psi^{(i)}(2^j x - k) f(x)dx$. The *wavelet leaders* and the *p-leaders* of $f$ are $d_\lambda = \sup_{i,\lambda'\subset 3\lambda} |c_{\lambda'}|$ and $d_\lambda^p = \left(\sum_{i,\lambda'\subset 3\lambda} |c_{\lambda'}^{(i)}|^p 2^{-(j'-j)}\right)^{1/p}$. Under weak global regularity assumptions, they are multiresolution quantities associated with the Hölder and the $p$-exponent, respectively. For multiresolution quantities for second generation exponents, see [4, 5].

*2.2. Derivation of the multivariate multifractal formalism*

Suppose that $m$ exponents $h_1(x), \cdots, h_m(x)$ are given, which can be pointwise exponents of one or of several functions or measures (we drop the dependency on $f$ in the notation); we assume that each of these exponents is admissible, and thus can be derived from a corresponding multiresolution quantity $d^i_\lambda$, $i = 1, \cdots, m$. A *grandcanonical multifractal formalism* allows to estimate the joint spectrum $D(H_1, \cdots H_m)$ of the collection of exponents $h_1(x), \cdots, h_m(x)$ as proposed in [6]. In the general setting provided by multiresolution quantities, it is derived as follows.

The *multivariate structure functions* associated with the $m$-tuple $(d^1_\lambda, \cdots d^m_\lambda)$ are defined by

$$\forall r = (r_1, \cdots, r_m) \in \mathbb{R}^m, \qquad S_f(r, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d^1_\lambda|^{r_1} \ldots |d^m_\lambda|^{r_m}. \tag{7}$$

The corresponding *scaling function* is

$$\eta(r) = \liminf_{j \to +\infty} \ \log\left(S_f(r, j)\right) / \log(2^{-j}). \tag{8}$$

The *joint Legendre spectrum* is obtained through a several-variable Legendre transform

$$\forall H = (H_1, \cdots H_m) \in \mathbb{R}^m, \qquad \mathcal{L}(H) = \inf_{r \in \mathbb{R}^m} \left(d - \eta(r) + H \cdot r\right), \tag{9}$$

where $H \cdot r$ denotes the usual scalar product in $\mathbb{R}^m$. Apart form [6], this formalism has been investigated in a wavelet framework for joint Hölder and oscillation exponents in [12], in an abstract general framework in [7] and on wavelet leader and $p$-leader based quantities in [4, 5]. An application to financial data, also based on wavelet leaders, has been worked out in [13].

*2.3. General properties of the multivariate multifractal formalism*

We derive relationships between the joint spectrum and the corresponding marginal (i.e., one variable) spectra. For the sake of clarity, the arguments are developed in 2 variables; extensions to more variables are straightforward. Consider admissible exponents $h_i(x)$ with associated sets $E_i(H_i)$ and marginal spectra $D_i(H_i)$, $i = 1, 2$, and with joint spectrum $D(H_1, H_2)$. Thus

$$E_1(H_1) = \bigcup_{H_2} E_{H_1, H_2} \quad \text{ and } \quad E_2(H_2) = \bigcup_{H_1} E_{H_1, H_2},$$

so that

$$D_1(H_1) \geq \sup_{H_2} D(H_1, H_2) \quad \text{ and } \quad D_2(H_2) \geq \sup_{H_1} D(H_1, H_2).$$

In general, equality needs not hold, because the supremum usually is taken on a non-countable set. However, the following result shows that equality holds for the Legendre spectra.

**Proposition 2.1.** *Two-variable Legendre spectra associated with admissible exponents satisfy*

$$\mathcal{L}_1(H_1) = \sup_{H_2} \mathcal{L}(H_1, H_2) \quad \textit{and} \quad \mathcal{L}_2(H_2) = \sup_{H_1} \mathcal{L}(H_1, H_2).$$

**Proof.** Since (with obvious notations) $S(r_1, r_2, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d^1_\lambda|^{r_1} |d^2_\lambda|^{r_2}$, it follows that $\eta_1(r_1) = \eta(r_1, 0)$ and $\eta_2(r_2) = \eta(0, r_2)$. We can rewrite the 2-variable Legendre spectrum

$$\mathcal{L}(H_1, H_2) = \inf_{r_1} \left( \inf_{r_2} (d - (\eta(r_1, r_2)) + H_2 r_2) + H_1 r_1 \right).$$

For a fixed value of $r_1$, the infimum in $r_2$ is a one-variable Legendre transform in the $H_2$ variable, and the value of its supremum is attained when this infimum is attained for $r_2 = 0$, see e.g., [14]. It follows that

$$\sup_{H_2} \mathcal{L}(H_1, H_2) = \inf_{r_1} \left(d - \eta(r_1, 0) + H_1 r_1\right) = \inf_{r_1} \left(d - \eta_1(r_1) + H_1 r_1\right) = \mathcal{L}_1(H_1). \qquad \square$$

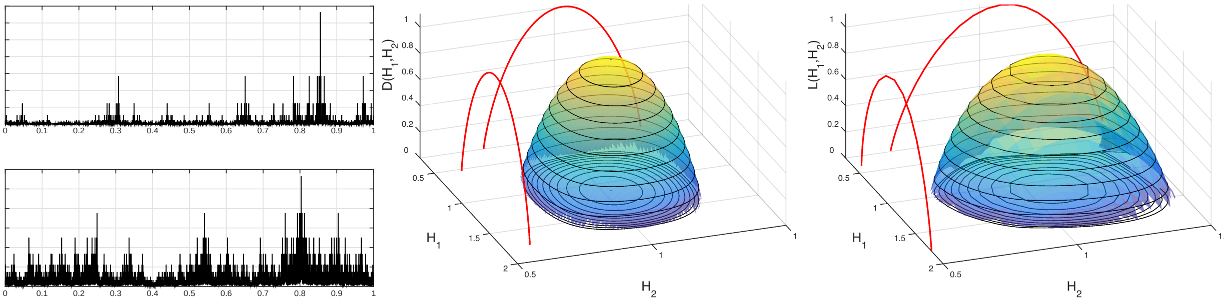


Figure 1: **Independent stochastic binomial measures.** Time series (left), theoretical multivariate spectrum (center) and estimated multivariate spectrum (right; red solid lines show the 1-variable marginal spectra).

### 2.4. *Independent signals*

The above general results did not require assumptions on correlations between the exponents $h_1$ and $h_2$. We now investigate the implications of such correlations on the joint spectrum. If the data from which the exponents are derived are independent, generic formulas for the intersection of the corresponding Hausdorff dimensions apply and, generically, codimensions add up [15], i.e.

$$D(H_1, H_2) = D(H_1) + D(H_2) - d; \tag{10}$$

note that the derivation of the corresponding formula for Legendre spectra is obtained in [6] under an independence assumption, and using the ergodic interpretation of structure functions as moments of identically distributed random variables. This is illustrated in Fig. 1 for two independent binomial cascades with parameters $0 < p, p' < \frac{1}{2}$, respectively. Binomial cascades are simple examples of measures supported by $[0, 1]$ which are obtained by a recursive construction on the dyadic intervals (defined in Section 3). Here, the weight $p$ (resp. $p'$) randomly multiplies with equiprobability, and independently for each cascade, either the left or right subinterval $\lambda_l$ or $\lambda_r$ for each dyadic interval, yielding independent signals.

### 2.5. *Fully dependent spectra*

A deterministic relationship between exponents (e.g., if there exists a function $\theta$ such that $\forall x$, $h_2(x) = \theta(h_1(x))$ clearly leads to spectra supported by the curve $H_2 = \theta(H_1)$. We now show how to construct pairs of functions satisfying this property in the setting supplied by the Hölder exponent.

**Proposition 2.2.** *Let $f \in C^\varepsilon$ for an $\varepsilon > 0$, and denote by $c_\lambda$ its wavelet coefficients and by $d_\lambda$ its wavelet leaders. Let $\theta : \mathbb{R}^{*,+} \to \mathbb{R}^{*,+}$ be a continuous increasing function.*

$$\textit{If} \quad \nu_\lambda = \log(d_\lambda)/\log(2^{-j}) \quad \textit{let} \quad e_\lambda = 2^{-\theta(\nu_\lambda)j}. \tag{11}$$

*The function $g$ whose wavelet coefficients are the $e_\lambda$ satisfies*

$$\forall x \in \mathbb{R}^d, \qquad h_g(x) = \theta(h_f(x)). \tag{12}$$

**Proof.** First note that $g \in C^{\varepsilon'}$ for an $\varepsilon' > 0$ because of the assumptions on $\theta$, so that (6) holds for $g$; $e_\lambda$ is hierarchical because $\theta$ is increasing. One can interpret (11) as stating that, if $h$ is such that $d_\lambda = 2^{-hj}$, then $e_\lambda = 2^{-\theta(h)j}$, so that (12) follows from (6) and (11). □

This construction yields the following joint spectrum for the couple $(f, g)$:

$$D(H_1, H_2) = D(H_1) 1_{H_2 = \theta(H_1)}(H_1, H_2). \tag{13}$$

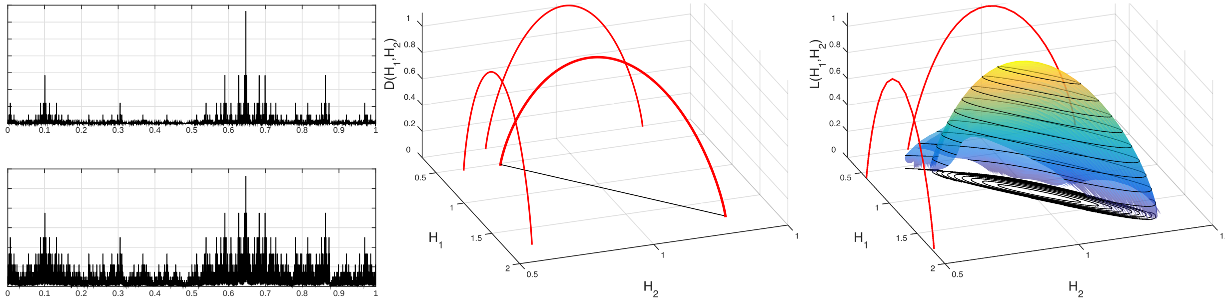


Figure 2: **Fully dependent stochastic binomial measures.** Time series (left), theoretical multivariate spectrum (center) and estimated multivariate spectrum (right; red solid lines show the 1-variable marginal spectra).

**Remark:** *The scaling function for functions satisfying* (12) *is derived in [6]. Yet when $\theta(H)$ is not affine, $D(H_1, H_2)$ needs not be concave and cannot be obtained as its Legendre transform* (9) *(see however [16] for a numerical procedure inspired by the multifractal formalism, which allows to estimate non-concave spectra).*

Fig. 2 illustrates a situation similar to Proposition 2.2 and shows the joint analysis of two binomial cascades as in Sec. 2.4, yet when choosing $p$ for a subinterval on one cascade implies the choice $p'$ for this subinterval on the other. This leads to an affine relationship between exponents:

$$H_1(s) = \big(s\log p + (1-s)\log(1-p)\big)/\log 2, \quad H_2(s) = -\big((s\log p' + (1-s)\log(1-p')\big)/-\log 2.$$

It follows that the joint spectrum is supported by a segment parametrized by $s \in [0,1]$; the numerically estimated spectrum is supported by a very thin ellipsoid-type area.

Several other subcases or variants of the construction in Proposition 2.2 can be mentioned:

• The joint multifractal spectrum of a function $f$ and its fractional integral of order $s$, denoted by $f^{(-s)}$ yields an important information on the nature of the singularities of $f$. Indeed, if $f$ only has *cusp singularities* (i.e. the Hölder exponent of $f^{(-s)}$ satisfies $\forall x_0$, $h_{f^{(-s)}}(x_0) = h_f(x_0) + s$ [4]), then this joint spectrum is supported by the line $H_2 = H_1 + s$. A joint spectrum which is not supported by this line is the signature of *oscillating singularities* in the data. A typical example is supplied by lacunary wavelet series, see [17] (at scale $j$, a proportion of $2^{(\eta-1)j}$ locations are drawn at random on $2^{-j}\mathbb{Z}$, with wavelet coefficients of size $2^{-\alpha j}$; all other coefficients vanish). An immediate verification shows that this model yields a joint spectrum for $(f, f^{(-s)})$ which is supported by a straight line of ends $(\alpha, \alpha+s)$ and $(\alpha/\eta, (\alpha+s)/\eta)$ where $D(H_1, H_2) = H_1\eta/\alpha$. Another example, the Riemann function $R(x) = \sum \sin(\pi n^2 x)/n^2$, leads to a joint spectrum of $(R, R^{(-s)})$ that is supported by the union of a segment where $H_2 = H_1 + s$ and $1/2 \le H_1 \le 3/4$ (where $D(H_1, H_2) = 4H_1 - 2$) and of the point $(3/2, 3/2 + 2s)$ (where $D(H_1, H_2) = 0$); this additional point corresponds to the chirps at rationals, see [18].

• Let $\mu$ be a multifractal probability measure on $[0,1]$ and $f$ its distribution function. The joint spectrum of the couple $(B_{\theta_1} \circ f, B_{\theta_2} \circ f)$, where $B_{\theta_1}$ and $B_{\theta_2}$ are two independent fractional Brownian motions (FBM) of exponent $\theta_1$ and $\theta_2$, respectively, is carried by the line $\theta_1 H_2 = \theta_2 H_1$ where $D(H_1, H_2) = f(H_1/\theta_1)$. This is because the effect of applying an FBM of exponent $\theta$ to an increasing function $f$ is to multiply everywhere the Hölder exponent of $f$ by $\theta$.

## 3. Joint multifractal spectra of correlated binomial cascades

The situations considered in Sections 2.4 and 2.5 are extreme cases corresponding to perfect correlation vs. independence. Joint spectra which are intermediate between these two situations are the signature of *multifractal correlations* between signals. Here, we consider the example of pairs of multiplicative cascades whose correlation can be tuned. The model is similar to the one considered in [6], where the corresponding scaling function was derived. We complement this result by determining pointwise exponents everywhere and deducing the joint multifractal spectrum.

The binomial measure is a probability measure constructed iteratively on the dyadic intervals. A standard way to index dyadic intervals is by "words" (i.e., finite sequences) with letters $\varepsilon_i \in \{0,1\}$: Let $\Sigma_j = \{0,1\}^j$, $j \geq 1$, denote the set of words $w = \varepsilon_1\varepsilon_2 \cdots \varepsilon_j$ of length $|w| = j$. The corresponding dyadic interval of generation $j$ is $\lambda_w = \left[\sum_{k=1}^{j} \varepsilon_k 2^{-k}, \sum_{k=1}^{j} \varepsilon_k 2^{-k} + 2^{-j}\right)$. We associate to an infinite word $w = \varepsilon_1\varepsilon_2 \cdots$ the real number $x_w = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$; this defines a mapping between the set $\Sigma = \{0,1\}^{\mathbb{N}^*}$ of infinite words and $[0,1]$. If $w$ is a (possibly infinite) word longer than $j$, we define $w_{|j} = \varepsilon_1 \cdots \varepsilon_j$. Finally, if $w$ is a word of length at least $j$, we define

$$N_j^0(w) = \#\{\varepsilon_i : \ 1 \leq i \leq j \text{ and } \varepsilon_i = 0\} \in \{0,1,...,j\},$$

i.e., the number of zeros in the first $j$ digits of $w$. If $x \in [0,1]$ is not dyadic, then $N_j^0(x) = N_j^0(x_w)$.

Let $p \in [0,1]$. The binomial measure $\mu_p$ is constructed iteratively as follows: $\mu_p([0,1]) = 1$; if $\lambda$ is a dyadic interval, denote by $\lambda_l$ and $\lambda_r$ respectively its left and right half; then $\mu_p(\lambda_l) = p\mu_p(\lambda)$ and $\mu_p(\lambda_r) = (1-p)\mu_p(\lambda)$; $\mu_p$ is thus defined on all dyadic intervals and extends to Borel sets of $[0,1]$. Note that $\mu_{1/2}$ is the Lebesgue measure and $\mu_0, \mu_1$ are Dirac masses. If $w$ has length $j$, then

$$\mu_p(\lambda_w) = p^{N_j^0(w)}(1-p)^{j-N_j^0(w)}. \tag{14}$$

**Remark:** *Let $\mu$ be a measure; as a consequence of (6), the wavelet series $F_\mu$ whose wavelet coefficients are $c_\lambda = \mu(\lambda)$ has the same pointwise Hölder exponent as $\mu$ at every $x \in [0,1]$. So, we can work equivalently with binomial measures $\mu_p$ or with $F_{\mu_p}$. More generally, we can also pick for wavelet coefficients $c_\lambda = 2^{-\alpha j}\mu(\lambda)^s$. If $\alpha$ and $s$ are nonnegative, then the wavelet coefficients form a hierarchical sequence, and the relationship between the lower dimension of the measure $\mu$ and the Hölder exponent of the associated function $f$ is given for any $x$ by $h_f(x) = \alpha + \beta h_\mu(x)$.*

An easy computation yields that the scaling function of $\mu_p$ is $\eta_{\mu_p}(r) = -\log_2(p^r + (1-p)^r)$. The following theorem lists known multifractal properties for one cascade $\mu_p$ needed below, see [19]. For $p < 1/2$, let $H_{p,\min} = -\log_2(1-p)$, $H_{p,m} = (-\log_2(p) - \log_2(1-p))/2$ and $H_{p,\max} = -\log_2(p)$.

**Theorem 1.** *The multifractal spectrum of $\mu_p$ is the Legendre transform of $\eta_{\mu_p}$, $D_{\mu_p}(H) = \inf_{r\in\mathbb{R}}(1 - \eta_{\mu_p}(r) + rH)$. Furthermore, for every $H, H' \in [H_{p,\min}, H_{p,m}]$, if $H \leq H'$ then*

$$\dim\{x : H_{\mu_p}(x) = H, \overline{H_{\mu_p}}(x) = H'\} = \dim\{x : H_{\mu_p}(x) \leq H, \overline{H_{\mu_p}}(x) \leq H'\} = D_{\mu_p}(H').$$

**Remark:** *It follows that the support of $D_{\mu_p}(H)$ is $[H_{p,min}, H_{p,\max}]$; $D_{\mu_p}$ is increasing on the interval $[H_{p,\min}, H_{p,m}]$, its maximum is reached at $H_{p,m}$ and it is decreasing on $[H_{p,m}, H_{p,\max}]$. The same $D_{\mu_p}(H)$ is obtained when $p$ multiplies randomly either the left or right subinterval [20].*

The following result gives the joint spectrum of $(\mu_p, \mu_q)$ for correlated $(p, q < 1/2)$ and anti-correlated $(p < 1/2 < q)$ cascades, respectively. We define the affine function

$$G_{p,q}(x) = (x + \log_2(1-p))\frac{-\log_2(q) + \log_2(1-q)}{-\log_2(p) + \log_2(1-p)} - \log_2(1-q);$$

$G_{p,q}$ has positive slope when $p, q < 1/2$, and negative slope when $p < 1/2 < q$.

**Theorem 2.** *Let $p, q \in (0,1)$.*

- **(a)** *If $0 < p, q < 1/2$, $\forall x \in [0,1]$, $H_{\mu_q}(x) = G_{p,q}(H_{\mu_p}(x))$, so that the joint spectrum $D_{(\mu_p,\mu_q)}$ of $\mu_p$ and $\mu_q$ is supported by the segment $\{(H, G_{p,q}(H)) : H \in [H_{p,\min}, H_{p,\max}]\}$ where*

$$D_{(\mu_p,\mu_q)}(H, G_{p,q}(H)) = D_{\mu_p}(H) \ (= D_{\mu_q}(G_{p,q}(H)). \tag{15}$$

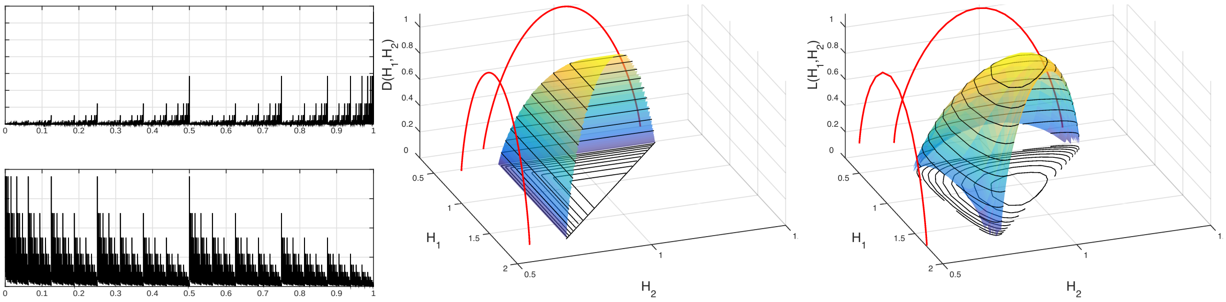


Figure 3: **Anti-correlated binomial measures.** Time series (left), theoretical multivariate spectrum (center), estimated multivariate spectrum and projections (right), illustrating Theorem 2 when $p < 1/2 < q$.

**(b)** *If* $0 < p < 1/2 < q < 1$*, then the joint spectrum* $D_{(\mu_p,\mu_q)}$ *is supported by the triangle* $H \in [H_{p,\min}, H_{p,\max}]$*,* $H' \in [H_{q,\min}, G_{p,q}(H)]$ *where*

$$D_{(\mu_p,\mu_q)}(H,H') = \min(D_{\mu_p}(H), D_{\mu_q}(H')).$$

The situation (b) (anti-correlated cascades) is illustrated in Fig. 3.

**Proof of Theorem 2 (a).** Let $\lambda_j(x)^{\pm} = \lambda_j(x) \pm 2^{-j}$ denote the left and right dyadic neighbor intervals of $\lambda_j(x)$ of scale $j$. The definition of the local dimension of a measure and (6) yield

$$\forall x, \qquad H_{\mu_p}(x) = \liminf_{j\to+\infty} \log_2(\max(\mu_p(\lambda_j(x)), \mu_p(\lambda_j(x))^+, \mu_p(\lambda_j(x)^-)))/j$$

and the same holds for $\overline{H_{\mu_p}}(x)$ by replacing the liminf by a limsup.

Let $N_j^{0,+}(x)$ denote the number of 0's in the unique word $w^+ \in \Sigma_j$ such that $\lambda_{w^+} = \lambda_j^+(x)$, and the analog definition for $N_j^{0,-}(x)$. It follows from (14) that

$$\begin{aligned} H_{\mu_p}(x) = \liminf_{j\to+\infty} \max\Big(&-\big(\log_2(p)N_j^0(x) + \log_2(1-p)(j - N_j^0(x))\big)/j, \\ &-\big(\log_2(p)N_j^{0,+}(x) + \log_2(1-p)(j - N_j^{0,+}(x))\big)/j, \\ &-\big(\log_2(p)N_j^{0,-}(x) + \log_2(1-p)(j - N_j^{0,-}(x))\big)/j\Big). \end{aligned} \tag{16}$$

Observe that $-\log_2(p), -\log_2(1-p) > 0$, but the way they are ordered depends on $p$: if $p \in (0, 1/2)$, $0 < -\log_2(1-p) < 1 < -\log_2(p)$, and the converse inequality holds when $p \in (1/2, 1)$.

Without loss of generality, we can assume from now on that $0 < p < 1/2$.
Since $0 < -\log_2(1-p) < 1 < -\log_2(p)$, the maximum in (16) is given by $-\big(\log_2(p)N^0_{j,\max}(x) + \log_2(1-p)(j - N^0_{j,\max}(x))\big)/j$, where $N^0_{j,\max}(x) = \max(N_j^0(x), N_j^{0,+}(x), N_j^{0,-}(x))$, hence

$$H_{\mu_p}(x) = -\log_2(1-p) + \big(-\log_2(p) + \log_2(1-p)\big) \liminf_{j\to+\infty} (N^0_{j,\max}/j). \tag{17}$$

Since (17) also holds for $\mu_q$ when $0 < p, q < 1/2$, $H_{\mu_q}(x) = G_{p,q}(H_{\mu_p}(x))$ and (15) follows. $\square$

We now consider the case where $q \in (1/2, 1)$. Then $0 < -\log_2(q) < 1 < -\log_2(1-q)$, and the maximum in (16) is now given by $-\Big(\log_2(q)N^0_{j,\min}(x) + \log_2(1-q)(j - N^0_{j,\min}(x))\Big)/j$, where $N^0_{j,\min}(x) = \min(N_j^0(x), N_j^{0,+}(x), N_j^{0,-}(x))$. For this case, (16) can be rewritten

$$H_{\mu_q}(x) = -\log_2(1-q) + \big(-\log_2(q) + \log_2(1-q)\big) \limsup_{j\to+\infty} (N^0_{j,\min}(x)/j). \tag{18}$$

Let us introduce the quantities

$$\underline{N^0_{\max}}(x) = \liminf_{j\to+\infty} N^0_{j,\max}(x)/j \qquad \underline{N^0_{\min}}(x) = \liminf_{j\to+\infty} N^0_{j,\min}(x)/j$$
$$\overline{N^0_{\max}}(x) = \limsup_{j\to+\infty} N^0_{j,\max}(x)/j \qquad \overline{N^0_{\min}}(x) = \limsup_{j\to+\infty} N^0_{j,\min}(x)/j.$$

When $\underline{N^0_{\max}}(x) = \overline{N^0_{\min}}(x) = H$, one has $H_{\mu_q}(x) = G_{p,q}(H_{\mu_p}(x))$, but now $G_{p,q}$ is affine with a negative slope. This situation is the most current one, as shown by the following result of [14, 21]

**Lemma 3.1.** *For every $H \in [H_{\min}, H_{\max}]$, $\dim\{x : \underline{N^0_{\max}}(x) = \overline{N^0_{\min}}(x) = H\} = D_\mu(H)$.*

In general, there is no equality. However, the following holds.

**Lemma 3.2.** *For every $x \in [0,1]$ , $\underline{N^0_{\max}}(x) \le \overline{N^0_{\min}}(x)$.*

**Proof.** If $x$ is a dyadic number, the word $w$ associated with $x$ is eventually constituted only by 0's. So the statement is immediate, since $N^0_{j,\max}(x) \sim N^0_{j,\min}(x) \sim j$ when $j$ tends to infinity.

For $x$ not a dyadic number, let $\alpha = \underline{N^0_{\max}}(x) \in [0,1]$. When $\alpha = 0$, the statement is obvious. So we can assume that $\alpha > 0$. By definition, given $\varepsilon > 0$ small enough,

$$\exists J_\varepsilon : \forall j \ge J_\varepsilon, \quad N^0_{j,\max}(x) \ge j(\alpha - \varepsilon). \tag{19}$$

Let $w$ denote the unique (infinite) dyadic word that encodes $x$. Assume that $\overline{N^0_{\min}}(x) < \underline{N^0_{\max}}(x) - 2\varepsilon = \alpha - 2\varepsilon$. This implies that for every $j \ge J_\varepsilon$

$$N^0_{j,\min}(x) < j(\alpha - 2\varepsilon). \tag{20}$$

**1.** Assume that $\lambda_j(x) \cup \lambda_j^+(x)$ forms a dyadic interval of generation $j-1$. Hence, if $w_j^+$ stands for the unique word of length $j$ that encodes $\lambda_j^+(x)$, then $w_{|j}$ and $w_j^+$ differ only by their last digit, which is 0 for $w_{|j}$ and 1 for $w_j^+$. In particular, $N_j^{0,+}(x) = N_j^0(x) + 1$, so $N_j^{0,+}(x)/j$ and $N_j^0(x)/j$ have approximately the same value (since $j$ is large). This also implies that $N^0_{j,\min}(x) \ne N_j^{0,+}(x)$.

Denote by $w_j^-$ the unique word of length $j$ that encodes $\lambda_j^-(x)$. Since the union $\lambda_j(x) \cup \lambda_j^+(x)$ is a dyadic interval of generation $j-1$, $N_j^{0,-}(x) \le N_j^0(x)$. As a conclusion, $N^0_{j,\min}(x) = N_j^{0,-}(x)$.

Denote by $J$, the length of the largest common prefix between $w^-$ and $w$. Obviously, $J < j$, and since $\lambda_j^-(x), \lambda_j(x), \lambda_j^+(x)$ are neighbors, the digits of $w^-$ located between the $J+1$-th digit and the $j$-th digit are only 1's, while the digits of $w$ at the same positions are only 0's.

Let us consider the dyadic intervals $\lambda_m^-(x), \lambda_m(x)$, and $\lambda_m^+(x)$ for $J < m < j$. First, $\lambda_m(x)$ and $\lambda_m^+(x)$ are always the two subintervals of the same dyadic interval $\lambda_{m-1}(x)$, so $N_m^{0,+}(x) = N_m^0(x)+1$. Also, $\lambda_{J+1}(x)$ and $\lambda_{J+1}^-(x)$ belong to the same dyadic interval $\lambda_J(x)$, so $N_{J+1}^{0,-}(x) = N_{J+1}^0(x) - 1$. From the arguments above, $N_{J+1}^{0,-}(x) = N_{J+1}^0(x) - 1 = N_{J+1}^{0,+}(x) - 2$. Thus, if $J \ge 4/\varepsilon$, then, using (19), $\frac{N_{J+1}^{0,-}(x)}{J+1} \ge \frac{N_{J+1}^0(x)}{J+1} - \frac{2}{J+1} \ge \frac{N_{J+1}^0(x)}{J+1} - \varepsilon/4 \ge \alpha - 5\varepsilon/4$, which contradicts (20).

**Remark:** *The contradiction does not hold if $J \le J_\varepsilon$. But the points $x$ for which $J \le J_\varepsilon$ occurs infinitely many times are dyadic numbers.*

**2.** Assume that the union $\lambda_j^-(x) \cup \lambda_j(x)$ forms a dyadic interval of generation $j-1$. The same argument yields that $N_j^0(x) = N_j^{0,-}(x) - 1$, and $N^0_{j,\min}(x) \ne N_j^0(x)$, so that $N_j^{0,+}(x) \ge N_j^0(x)$, and $N^0_{j,\min}(x) = N_j^{0,-}(x)$. Let $J$ denote the length of the largest common prefix between $w$ and $w^+$. The digits of $w$ located between the $J+1$-th digit and the $j$-th digit are only 1's, while the digits of $w^+$ at the same positions are only 0's. The same arguments as above give that, if $J \ge J_\varepsilon$, then $\frac{N_{J+1}^{0,-}(x)}{J+1} = \frac{N_{J+1}^0(x)}{J+1} - 1/(J+1) \ge \frac{N_{J+1}^0(x)}{J+1} - \varepsilon/4 \ge \alpha - 5\varepsilon/4$, which contradicts (20). $\square$

We are now in position to prove Part (b) of Theorem 2.

**Proof of Theorem 2 (b).** First, $\forall x$

$$H_{\mu_p}(x) = -\log_2(1-p) + \big(-\log_2(p) + \log_2(1-p)\big)\underline{N^0_{\max}}(x)$$
$$H_{\mu_q}(x) = -\log_2(1-q) + \big(-\log_2(q) + \log_2(1-q)\big)\overline{N^0_{\min}}(x).$$

By Lemma 3.2, $H_{\mu_q}(x) \in [H_{q,\min}, G_{p,q}(H_{\mu_p}(x))]$ (observe that the negative sign of $-\log_2(q) + \log_2(1-q)$ is important). So, in general, $D_{F_{p,q}}(H, H') \neq -\infty$ only if $H' \in [H, G_{p,q}(H)]$.

Now, fix $(H, H')$ such that $H \in [H_{p,\min}, H_{p,\max}]$ and $H' \in [H, G_{p,q}(H)]$. We denote by $\alpha \in [0,1]$ and $\beta \in [\alpha, 1]$ the two unique real numbers such that

$$H = -\log_2(1-p) + \big(-\log_2(p) + \log_2(1-p)\big)\alpha, \quad H' = -\log_2(1-q) + \big(-\log_2(q) + \log_2(1-q)\big)\beta.$$

A direct extension of the famous Besicovich-Eggleston formula

$$\dim\{x \in [0,1] : \underline{N^0_{\max}} = \underline{N^0_{\min}} = \overline{N^0_{\max}} = \overline{N^0_{\min}} = \alpha\} = -\alpha \log_2(\alpha)$$

yields that

$$\dim\{x \in [0,1] : \underline{N^0_{\max}} = \underline{N^0_{\min}} = \alpha, \overline{N^0_{\max}} = \overline{N^0_{\min}} = \beta\} = \min(-\alpha \log_2(\alpha), -\beta \log_2(\beta)). \tag{21}$$

Since $\dim\{x \in [0,1] : H_{\mu_p}(x) = H\} = \dim\{x \in [0,1] : \underline{N^0_{\max}} = \alpha\} = D_{\mu_p}(H)$, it follows that $-\alpha\log_2(\alpha) = D_{\mu_p}(H)$ and similarly, $-\beta \log_2(\beta) = d_{\mu_q}(H')$. From (21) we finally deduce that

$$D_{F_{p,q}}(H, H') = \min(D_{\mu_p}(H), d_{\mu_q}(H')).$$ □

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