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LETTER TO THE EDITOR

The Wavelet-Based Synthesis for Fractional Brownian Motion Proposed by F. Sellan and Y. Meyer: Remarks and Fast Implementation

PATRICE ABRY[1,3] AND FABRICE SELLAN[2,3]

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Abstract—The aim of this communication is to propose some complementary remarks and interpretation on the wavelet-based synthesis technique for fractional Brownian motion proposed by Sellan in 1995. These comments will lead us to propose a fast and efficient pyramidal filter bank-based Mallat-type algorithm, which permits an easy and efficient implementation of this synthesis technique. ©1996 Academic Press, Inc.

1. MOTIVATION

Fractional Brownian motion. Fractional Brownian motion (hereafter fBm) is a continuous-time random process proposed by Mandelbrot and Van Ness [11]. Basically, it consists in a fractional integration of a white Gaussian process and is therefore a generalization of Brownian motion (as defined by P. Lévy), which consists *simply* in a standard integration of a white Gaussian process. Because it presents deep connections with the concepts of self-similarity, fractal, long-range dependence or $1/f$-processes, fBm quickly became a major tool for the various fields where such concepts are relevant. Many efforts have therefore been devoted to the possibility of performing numerical simulation for such a process (for a review, see [14]). None of these methods, however, was able to produce a process that possesses all the properties of fBm.

Very recently, Sellan proposed [14, 15] a powerful wavelet-based analysis of fBm which also provides us with a general scheme to synthesize it.

Scope of the communication. Our aim here is to propose a fast and efficient implementation of this synthesis technique that relies on the use of a fast filter bank-based Mallat-type pyramidal algorithm as well as to propose further remarks that give complementary viewpoints on this technique. While the next section restates the main ideas of the construction and theorems presented in [14], Section 3 clearly details how to derive the coefficients of the filter bank involved in the fast pyramidal algorithm. Section 4 addresses both practical issues and interpretation questions. Matlab routines can be obtained at the *ACHA* software ftp site, and are available upon request.

2. WAVELET-BASED SYNTHESIS FOR fBm

DEFINITION. Let us first recall the commonly used definition for fBm [11],

$$B_H(t) = \int_{-\infty}^{+\infty} [K_H(t-s) - K_H(-s)]dB(s), \tag{1}$$

where

$$0 \leqslant H \leqslant 1, \qquad \begin{array}{ll} K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, & t \geqslant 0 \\ K_H(t) = \quad 0, & t < 0 \end{array}$$

and $B(s)$ is ordinary Brownian motion (i.e., the integration of a white Gaussian process $W(s)$). Such a definition should be understood as the mathematically correct formulation (converging difference of two diverging integrals) for the intuitive definition of a fractional integration of a white Gaussian process [13]

$$\int_{-\infty}^{t} |t-s|^{H-1/2} dB(s).$$

Moreover, from this definition, one can obtained the autocovariance structure for fBm

$$E(B_H(t)B_H(s)) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{2}$$

where

$$\sigma^2 = \Gamma(1-2H)\frac{\cos \pi H}{\pi H}.$$

[1] CNRS URA 1325, Laboratoire de Physique, Ecole Normale Supérieure de Lyon, 46, allée d'Italie 69 364 Lyon Cedex 07, France. Fax: (+ 33) 72 72 80 80. E-mail: pabry@physique.ens-lyon.fr.
[2] CNRS URA 1378, Laboratoire de Traitement des Images et du Signal, MATRA CAP SYSTEMES, BP 14, 78142 Vélizy Villacoublay Cedex, France. E-mail: fabrice.sellan@matra-ms2i.fr.
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Wavelet representation. The basic idea underlying the construction in [14] consists in the fact that the coefficients of the expansion of a white process over an orthonormal basis will constitute a collection of uncorrelated coefficients. Starting with an orthonormal set of wavelets and scaling functions

$$\{\{\phi_o(t-k), k \in \mathscr{Z}\},$$
$$\{\psi_{j,k}(t) = 2^{-j/2}\psi_0(2^{-j}(t-k)), (j,k) \in (\mathscr{Z}^+, \mathscr{Z})\}\}$$

constructed from a multiresolution analysis (MRA) [12, 6], this can be written

$$W(t) = \sum_k \lambda(k)\phi_0(t-k) + \sum_{j\leqslant 0,k} \gamma_j(k)\psi_{j,k}(t),$$

where $\lambda(k)$ and $\gamma_j(k)$ are samples of i.i.d. white Gaussian processes. To obtain a fBm, the key idea in [14] is to fractionally integrate each vector of the expansion basis:

$$B_H(t) = \sum_k \lambda(k)(D^{(-s)}\phi_0)(t-k) + \sum_{j\leqslant 0,k} \gamma_j(k)(D^{(-s)}\psi_{j,k})(t),$$

where $D^{(s)}$ stands for the fractional differentiation operator of order s. However, such an idea raises a difficulty of major importance: whereas the fractional integration of a wavelet still provides us with a wavelet, it completely delocalizes the scaling function and therefore kills the multiresolution nature of the synthesis. To overcome such a difficulty, Meyer and Sellan proposed to perform a generalized Abel transform for the summation $\sum_k \lambda(k)D^{-s}\phi_0(t-k)$ (see [15] for full details). They ended up with the wavelet representation for fBm

$$B_H(t) - b_0 = \sum_k b_h(k)\phi_{0,k}^{(s)}(t) + \sum_{j\leqslant 0,k} \gamma_j(k)4^{-s}2^{-js}\psi_{j,k}^{(s)}(t), \quad (3)$$

here $s = H + 1/2$, b_o is an arbitrary constant, γ_j are independent identically distributed Gaussian random variables, $b_H(k)$ is a fractional ARIMA$(0, s, 0)$ process, and $\phi^{(s)}$ and $\psi^{(s)}$ are suitably defined *fractional* scaling function and wavelet.

Fractional wavelets. It is shown in [14] that for the above decomposition to hold, the scaling function $\phi^{(s)}$ and the wavelet $\psi^{(s)}$ are to be designed starting from an MRA whose regularity r [12] has to satisfy $r > s > 0$. More precisely, it is proven that starting from an MRA $V_0(\phi_0)$ (in which the orthonormal scaling function ϕ_0 and the associated orthonormal wavelet ψ_0 have been selected), one can design two new MRAs, $V_0^{(s)}(\phi_0^{(s)})$ and $V_0^{(-s)}(\phi_0^{(-s)})$, whose scaling functions are defined by

$$\begin{aligned} &V_0^{(s)}(\phi_0^{(s)}) &&\text{with } \phi_0^{(s)} = U_s(\phi_0) \\ &V_0^{(-s)}(\phi_0^{(-s)}) &&\text{with } \phi_0^{(-s)} = \bar{U}_{-s}(\phi_0), \end{aligned} \quad (4)$$

where $g = U_s(f)$ has for Fourier transform $\hat{g}(\nu) = (i2\pi\nu)^{-s}(1 - \exp(i2\pi\nu))^2\hat{f}(\nu)$. It is then possible to construct, for each MRA respectively, a wavelet $\psi^{(s)}$ and $\psi^{(-s)}$,

$$\begin{aligned} \psi_0^{(s)} &= 4^sD^{-s}(\psi_0) \\ \psi_0^{(-s)} &= 4^{-s}\bar{D}^s(\psi_0) \end{aligned} \quad (5)$$

such that they constitute a pair of biorthogonal wavelets in the sense of Cohen *et al.* [5].

Fractional ARIMA process. A fractional ARIMA(0, s, 0) process [8, 9] is the generalization to fractional integration of the ARIMA$(0, n, 0)$ process defined in [4], which consisted in the standard integration of a discrete-time white Gaussian noise. Its z-transform therefore reads

$$X(z) = (1 - z^{-1})^{-s}W(z),$$

where $W(z)$ stands for the z-transform of a discrete-time white Gaussian noise $W(t)$.

Interpretation. Equation (3) proposes to describe fBm as a trend $(\sum_k b_H(k)\phi_{0,k}^{(s)}(t))$ over which are superimposed a succession of details or refinements $(\sum_{j\leqslant 0,k} \gamma_j(k) 4^{-s}2^{-js}\psi_{j,k}^{(s)}(t))$. Note, moreover, to complete the scheme proposed in Section 3 of [14], that if the coefficients of the expansion on an orthonormal basis of the *orthogonal* projection of a continuous white noise are again a discrete white noise, the ARIMA(0, s, 0) (i.e., the trend of the fBm) results from the expansion on a Riesz (nonorthonormal) basis for the first MRA $V^{(s)}$ of the *oblique* projection of the continuous fBm *along a direction* orthogonal to the second MRA $V^{(-s)}$.

Let us now try to figure out which features of the fBm are carried by the trend and the details. When expanding a signal over a basis, one attributes, in some sense, some of its properties to the coefficients of the representation while its remaining features are carried by the elements of the bases themselves. In Eq. (3), if one considers first the parts of the fBm conveyed by the wavelet (or detail) coefficients, the uncorrelated Gaussian variables γ_j carry only the Gaussian nature of the fBm while the wavelet basis has been given by construction of the exact amount of correlation to catch the short-term (or high-pass) correlation structure of the fBm. The fractional wavelet basis therefore acts as a Karhunen–Loève basis for the high-pass part of the spectrum of the fBm and conveys its scale-invariant nature. For the lower-frequency part of the fBm—the trend—the situation is different. Although both the coefficients and the elements of the basis carry some correlation, the long-term correlation (or long-range dependence, i.e., the power-law

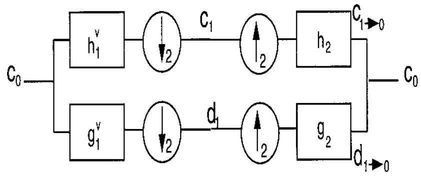

FIG. 1. Two-band analysis/synthesis filter bank. Scheme of the classical two-band decomposition/reconstruction filter band underlying the fast implementation of the dyadic wavelet analysis/synthesis.

decrease of the autocovariance function) is contained in the ARIMA process b_H, that is, in the coefficients of the expansion rather than in the basis itself. The presence of this important feature of the fBm (which very often was missing in other standard synthesis techniques for the fBm) can be observed in the plots of Fig. 4.

Fast implementation. Equation (3) leads to the straightforward pyramidal filter bank-based recursive implementation sketched in Fig. 3, provided that (i) one can produce the ARIMA process b_H and (ii) one can compute the coefficients of the synthesis lowpass (h_2) and highpass (g_2) related to the fractional scaling function $\phi_0^{(s)}$ and wavelet $\psi_0^{(s)}$, involved in the synthesis. This will be explained in detail in the next section.

3. FAST IMPLEMENTATION

3.1. Fractional Wavelets

MRA, scaling function, and wavelet. Let us briefly recall that an MRA is fully defined by its scaling function ϕ_0 [6, 12] or equivalently by the generating sequence u [3] of ϕ_0:

$$\phi_1(t) = \phi_0(t/2)/\sqrt{2} = \sum_k u(k)\phi_0(t-k). \tag{6}$$

Equivalently, the wavelet itself can be defined through its generating sequence v:

$$\psi_1(t) = \psi_0(t/2)/\sqrt{2} = \sum_k v(k)\phi_0(t-k). \tag{7}$$

Orthonormal basis. For the $\{\phi_0(t-k)\}_{k\in\mathcal{Z}}$ to constitute an orthonormal basis for V_0, the generating sequence has to satisfy

$$\downarrow_2 [u * u^\vee] = \delta_0,$$

where $\downarrow_2 [x](k) = x(2k)$ stands for the decimation operator that drops one sample of x out of 2, $x^\vee(k) = x(-k)$ stands for the time-reversal operator, and $\delta_n(k)$ is 1 for $k = n$ and 0 elsewhere. Moreover, the collection $\{\psi_{j,k} = 2^{-j/2}\psi_0(2^{-j}t - k)\}_{(j,k)\in(\mathcal{Z},\mathcal{Z})}$, derived from the mother wavelet ψ_0, defines an orthonormal basis if

$$\begin{aligned} \downarrow_2 [u * u^\vee] &= \delta_0 \\ v &= \delta_1 * \tilde{u}^\vee, \end{aligned} \tag{8}$$

where $\tilde{x}(k) = (-1)^k x(k)$. Such series are called quadrature miror filters (QMF) series. Let us add, moreover, that it is possible to derive a wavelet orthonormal basis from any arbitrarily chosen multiresolution using the orthonormalization technique described in [3, 1].

Fractional MRAs. To fully characterize the fractional scaling functions and wavelets defined in [14], whose construction has been recalled above, we need to derive their generating sequences:

$$\begin{aligned} \phi_0^{(s)}(t/2) &= \sqrt{2}\sum_k u^{(s)}(k)\phi_0^{(s)}(t-k) \\ \psi_0^{(s)}(t/2) &= \sqrt{2}\sum_k v^{(s)}(k)\phi_0^{(s)}(t-k) \\ \phi_0^{(-s)}(t/2) &= \sqrt{2}\sum_k u^{(-s)}(k)\phi_0^{(-s)}(t-k) \\ \psi_0^{(-s)}(t/2) &= \sqrt{2}\sum_k v^{(-s)}(k)\phi_0^{(-s)}(t-k). \end{aligned} \tag{9}$$

Some cumbersome but not difficult calculations enabled us to show that

$$\begin{aligned} u^{(s)} &= f^{(s)} * u, & F^{(s)}(z) &= 2^{-s}(1+z^{-1})^s \\ v^{(s)} &= g^{(s)} * v, & G^{(s)}(z) &= 2^{s}(1-z^{-1})^{-s} \\ u^{(-s)} &= f^{(-s)} * u, & F^{(-s)}(z) &= 2^{s}(1+z)^{-s} \\ v^{(-s)} &= g^{(-s)} * v, & G^{(-s)}(z) &= 2^{-s}(1-z)^{s}, \end{aligned} \tag{10}$$

where u and v are the generating sequences for the orthogonal scaling function and wavelet (uppercase letters obviously stand for the z-transforms of the corresponding lowercase ones).

Biorthogonality and exact reconstruction. It is not difficult to check that these sequences satisfy the perfect reconstruction filter bank equations related to the standard structure, underlying the dyadic wavelet analysis/synthesis algorithm, sketched in Fig. 1, with

$$\begin{aligned} h_1 &= u^{(-s)\vee}, & g_1 &= v^{(-s)\vee} \\ h_2 &= u^{(s)}, & g_2 &= v^{(s)}. \end{aligned} \tag{11}$$

It is also easy to check that these sequences satisfy the cross QMF conditions,

$$\begin{aligned} \downarrow_2 [u^{(-s)} * (u^{(s)})^\vee] &= \delta_0 \\ v^{(s)} &= \delta_1 * (\tilde{u}^{(-s)})^\vee \\ v^{(-s)} &= \delta_1 * (\tilde{u}^{(s)})^\vee, \end{aligned} \tag{12}$$

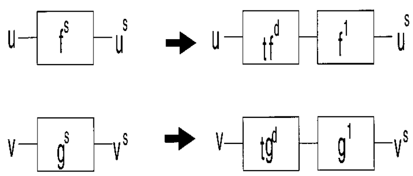

FIG. 2. Fractional wavelets. Approximation scheme for the computation of the generating sequences of the fractional wavelets.

which simply restate that the designed scaling function and wavelet constitute dual biorthogonal pairs as defined by the inventors of the biorthogonal wavelet transform [5, 6].

Implementation. The sequences $f^{(s)}, g^{(s)}, f^{(-s)}$, and $g^{(-s)}$ in general have infinite support. Moreover, for fBm, the defining parameter H ranges from 0 to 1, which means that for $s = H + 1/2$, the sequences are diverging [8]. Let $d = H - 1/2$, we propose to get numerically the generating sequences through the convergent sequences $f^{(d)}$ and $g^{(d)}$,

$$\begin{aligned} u^{(s)} &= u * f^{(1)} * tf^{(d)} \\ v^{(s)} &= v * g^{(1)} * tg^{(d)} \\ u^{(-s)} &= -\delta_{-1} * (\tilde{v}^{(s)})^{\vee} \\ v^{(-s)} &= \delta_1 * (\tilde{u}^{(s)})^{\vee}, \end{aligned} \tag{13}$$

here $tf^{(d)}$ and $tg^{(d)}$ consist in versions of $f^{(d)}$ and $g^{(d)}$ truncated up to an order chosen a priori. Such a procedure is depicted in Fig. 2.

3.2. *ARIMA Process*

Formally an ARIMA(0, s, 0) process can be obtained from a zero-mean white Gaussian noise by discrete-time convolution with an I.I.R. filter $\alpha^{(s)}$ whose z-transform reads $A(z) = (1 - z^{-1})^{-s}$. Up to the scaling factor 2^s, this sequence is $g^{(s)}$. We propose therefore to obtain the ARIMA process b_H from the same approximation,

$$b_H = \gamma * \alpha^{(1)} * t\alpha^{(d)},$$

where $t\alpha^{(d)}$ is a version of $\alpha^{(d)}$ truncated to an order chosen a priori which is still I.I.R. yet convergent.

3.3. *Fast Algorithm*

A LA MALLAT ALGORITHM. We end up with the implementation of Eq. (3) thanks to a recursive filter band-based Mallat-type algorithm, as sketched in Fig. 3. It is well known that the coefficients of the low-pass and high-pass filters simply consist in the coefficients of the generating sequences of the synthesis scaling function and wavelet:

$$\begin{aligned} h_2 &= u^{(s)} \\ g_2 &= v^{(s)}. \end{aligned} \tag{14}$$

The inputs of this filter bank consist in a collection of samples of independent zero-mean white Gaussian processes whose variances are σ_w^2 for the approximation and $\{\sigma_w^2 2^{js}, j = 0, 1, 2, \ldots\}$ for the details.

Border effects. The border effects are treated in a very simple way. The number of computed samples of the fBm is larger than the desired number. The extra number of computed coefficients is such that the number of coefficients unpolluted by border effects is larger than or equal to the desired number.

4. FURTHER COMMENTS

Number of samples and number of octaves. The practical use of Eq. (3) implies the selection of a finite number J of octaves for the synthesis. Because of this limitation of the summation over a finite number of octaves, the output of the algorithm does not consist in a collection of actual samples of the fBm, but rather in coordinates of the expansion of B_H over the basis of scaling functions $\{\phi^{(s)}_{-J,k}(t), k \in \mathscr{Z}\}$,

$$\begin{aligned} B_H^{-J}(t) &= \sum_k b_H(k)\phi^{(s)}_{-J,k}(t) + \sum_{j=0}^{J}\sum_k \gamma_j(k)4^{-s}2^{-js}\psi^{(s)}_{-j,k}(t) \\ &= \sum_k a_H(-J,k)\phi^{(s)}_{-J,k}(t). \end{aligned} \tag{15}$$

$B_H^{-J}(t)$ only is an approximation for $B_H(t)$, obtained as the oblique projection (along a direction orthogonal to the sec-

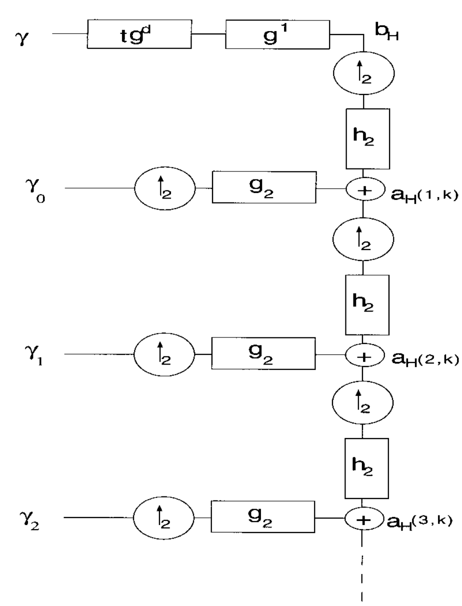

FIG. 3. Fast pyramidal Mallet-type filter bank algorithm for the synthesis of the fBm. The synthesis filters are given by the generating sequences of the fractional scaling function and wavelet ($h_2 = u^{(s)}$ and $g_2 = v^{(s)}$). The input consists in a collection of samples of i.i.d. zero-mean Gaussian processes whose variances follow a power law of parameter $H + 1/2$.

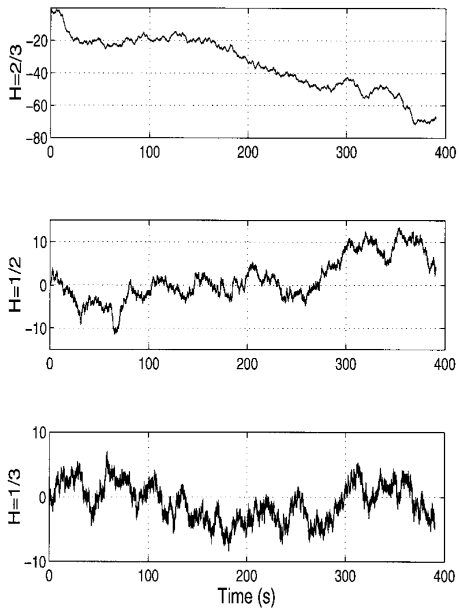

FIG. 4. Synthesized fBm. Examples of fBm synthesized with the procedure given; from top to bottom, $H = 2/3, 1/2, 1/3$.

ond MRA $V^{(-s)}$) in the subspace $V_{-J}^{(s)}$ of the first MRA. Therefore, the $a_H(-J, k)$ constitute an ARIMA process as well as the $b_H(k)$. Such an ARIMA is, however, used in the usual way since its samples are approximations of those of $B_H(t)$ at noninteger times $t = 2^{-J}k$, whereas $b_H(k)$ were approximations at the much coarser resolution labeled 0. Therefore, the higher the J, the better the approximation. In practice, $J = 5, 6$ already provides us with relevant approximations.

Moreover, practical syntheses are performed over a finite number N of samples. To remain coherent with the definition and properties of the fBm (mainly its variance σ^2), obtaining N samples through J octaves amounts to synthesizing a fBm over a time duration $T = 2^{-J}N$. Let us insist again on the fact that border effects are handled in such a way that one gets N samples unaffected by this problem.

Choosing the starting orthonormal MRA. For Eq. (3) to be valid, one has to start with an orthonormal MRA whose regularity is larger than $s = H + 1/2$. We propose in the present implementation of the algorithm to work either with the standard *Daubechies 10* MRA [6], whose generating sequence has 20 nonzero coefficients and whose regularity is close to 2, or with the spline of order 5 MRA. This latest MRA is one of those leading to the celebrated Battle–Lemarié orthonormal spline wavelets [10]. Although they have a closed-form formulation in the frequency domain, such wavelets can be very efficiently and easily approximated by the following construction in the time domain. A generating sequence for this spline MRA can be obtained by convolving with itself the generating sequence $u_0 = [1\ 1]/\sqrt{2}$ of the Haar (or spline of order 0) wavelet [3]:

$$u_5 = 2^{-5/2} u_0 * u_0 * u_0 * u_0 * u_0.$$

To obtain orthonormal scaling functions and wavelets, one has to use the orthonormalization method proposed in [3]. Compute

$$u_5^{(o)} = \uparrow_2 [a^{-1/2}] * u_5 * a^{1/2},$$

where x^α denotes the sequence whose z-transform $X^\alpha(z)$ is equal to $(X(z))^\alpha$. Moreover, a is the autocorrelation sequence of u_5 which can be obtained [3] from

$$a = \downarrow_2 [u_5 * a * \overset{\vee}{u}_5].$$

This last equation can be solved very easily through the determination of the eigenvectors of a matrix designed from u_5 [1, 2]. This MRA is very regular but the support of the generating sequence $u_5^{(o)}$ is not compact, so that a truncation, easy to handle in practice, is needed.

These two MRAs (Daubechies 10 or spline 5) present enough regularity to cover the range of variation of parameter H for fBm, $0 < H < 1$. However, if one wants to experiment with the synthesis of processes with higher H (which will no longer be fBm), one can produce orthonormal scaling functions and wavelets of arbitrary high regularity, by convolving MRAs with themselves and performing the orthonormalization trick described above [3, 1, 2].

Controlling the variance. When synthesizing fBms, one can be interested in controlling not only the long-range parameter H, which is proven to be perfectly achieved with this algorithm, but also the variance σ^2. Since the theoretical variance for the fBm reads (with the commonly used convention $B_H(0) = 0$)

$$EB_H(t)^2 = \sigma^2 |t|^{2H},$$

it is not difficult to check that the variance of the coefficients $d_H(j, k)$ of a wavelet decomposition of the fBm reads (see, for instance, [7])

$$Ed_H(j,k)^2 = -\frac{\sigma^2}{2} 2^{2js} \int |u|^{2H} \gamma_0(u) du,$$

where γ_0 is the autocorrelation function of the mother wavelet ϕ_0,

$$\gamma_0(u) = \int \phi_0(t)\phi_0(t+u) dt.$$

Using Parseval identity, this can be rewritten as

$$Ed_H(j,k)^2 = \sigma^2 2^{j(2H+1)} \Gamma(2H+1) \sin(\pi H)(2\pi)^{-(2H+1)} \times \int |\nu|^{-(2H+1)} |\Psi_0(\nu)|^2 d\nu.$$

In the very special case where $d_H(j,k)$ is the coefficient of the expansion of the fBm on the fractionally integrated wavelet $\psi^{(s)}$ defined above, it results from the inner product between B_H and the dual $\psi^{(-s)}$ of $\psi^{(s)}$:

$$d_H(j,k) = \langle B_H, \psi^{(-s)} \rangle.$$

Since $\hat{\psi}^{(-s)}(\nu) = 4^{-s}\overline{(i2\pi\nu)^s}\hat{\psi}_0(\nu)$, we obtain that

$$\begin{aligned} &Ed_H(j,k)^2 \\ &\quad = \sigma^2 2^{j(2H+1)}\Gamma(2H+1)\sin(\pi H)4^{-(2H+1)} \\ &\qquad \times \int |\nu|^{-(2H+1)}|\nu|^{(2H+1)}|\Psi_0(\nu)|^2 d\nu. \end{aligned}$$

Since ψ_0 is an orthonormal wavelet, it yields

$$Ed_H(j,k)^2 = 4^{-(2H+1)}2^{j(2H+1)}\sigma^2\Gamma(2H+1)\sin(\pi H).$$

Moreover, we know by construction that $d_H(j,k) = 4^{-s}2^{js}\gamma_j(k)$ (with $\gamma_j(k)$ samples of a white Gaussian process), and hence

$$Ed_H(j,k)^2 = 4^{-(2H+1)}2^{j(2H+1)}\sigma_w^2.$$

We therefore see that the variance of the fBm can be controlled by the variance of the Gaussian processes with which the inputs of the algorithm are designed:

$$\begin{aligned} \sigma_w^2 &= \sigma^2\Gamma(2H+1)\sin(\pi H) \\ &= \Gamma(1-2H)\Gamma(1+2H)\frac{\sin(\pi H)\cos(\pi H)}{\pi H}. \end{aligned} \tag{16}$$

One can easily check that this last relation shows that

$$0 < H < 1, \quad \sigma_w^2 \equiv 1.$$

Comparison with an earlier wavelet-based synthesis for fBm. An earlier attempt to synthesize the fBm using wavelets was made in [16]. The basic idea was to write the fBm as a weighted summation of orthonormal wavelets, weights being samples of independent zero-mean Gaussian processes, whose power (or variance) was power-law scale dependent: $\sigma_j^2 = \sigma_0^2 2^{j(2H+1)}$,

$$B_H(t) \simeq \sum_{j,k} \gamma_j(k)\psi_{j,k}^{(o)}(t).$$

When one compares our method with the method proposed by Meyer and Sellan (i.e. with Eq. (3)), one sees that their technique contains two major drawbacks as far as fBm synthesis is concerned. First, because the coarsest level approximation is arbitrarily set to zero, this method cannot produce the trend of the fBm, which is one of its important features, since it is related to its fractal or self-similar or long-range dependent nature. Second, because it makes use of a set of orthonormal wavelets, this method gives only a *nearly* $1/f$ spectral behavior instead of an exact one (i.e., the spectrum $S(\nu)$ is squeezed between two power laws, $k_1/|\nu|^\alpha < S(\nu) < k_2/|\nu|^\alpha$). The coefficients of the expansion of fBm over an orthonormal wavelet basis would indeed remain slightly correlated [7]; therefore, an orthonormal wavelet set does not act as a Karhunen–Loève basis for the high-frequency part of the fBm. Hence one cannot obtain the exact correlation structure of the fBm from the combination of independent coefficients with orthonormal wavelets. On the contrary, the fractional wavelet, designed from parameter H, achieves perfect decorrelation between the wavelet coefficients of the fBm, and therefore exactly carries the correlation structure of the fBm. Although it is not strictly exact, this orthonormal wavelet-based technique has remained one of the most frequently used fBm synthesis techniques up to now.

5. CONCLUSION

In [14], a wavelet-based technique for the synthesis of the fBm is proposed which by far outperforms any previously proposed techniques since it ensures that all of its properties (Gaussianity, self-similarity, correlation structure, etc.) are satisfied by construction. The multiresolution framework underlying this technique *naturally* leads us to the fast and efficient implementation algorithm we described above. Some examples are shown in Fig. 4.

We also believe that both the theoretical results obtained in [14, 15] and the possibility of making experiments with an easy-to-handle algorithm will enable us to enquire into processes whose properties depart from those of fBm (H exceeding one or varying along time, non-Gaussianity, etc.).

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