Warped infinitely divisible cascades: beyond power laws

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Résumé – Nous présentons les définitions et synthèses de processus stochastiques respectant des lois d'échelles *voilées*, qui s'écartent de façon contrôlée d'un comportement en loi de puissance. Nous définissons des bruit, mouvement et marche aléatoire issus de cascades infiniment divisibles (IDC) voilées. Nous étudions analytiquement le comportement des moments des accroissements de ces processus à travers les échelles. Ces résultats théoriques sont illustrés sur l'exemple d'une cascade log-Normale voilée. Les algorithmes de synthèse et les fonctions Matlab utilisés sont disponibles sur nos pages web.

Abstract – We address the definitions and synthesis of stochastic processes which possess *warped* scaling laws that depart from power law behaviors in a controlled manner. We define warped infinitely divisible cascading (IDC) noise, motion and random walk. We provide a theoretical derivation of the scaling behavior of the moments of their increments. We provide numerical simulations of a warped log-Normal cascade to illustrate these results. Algorithms for synthesis and Matlab functions are available from our web pages.

1 Introduction

Scaling has been observed for many years in a large number of fields including natural phenomena: turbulence in hydrodynamics, rhythm of human heart in biology, spatial repartition of faults in geology and others such as computer networks and financial markets. The multifractal formalism[1, 12, 20] has become one of the most popular frameworks to analyse signals that exhibit *power law scaling*. In current verbage, this latter term refers to the power law behavior of the absolute moments of increments $\delta_\tau X(t) = X(t+\tau) - X(t)$ of a process X. Then, power law scaling is to be described by a set of multifractal exponents $\zeta(q)$ such that[1]

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q \tau^{\zeta(q)} \qquad \text{as } \tau \to 0. \tag{1}$$

For instance, statistically self-similar processes such as fractional Brownian motions [15] with Hurst exponent H fit into this framework with $\zeta(q) = qH$. The so-called multifractal formalism establishes conditions under which property (1) and multifractal are equivalent. The multifractal decomposition gives precious information on the presence of local singularities in the trajectories of processes. However, this framework is restrictive in at least two ways.

First, in real world applications one is usually confined to observing power laws in a given range of scales $\tau_{min} \leq \tau \leq \tau_{max}$ which we then prefer to call *multiscaling* to distinguish it from multifractals. Multiscaling is usually considered as a best approximation to (1) and as a first step towards the use of the multifractal formalism. However, while property (1) is sensitive only to the limiting behavior it might not capture some richness in the progression at all observable scales. Second, powerlaws may not provide an accurate description of the scaling behavior of data or models.

The need for an appropriate mathematical framework substituting (1) was met with the *infinitely divisible scaling* (for an overview see [7]). This setting allows for more flexible scaling and thus better fitting of data and honors the contribution of all scales in a range of interesting scales $\tau_{min} \leq \tau \leq \tau_{max}$ as follows:

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q \exp[-\zeta(q) n(\tau)], \; \tau_{min} \leq \tau \leq \tau_{max}, \tag{2}$$

where $n(\tau)$ is some monotonous function. Such a behavior is analysed in terms of a *cascading mechanism* through the scales from τ_{max} to τ_{min}. In terms of scale dependence, the infinitely divisible scaling framework generalizes (1) which is recovered by choosing $n(\tau) = -\ln \tau$. The difference in spirit lies in the fact that multifractal analysis applies to any process (compare footnote 1) and is concerned with local properties in the limit of fine scales, but not finite scales. Note that both, multifractal analysis and infinitely divisible scaling can be formulated using wavelet coefficients [19, 23].

While analysis tools for multiscaling and infinitely divisible scaling processes have been widely developed, only few recent works proposed tools for synthesis of processes with prescribed and controllable infinitely divisible scaling [3, 5, 11, 21, 22]. Multiplicative cascades have always

[1] A definition which works for any process is:

$$\zeta(q) = \liminf_{\tau \to 0} \log_\tau \mathbb{E}|\delta_\tau X(t)|^q.$$

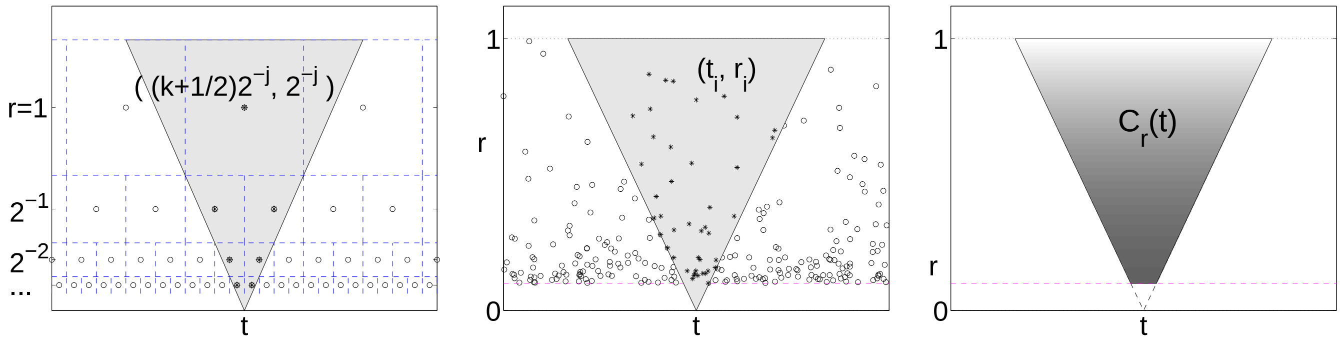

FIG. 1: **Comparison between the "time-scale" construction of multiplicative cascades.** **(left)** Binomial cascade, **(center)** Compound Poisson Cascade, **(right)** Infinitely Divisible Cascade. The grey region indicates the cone containing multipliers that determine the value of the density at time t.

played a central role to this purpose in intimate connection with multifractals. The synthesis of *Infinitely Divisible Cascades* (IDC) presented below can be seen as a generalized continuous multiplicative cascade. Following a work by Barral & Mandelbrot [5] and inspired by the densified multiplicative cascades by Schmitt & Marsan [22] and the Multifractal Random Walk by Bacry *et al.* [3], we recently discussed and studied the *Infinitely Divisible Cascading processes* [10]. Similar results were obtained independently and simultaneously by Bacry & Muzy [4, 18]. Here, we extend our previous work to the case of *warped IDC* [8, 9]. These continuous-time processes have stationary increments and exhibit continuous scaling laws with prescribed exponents (cf. $\zeta(q)$) as well as prescribed departures from power law behaviors (cf. $n(\tau) \neq -\ln\tau$). This article provides an easily accessible overview of their known properties as well as a thorough illustration via numerical simulations. The reader is referred to [9] for a more formal presentation and complete mathematical proofs.

2 IDC Noise

Intuitions towards continuous cascades. The original ancestor of multiplicative cascades is the binomial cascade introduced by Mandelbrot [14]. Under some convergence hypotheses, binomial cascades lead to positive densities that can display controlled multifractal properties. From a time-scale point of view, the construction of a binomial density relies on two basic ingredients: a dyadic grid $\{(t_{j,k}, r_{j,k}) = ((k+\frac{1}{2})2^{-j}, 2^{-j}), j \in \mathbb{N}, k \in \mathbb{N}\}$ in the time-scale plane and positive i.i.d. mean one random multipliers $W_{j,k}$ associated to dyadic grid points $(t_{j,k}, r_{j,k})$. Without loss of generality, let us fix the range of scales to $(0,1]$. Roughly speaking, the binomial cascade is defined as the limit of densities $Q_r(t)$ corresponding to resolution $1 > r = 2^{-n} \to 0$. While literature introduces Q_r often as an iterative redistribution of mass, an equivalent formulation is more useful here which writes Q_r as the product of precisely those multipliers which belong to a cone $\mathcal{C}_r(t) = \{(t', r') : r \le r' \le 1, t - r'/2 \le t' \le t + r'/2\}$ pointing to the time instant t, see FIG. 1(left):

$$Q_r(t) = \prod_{\{(j,k):1\le j\le n, k2^{-j}\le t<(k+1)2^{-j}\}} W_{j,k}, \tag{3}$$

Because of the dyadic structure, binomial cascades display *discrete scale invariance* only. Moreover, such a construction is not *time-shift invariant* so that it is *not stationary* in the strict sense.

The work by Barral & Mandelbrot[5] opened a door to overcome these drawbacks by introducing the *Multifractal Products of Cylindrical Pulses* (MPCP). Essentially, the key idea consists in replacing the dyadic grid by a well chosen *random Poisson point process* (t_i, r_i) in the time-scale plane, see FIG. 1(center):

$$Q_r(t) = \frac{\prod_{(t_i,r_i)\in\mathcal{C}_r(t)} W_i}{\mathbb{E}\left[\prod_{(t_i,r_i)\in\mathcal{C}_r(t)} W_i\right]} \tag{4}$$

Aiming at power law scaling, "well chosen" means that it has density $dm(t,r) = dtdr/r^2$. Thus, the density in points increases as $r \to 0$ in a way similar to a dyadic grid. Note that this density is time-shift invariant. Thus, MPCP are *stationary*. Moreover, scaling laws are observed over a continuous range of scales since no privileged scale ratio has been introduced. From a time-scale point of view, MPCP may be called *Compound Poisson Cascades* since the distribution of $Q_r(t)$ is a compound Poisson distribution. The Poisson distribution coming from the point process (t_i, r_i) is compound with the distribution of the random multipliers W_i.

Noting that compound Poisson distributions are infinitely divisible and that

$$\ln Q_r(t) \propto \ln \prod_{(t_i,r_i)\in\mathcal{C}_r(t)} W_i = \sum_{(t_i,r_i)\in\mathcal{C}_r(t)} \ln W_i \tag{5}$$

one may go one step further. Indeed, the right hand term above can be read as a specific (discrete) case of a random measure of the set $\mathcal{C}_r(t)$. This leads to the definition of a process $Q_r(t)$ based on the summation of a *continuous* random measure $dM(t,r)$ [10, 18] :

$$Q_r(t) \propto \exp \int_{\mathcal{C}_r(t)} dM(t', r') = \exp M(\mathcal{C}_r(t)). \tag{6}$$

It appears that the continuous random measure M needs to be defined from an *infinitely divisible distribution*. The idea of introducing an infinitely divisible random measure $dM(t,r)$ appeared in [22] where no systematic scaling analysis was performed. The multifractal random walk (MRW) introduced in [3] was built without using any explicit multiplicative construction but, interestingly, the MRW can be described as resulting from an infinitely divisible as well. Infinitely divisible cascades following the intuition given by (6) were simultaneously and independently introduced in [18] and [10] in the scale invariant case (with power law scaling). The purpose of our contribution below is to show how far infinitely divisible cascades may lead to non power law scaling behaviors.

Infinitely divisible cascades. Now, we give precise definitions. Let G be an infinitely divisible distribution with moment generating function $\tilde{G}(q)$ that can be written in the form $e^{-\rho(q)}$.

Let $dm(t,r) = g(r)dtdr$ a positive measure on the time-scale half-plane $\mathcal{P}^+ := \mathbb{R} \times \mathbb{R}^+$.

Let M denote an infinitely divisible, independently scattered random measure distributed by G, supported on the time-scale half-plane $\mathcal{P}^+$ and associated to its so-called *control measure* $dm(t,r)$. The random measure M is such that $\mathbb{E}[\exp[qM(\mathcal{E})]] = \exp[-\rho(q)m(\mathcal{E})]$; for all disjoint subsets $\mathcal{E}_1$ and $\mathcal{E}_2$, $M(\mathcal{E}_1)$ and $M(\mathcal{E}_2)$ are independent random variables and $M(\mathcal{E}_1 \cup \mathcal{E}_2) = M(\mathcal{E}_1) + M(\mathcal{E}_2)$.

Definition 1.
A cone of influence $\mathcal{C}_r(t)$ is defined[2] *for every $t \in \mathbb{R}$ as $\mathcal{C}_r(t) = \{(t', r') : r \le r' \le 1, t - r'/2 \le t' \le t + r'/2\}$ (see* FIG. 1*(right)). With an infinitely divisible randomly scattered measure M given, an* Infinitely Divisible Cascading noise *(IDC noise) is a family of processes $Q_r(t)$ parametrized by r of the form*

$$Q_r(t) = \frac{\exp[M(\mathcal{C}_r(t))]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]} \tag{7}$$

Possible choices for distribution G are the Normal distribution, Poisson distribution, compound Poisson distributions, Gamma laws, Stable laws...See FIG. 3(left) for a sample of a replication.

An immediate consequence of the definition is that Q_r is a *stationary positive random process* with:

$$\mathbb{E}Q_r = 1. \tag{8}$$

Stationarity is ensured by the time-invariance of both, control measure and cone of influence. Moreover, Q_r has a *log-infinitely divisible distribution*, that is $\ln Q_r$ has an infinitely divisible distribution.

Altogether, the measure M, the distribution G, the control measure m and the geometry of the cone of influence $\mathcal{C}_r(t)$ control the scaling structure as well as marginal and higher order distributions of the cascade. One major scaling property of IDC noises is:

$$\mathbb{E}[Q_r(t)^q] = \exp[-\varphi(q)\, m(\mathcal{C}_r)] \tag{9}$$

[2] Note that the large scale in the definition of $\mathcal{C}_r(t)$ has been arbitrarily set to 1 without loss of generality. Choosing a different large scale L would simply reduce to a change of units $t \to t \cdot L$, $r \to r \cdot L$.

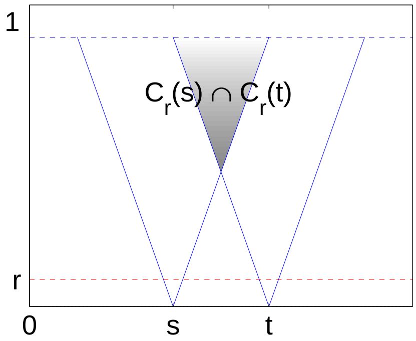

FIG. 2: **Dependence between $Q_r(t)$ and $Q_r(s)$**, in particular their correlation, stems entirely from the contribution of the intersection of two cones $\mathcal{C}_r(t)$ and $\mathcal{C}_r(s)$

where

$$\varphi(q) = \rho(q) - q\rho(1), \qquad (\varphi(1) = 0), \tag{10}$$

for all q for which $\rho(q) = -\ln \tilde{G}(q)$ is defined. Note the similarity between (9) and (2). Power laws are recovered when $m(\mathcal{C}_r)$ is proportional to $-\ln r$. The cascade is called *warped* when $m(\mathcal{C}_r)$ is not proportional to $-\ln r$.

A nice property of IDC noises lies in the geometrical interpretation of their correlations that are controlled by the intersections of cones $\mathcal{C}_r(t) \cap \mathcal{C}_r(s)$ in the time-scale plane $\mathcal{P}^+$ (see FIG. 2):

$$\mathbb{E}[Q_r(t)Q_r(s)] = \exp[-\varphi(2)m(\mathcal{C}_r(s) \cap \mathcal{C}_r(t))] \tag{11}$$

This highlights the fact that multiplicative cascades provide an easy way towards complex correlation structures: prescribing the autocorrelation function of Q_r is equivalent to choosing measure $dm(t,r)$ and cone $\mathcal{C}_r(t)$.

As explained in the previous section, the IDC-noise can be recognized as a "continuously iterative" multiplication (compare FIG. 1 (left) & (right)) where $m(\mathcal{C}_r(t))$ can be interpreted as the "average number of multipliers" that determine $Q_r(t)$. A causal definition can be proposed as well by simply defining $\mathcal{C}_r(t) = \{(t', r') : r \le r' \le 1,\ t - r' \le t' \le t\}$. For sake of simplicity in this presentation, we will keep the symmetric non causal definition while results presented below extend without restriction to the causal definition.

3 IDC Motion & Random Walk

Besides their nice scaling properties, the IDC have the distinct property of being positive. While this can provide an ideal match in some applications such as network traffic modeling, it is inappropriate in others such as the description of the velocity in a turbulent flow where data shows oscillations in both positive and negative directions. Two steps will permit to overcome this restriction. First, we define an increasing process $A(t)$ (IDC Motion). Then we define some process $V_H(t) = B_H(A(t))$ (IDC Random Walk) as a fractional Brownian motion B_H of which time has been replaced by the irregular time $A(t)$.

By analogy with binomial measures, we introduce the *Infinitely Divisible Cascading Motion* as the integral of $Q_r(t)$.

Definition 2.
An Infinitely Divisible Cascading Motion *(IDC-Motion) $A(t)$ is the limiting integral*[3] *of an IDC-noise $Q_r(t)$ (see* FIG. 3*):*

$$A(t) = \lim_{r\to 0} A_r(t), \tag{12}$$

where

$$A_r(t) = \int_0^t Q_r(s)ds. \tag{13}$$

The increment process $\delta_\tau A_r(t) = A_r(t+\tau) - A_r(t)$ of A_r inherits stationarity from Q_r since $\delta_\tau A_r(t) = \int_t^{t+\tau} Q_r(s)ds$. An IDC Motion $A(t)$ inherits scaling properties from its IDC Noise $Q_r(t)$ as shown below in Section 4.

By construction, A is a non-decreasing process which appears most natural in some real world contexts, but can be seen as a severe limitation in others. Following an idea which goes back to Mandelbrot [16] and to the *Brownian motion in multifractal time*, we define a *fractional Brownian motion in warped IDC time.* This process has stationary increments, continuous scaling, prescribed departures from power laws and prescribed scaling exponents as well as positive and negative fluctuations.

Definition 3. *Let A be an infinitely divisible cascading motion, and B_H the fractional Brownian motion with Hurst parameter H. The process*

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+, \tag{14}$$

is called an Infinitely Divisible Cascading Random Walk *(IDC Random Walk).*

The IDC Random Walk inherits stationary increments from both B_H and A. Above all, the precise scaling behavior of $A(t)$ is transferred to $V_H(t)$ thanks to the self-similarity of the fractional Brownian motion as explained below. A sample of infinitely divisible cascading processes $Q_r(t)$, $A_r(t)$ and $V_H(t)$ is shown on FIG. 3.

4 Scaling behavior of IDC

This section states our main results: it characterizes the scaling properties of certain IDC-Motions and their associated IDC-Random Walk. The reader is referred to Appendix A for the full theorem and an outline of its demonstration. See [9] for detailed proofs. Here only a corollary of the general results is stated.

Theorem (simplified version)

Let $q > 0$. Let A_r be either a CPC Motion with finite $\mathbb{E}[W^q]$ or a log-normal IDC motion. Assume that the control measure $g(r)dtdr$ is such that $g^{(n)}(r) := b^{2n}g(b^n r)\cdot \mathbf{1}_{[0,1]}$ converges as $n \to \infty$. Assume that A_r converges in $\mathcal{L}_q$; for $q < 2$, e.g., it suffices that $c\varphi(2) > -1$ and $g(r) \le 1/r^2$. Then, there exist constants $\overline{C}_q$ and $\underline{C}_q$ and $\overline{C}'_q$, $\underline{C}'_q$ such that for any $t < b$

$$\underline{C}_q \le \quad \mathbb{E}A(t)^q \cdot t^{-q} e^{\varphi(q)m(\mathcal{C}_t)} \quad \le \overline{C}_q, \tag{15}$$

$$\underline{C}'_q \le \quad \mathbb{E}|V_H(t)|^q \cdot t^{-qH} e^{\varphi(qH)m(\mathcal{C}_t)} \quad \le \overline{C}'_q. \tag{16}$$

[3]Conditions for the convergence of the positive martingale A_r as $r \to 0$ are detailed in [9].

The scaling behavior of V_H is a direct consequence of the self-similarity of a fractional Brownian motion B_H combined to the scaling behavior of an IDC Motion A [20]. Using the self-similarity of B_H, one finds that

$$\begin{aligned} \mathbb{E}[|V_H(t)|^q] &= \mathbb{E}\mathbb{E}[|B_H(A(t))|^q | A] \\ &= \mathbb{E}[|B(1)|^q]\cdot\mathbb{E}[A(t)^{qH}]. \end{aligned} \tag{17}$$

The fact that $A(t)$ and $V_H(t)$ have stationary increments and $A(0) = 0$ and $V_H(0) = 0$ yields, $\forall \tau \le 1$,

$$\left\{ \begin{array}{rcl} \mathbb{E}[\delta_\tau A^q] &=& C_q(\tau)\tau^q \exp\left[-\varphi(q)m(\mathcal{C}_\tau)\right], \\ \mathbb{E}[|\delta_\tau V_H|^q] &=& C'_q(\tau)\tau^{qH} \exp\left[-\varphi(qH)m(\mathcal{C}_\tau)\right], \end{array} \right. \tag{18}$$

where $C_q(\tau)$ and $C'_q(\tau)$ are bounded, see (15) and (16). In numerical experiments it turns out that both $C_q(\tau)$ and $C'_q(\tau)$ are close to constant for $\tau \ll 1$.

Moreover, one expects that $\mathbb{E}[\delta_\tau A^q] \sim \tau^q$ and $\mathbb{E}[\delta_\tau V_H^q] \sim \tau^{qH}$ for large $\tau \gg 1$. This can be understood thanks to a central limit theorem argument under some technical assumptions [9].

A key property of these scaling behaviors (15) or (18) is that they hold continuously through the scales, not only for a particular set of discrete scales. Again, we put the emphasis as well on the fact that the construction of Q_r and A enables a full control of the way the cascading process develops along scales and not only of the multifractal behavior obtained in the limit $\tau \to 0$. As far as applications and real world data modeling are concerned, we believe that the control of the entire cascade process is probably more relevant than that of the asymptotic behavior as $\tau \to 0$ only.

5 Evolution of the increments distributions

IDCs and infinitely divisible scaling. Let us note that previous work in this area [6, 7, 23] inspired a priori the search for non power law scaling as in (2) of the form $\exp[-\zeta(q)n(\tau)]$ by analysis and measurement. This approach has been referred to as *log-infinitely divisible cascades* in the past. To avoid confusion between synthesis and analysis, we prefer to reserve the word "cascade" to describe a construction and to talk of *Infinitely Divisible Scaling* as far as the analysis is concerned below.

In this paper, we have focussed on the *construction* of processes with such prescribed properties. On one hand, this is achieved as far as the behavior of Q_r with r or the behavior of $\delta_\tau A/\tau$ are concerned. On the other hand, we are naturally led in (15) (16) and (18) to a mixture of a power law and a non power law behavior of the form $\tau^q \cdot \exp[-\varphi(q)m(\mathcal{C}_\tau)]$. This result is inherent to the use of an integral to define $A(t)$. The τ^q term is due to the fact that an IDC-Motion is obtained by *integration* of an IDC-Noise. The $\exp[-\varphi(q)m(\mathcal{C}_\tau)]$ term is related to the underlying IDC-Noise $Q_r(t)$. Equation (18) does not reduce to (2) unless $m(\mathcal{C}_\tau) = n(\tau) = -\ln\tau$. Even though the processes presented here do not exactly match the framework of the traditional *infinitely divisible scaling analysis*,

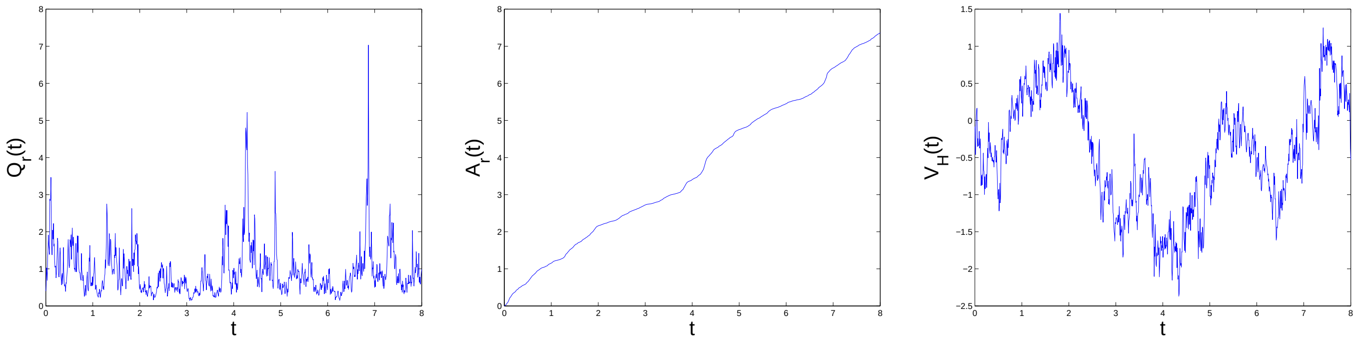

FIG. 3: Sample of a realization of **(left)** $\boldsymbol{Q_r(t)}$, **(middle)** $\boldsymbol{A(t)}$ and **(right)** $\boldsymbol{V_H(t)}$.

this approach provides us with a way to point out relevant quantity to look at when aiming at a precise description of IDC motion and IDC Random Walk introduced above. The content of next paragraph is inspired by the spirit of infinitely divisible scaling analysis but will mainly focus on the particular properties of the IDC Motion and Random Walk.

Evolution of probability density functions. Self-similar processes such as fractional Brownian motion and Lévy motion are bound to have linear exponents $\zeta(q) = qH$. A non-linear dependence of scaling exponents on q ($\zeta(q) \neq qH$) on the other hand has its bearing on at least two approaches to the analysis of process with complex scaling structure.

First, in multifractal analysis the presence of a non-linear function $\zeta(q)$ is usually taken as an indication of a rich and highly interwoven local regularity structure, though the connection between the global ζ and the local Hölder regularity can be made precise only in the context of the multifractal formalism, which usually has to be established with much effort. Second, a non-linear function ζ can also be observed as well as an evolution of the probability density functions (PDF) of the increments of a process through the scales as we are about to explain.

For a self-similar process like a fractional Brownian motion, the PDF of the increments over small or large lags are identical up to some adapted renormalization (e.g., a fBm has Gaussian increments). In contrast, those PDFs for an IDC process display an evolution from Gaussian at large scales to non-Gaussian at small scales. We now briefly explain how those PDF of increments for IDC Motion and Random Walk evolve through the scales (see FIG. 9).

Let P_τ the probability density function of $Y = \ln|\delta_\tau X|$ at scale τ. Note that,

$$\mathbb{E}|\delta_\tau X|^q = \int e^{q\ln|\delta_\tau X|} P_\tau(\ln|\delta_\tau X|)\mathrm{d}\ln|\delta_\tau X| \qquad (19)$$

$$= \int_{-\infty}^{+\infty} e^{qY} P_\tau(Y)\,dY = \tilde{P}_\tau(q) \qquad (20)$$

where $\tilde{P}_\tau(q)$ is the *moment generating function* (analogous to a two-sided Laplace transform) of P_τ. If scaling laws (18) are power laws, one has for $0 < \tau_2 \le \tau_1 < 1$:

$$\mathbb{E}|\delta_{\tau_2} X|^q = \exp\{-\zeta(q).[(-\ln\tau_2) - (-\ln\tau_1)]\} \cdot \mathbb{E}|\delta_{\tau_1} X|^q$$
$$\tilde{P}_{\tau_2}(q) = \underbrace{\tilde{G}(q)^{[(-\ln\tau_2)-(-\ln\tau_1)]}}_{\tilde{G}_{\tau_1,\tau_2}(q)} \cdot \tilde{P}_{\tau_1}(q). \qquad (21)$$

Subjecting the last product to an inverse Laplace transform it turns into the following convolution in the "real" space:

$$\begin{aligned} P_{\tau_2}(Y) &= \underbrace{G^{*[(-\ln\tau_2)-(-\ln\tau_1)]}} * P_{\tau_1}(Y) \\ &= \quad G_{\tau_1,\tau_2} \quad * P_{\tau_1}(Y), \end{aligned} \qquad (22)$$

where G_{τ_1,τ_2} is the probability density function of a distribution that carries the whole information describing *the evolution of the probability density functions* $P_\tau(\ln|\delta_\tau X|)$ *through the scales* τ. Note that G_{τ_1,τ_2} takes a special form with an exponent $\ln\tau$ when associated to a power law scaling.

Let us now remark that the general form of the last line of (22) may suit more general scaling processes like IDC Motion and Random Walk. Indeed, using (18), $\tilde{G}_{\tau_1,\tau_2}(q)$ in (21) becomes for A and V_H respectively:

$$\begin{cases} \tilde{G}^{A}_{\tau_1,\tau_2}(q) = \exp[q\ln(\frac{\tau_2}{\tau_1}) - \varphi(q)(m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1}))] \\ \tilde{G}^{V_H}_{\tau_1,\tau_2}(q) = \exp[qH\ln(\frac{\tau_2}{\tau_1}) - \varphi(qH)(m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1}))]. \end{cases} \qquad (23)$$

Cumulants. Since the evolution of the PDF is described by a convolution, a description in terms of the *cumulants* of distributions is enlightening[4]:

$$\ln\tilde{G}_{\tau_1,\tau_2}(q) = \sum_{k=1}^{\infty} \frac{C^G_k(\tau_1,\tau_2)}{k!} q^k. \qquad (24)$$

Thus, the cumulants $C^Y_k(\tau)$ of $Y = \ln\mid\delta_\tau X\mid$ obey:

$$C^Y_k(\tau_2) = C^G_k(\tau_1,\tau_2) + C^Y_k(\tau_1). \qquad (25)$$

Recall that C_1 and C_2 are respectively the mean and the variance of the corresponding distribution. Note that only the mean may vary for a self-similar process: the invariance by dilation on $\delta_\tau X$ becomes an invariance by translation on $Y = \ln|\delta_\tau X|$. The PDF of the increments of

[4] This description makes sense only under the assumption that the cumulants are well defined. This may not be true in some cases. For instance, only one singular cumulants C_α, $0 < \alpha \le 2$, may be defined for α-stable cascades.

$\delta_\tau X$ have the same shape at all scales; then the PDF of Y at scale τ_2 simply results from a translation of the PDF at scale τ_1. As a consequence, all the cumulants C_k^G of order $k \geq 2$ are zero in the self-similar case. As soon as there exists some non zero cumulant $C_k^G(\tau_1, \tau_2)$ of order $k \geq 2$, one observes an evolution of the PDF of the increments through the scales.

One of the most simple example of multiscaling process is the power law scaling log-normal cascade for which $\varphi(q) = \frac{\sigma^2}{2} q(1-q)$, and

$$\tilde{G}^{V_H}(q) = \exp(-(1+\frac{\sigma^2}{2})qH + \sigma^2 H^2 q^2/2) \qquad (26)$$

so that

$$G_{\tau_1,\tau_2} = \mathcal{N}(-(1+\frac{\sigma^2}{2})H\ln(\frac{\tau_1}{\tau_2}), \sigma^2 H^2 \ln(\frac{\tau_1}{\tau_2})). \qquad (27)$$

Then

$$\begin{cases} C_1^G & = (1+\frac{\sigma^2}{2})H\ln(\frac{\tau_2}{\tau_1}) \\ C_2^G & = -\sigma^2 H^2 \ln(\frac{\tau_2}{\tau_1})). \end{cases} \qquad (28)$$

Thus $C_1^Y(\tau)$ (resp. $C_2^Y(\tau)$) is expected to be an increasing[5] (resp. decreasing) function of $\ln\tau$ (see FIG. 10). The log-normal cascade corresponds to Kolmogorov's 1962 model of turbulence[13] and is usually referred to as the simplest model to *describe* the evolution of the PDF of the increments of a multiscaling process. Here a *synthetic* (not *analytic*) model is provided.

In general, we get for IDC Motion A:

$$C_1^G(\tau_1,\tau_2) = \ln(\tfrac{\tau_2}{\tau_1}) - \varphi'(0)[m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1})], \qquad (29)$$
$$C_k^G(\tau_1,\tau_2) = -\varphi^{(k)}(0)[m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1})],\ k \geq 2.$$

For IDC Random Walk V_H, we get:

$$C_1^G(\tau_1,\tau_2) = H\ln(\tfrac{\tau_2}{\tau_1}) - H\varphi'(0)[m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1})], (30)$$
$$C_k^G(\tau_1,\tau_2) = -H^k\varphi^{(k)}(0)[m(\mathcal{C}_{\tau_2}) - m(\mathcal{C}_{\tau_1})],\ k \geq 2.$$

The next section will show that properties (29) and (30) can be checked on synthesized processes A and V_H. As soon as $\varphi(q)$ is a non-linear function of q (which is always the case, otherwise resulting processes are trivial), the PDF of the increments evolve from large scales to smaller scales from Gaussian to non-Gaussian.

6 Numerical validations

6.1 A model from hydrodynamics

To give some pictures of these processes, we describe the numerical examples of two IDC with respectively power law scaling and warped scaling behaviors. We propose to consider a Log-Normal cascade, i.e., distribution G is $\mathcal{N}(\mu, \sigma^2)$ and $\varphi(q) = \frac{\sigma^2}{2}q(1-q)$. For the warped IDC we choose the control measure $dm(t,r) = 1/r^{2+\beta}dtdr$ with $\beta < 0$, which leads to the function $m(\mathcal{C}_\tau(0)) = (\tau^{-\beta} - 1)/\beta$. This choice provably satisfies the conditions of the theorem above, e.g., there are no convergence problems, and corresponds to the model known as the Castaing model [6] in hydrodynamic turbulence. Note that

[5] Recall that in the log-normal case, $\varphi(q) = -\mu q - \sigma^2/2q^2$ and $\varphi(1) = 0 \Rightarrow \mu = -\sigma^2/2 < 0$.

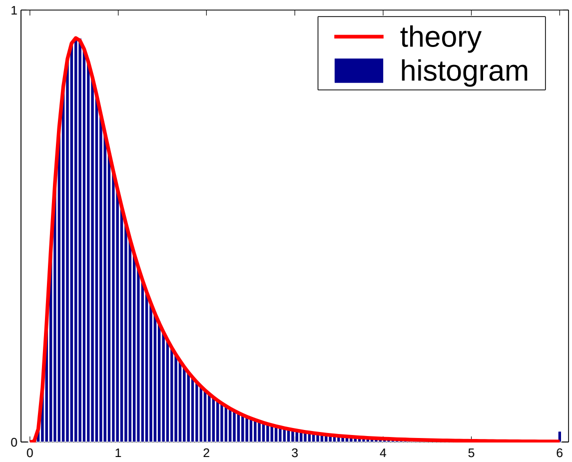

FIG. 4: **Histogram of $\mathbf{Q_r}$** compared to its theoretical log-Normal probability density function: the agreement is perfect.

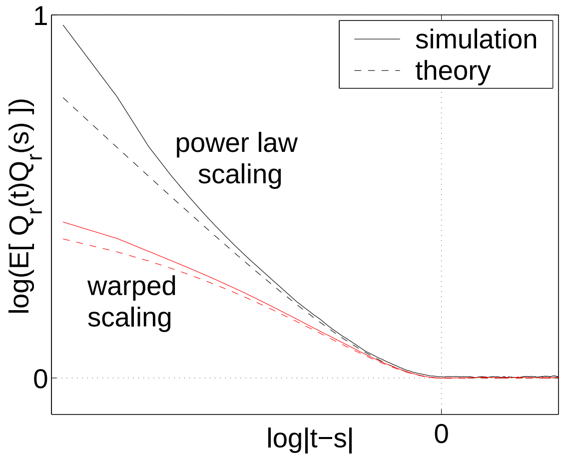

FIG. 5: **Theoretical and estimated autocorrelation functions of $\mathbf{Q_r}$** respectively in a power law scaling case $(dm(t,r) = dtdr/r^2)$ and a warped scaling case $(dm(t,r) = dtdr/r^{2+\beta})$: it is sensitive to a departure from the reference power law behavior in a controlled manner.

$\beta = 0$ reduces to the well-known power law scaling case $(m(\mathcal{C}_\tau) = -\ln\tau)$ [3, 5, 10]. Parameters of the simulation are $\mu = -0.1$, $\sigma^2 = 0.2$ and $\beta = -0.4$. The Hurst exponent H of the fractional Brownian motion B_H used to build $V_H(t)$ has been set to $H = 1/3$.

The next sections will illustrate with some graphics that the numerically synthesized processes have the prescribed properties described in previous sections.

6.2 Scaling of IDC Noise

Marginal distribution. A very basic property of the IDC noise under study is that $Q_r(t)$ has a log-Normal distribution with known parameters $\mu(\tau) = \mu m(\mathcal{C}_\tau)$ and $\sigma^2(\tau) = \sigma^2 m(\mathcal{C}_\tau)$. The log Normal nature of this distribution is independent of the precise form of $m(\mathcal{C}\tau)$; only parameters $\mu(\tau)$ and $\sigma^2(\tau)$ are sensitive to $m(\mathcal{C}\tau)$. FIG. 4 shows that the estimated normalized histogram

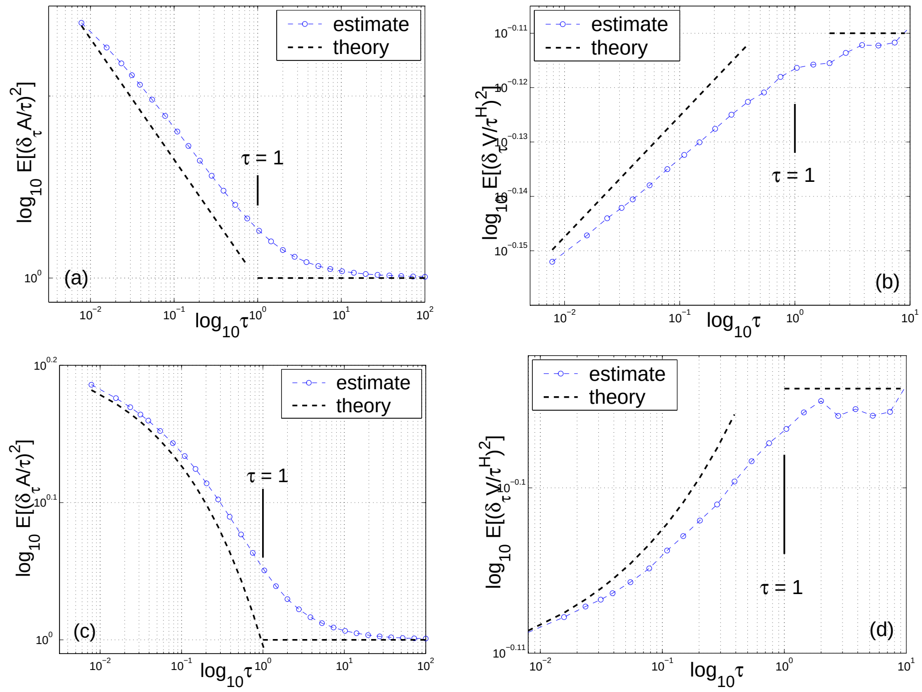

FIG. 6: **A power law scaling cascade: (a)** $\ln \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $c\varphi(2)\ln\tau + Cte$. **(b)** $\ln \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $c\varphi(2H)\ln\tau + Cte$. **A warped cascade deviates from power laws: (c)** $\ln \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $-\varphi(2)m(\mathcal{C}_\tau)$; **(d)** $\ln \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $-\varphi(2H)m(\mathcal{C}_\tau)$. Power law scaling is associated to straight lines in log-log diagrams.

and the theoretical probability density function are in perfect agreement.

Autocorrelation. From (11), we get in the power law scaling case ($dm(t,r) = dtdr/r^2$) for $r \leq |t-s| \leq 1$:

$$\mathbb{E}[Q_r(t)Q_r(s)] = |t-s|^{\varphi(2)}e^{-\varphi(2)(|t-s|-1)}. \quad (31)$$

Note that a power law behavior is expected at small scales: power law scaling is connected to the power law behavior of the autocorrelation of Q_r. In contrast, we get for the warped scaling case under study (recall that $\beta = -0.4$) for $r \leq |t-s| \leq 1$:

$$\mathbb{E}[Q_r(t)Q_r(s)] = \exp\left[-\varphi(2)\left(\frac{1-|t-s|^{-\beta}}{-\beta} + \frac{|t-s|-|t-s|^{-\beta}}{1+\beta}\right)\right]. \quad (32)$$

Figure 5 shows both theoretical and experimental autocorrelation functions obtained from a power law scaling ($dm(t,r) = dtdr/r^2$) and a warped ($dm(t,r) = dtdr/r^{2+\beta}$) cascades with identical parameters. The observed behaviors are clearly distinct and are in good agreement with theoretical computations.

6.3 IDC Motion & Random Walk

Scaling behaviors. Departures from powerlaw behaviors corresponding to the $\exp[-\varphi(q)m(\mathcal{C}_\tau)]$ term in (15) are expected. FIG. 6 and FIG. 7 shows the results obtained from the analysis of IDC processes respectively in the power law scaling and the warped scaling cases. The

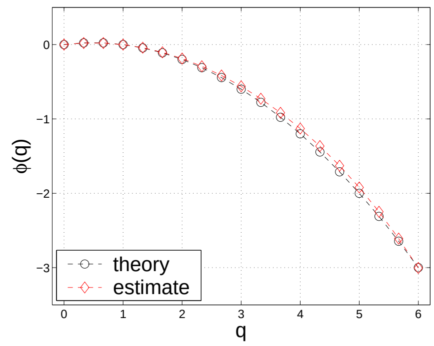

FIG. 7: **Estimated exponents $\varphi(\mathbf{q})$ correspond to prescribed theoretical ones.**

comparison between these results shows that such departures are observed on both $A(t)$ and $V_H(t)$. The performed analysis focuses on $\mathbb{E}[(\delta_\tau A/\tau)^q] \sim \exp[-\varphi(q)m(\mathcal{C}_\tau)]$, resp. $\mathbb{E}[(\delta_\tau V/\tau^H)^q] \sim \exp[-\varphi(qH)m(\mathcal{C}_\tau)]$. In a log-log plot, a curvature is clearly visible whereas the power law scaling case ($\beta = 0$) would have led to straight lines. Note that this warping is accurately controlled for $\tau < 1$ by the form of $m(\mathcal{C}_\tau) \neq -\ln\tau$. These numerical observations are perfectly consistent with our theoretical results. Exponents $\varphi(q)$ can be estimated as well from linear regressions in $\ln \mathbb{E}[(\delta_\tau A/\tau)^q]$ *vs* $m(\mathcal{C}_\tau)$ diagrams – FIG. 7: prescribed exponents are recovered.

Note that a trivial scaling behavior is observed for $A(t)$ as well as for $V_H(t)$ at large scales. For $\tau \geq 1$, $\mathbb{E}[\delta_\tau A^q]$ behaves as τ^q, while $\mathbb{E}[\delta_\tau V_H^q]$ behaves as τ^{qH} (see Section 4).

At small scales, the behavior of $\mathbb{E}[\delta_\tau A^q]$ is dominated by

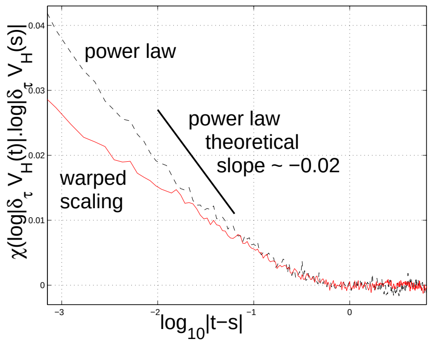

FIG. 8: **Autocorrelation of the log of increments of $\mathbf{V_H}$** from a warped cascade deviates from logarithmic behavior.

the term τ^q. As a consequence, log-log diagrams display close to linear behaviors if no renormalization is used. The warping of the power law, due to the term $\exp[-\varphi(q)m(\mathcal{C}_\tau)]$, may be subtle yet it is true functional dependence and cannot be subsumed by a constant error bound. One may also object that a trivial scaling may be observed at infinitely small scales. Again, we emphasize that the infinitely small scales limit remains out of reach from measurements in applications. Furthermore, there generally exists some finite smallest scale, e.g., the dissipation scale in turbulence. Thus, it should be clear that the purpose is not to control the scaling behavior over the whole range $\tau \in [0, 1]$: the control of a finite range of scales of several decades is sufficient for modeling in applications.

We emphasize that, as far as we are aware of, these are the first cascades displaying controlled non power law behaviors up to a large range of scales (two decades on FIG. 7).

Evolution of probability density functions. As explained in section 5, one expects that the probability density functions of the increments of an IDC Random Walk change from Gaussian at large scales ($\tau \geq 1$) to non Gaussian at smaller scales ($\tau \ll 1$). This is numerically observed on FIG. 9. Figure 9(a) shows this evolution in the power law scaling case ($m(\mathcal{C}_\tau) = -\ln \tau$) while FIG. 9(b) deals with the warped case ($m(\mathcal{C}_\tau) \neq -\ln \tau \Rightarrow$ non power laws scaling). From a qualitative viewpoint, the effect is the same even though it seems that this evolution is less important in the warped case – FIG. 9(b). This is actually true: the kurtosis varies from 3 to 3.6 for the warped case while it varies from 3 to 4.6 for the power law case. This is consistent with the cumulant analysis performed below.

Cumulants of $\ln|\delta_\tau V_H|$. We have seen in section 5 that the information describing the evolution of the PDF of the increments from large scales to smaller scales was held by some distribution G_{τ_1,τ_2} (see (22)). Moreover, the cumulants of this distribution or equivalently the cumulants C_k^Y of $Y = \ln|\delta_\tau X|$ appeared as relevant quantities to look at to precisely describe this evolution. Cumulants of order 1 and 2 are shown on FIG. 10 for both a power law scaling and a warped scaling processes. We emphasize the fact that the comparison with the expected theoretical behaviors is rather satisfactory in both cases. This is an evidence of the quality of the synthesis method. Note that it can be proven that if v is some Gaussian random variable, the second order cumulant of $\ln|v|$ is some universal constant close to 1.23. As a consequence, one expects that $C_2^Y \simeq 1.23$ at large scales as observed on FIG. 10(b).

Autocorrelation of $\ln|\delta_\tau V_H|$. A last quantity people often look at is the autocorrelation function of $\ln|\delta_\tau V_H|$. Indeed, its functional form is fundamentally linked to the type of scaling the moments of the increments $\mathbb{E}|\delta_\tau V_H|^q$ obey. Power law scaling is intimately connected to a logarithmic dependence on τ [2, 3]. Figure 8 shows that this is indeed the case for the power law scaling IDC Random Walk while a departure from this canonical behavior is clearly observed for the warped one. Again, the departure from a power law is visible where it was expected to be.

7 Conclusion

In the present work, we gave an overview of the definitions and main properties of continuous time processes with controlled continuous multiscaling behavior. Most importantly, scaling laws exist continuously through the scales and possible departures from a power law behavior are taken into account. We have shown that numerical replications of such processes satisfied the expected theoretical properties that can be consistently studied from various viewpoints (scaling of the moments $\mathbb{E}|\delta_\tau X|^q$, autocorrelation functions, probability density functions, cumulants of $\ln|\delta_\tau X|$...). Reference [9] gives a detailed presentation of synthesis algorithms and theoretical results. Up to our knowledge, Infinitely Divisible Cascading processes are the first continuous multiplicative cascades displaying controlled non power law scaling behaviors. Potential fields of application range from hydrodynamic turbulence to computer network traffic. **Matlab** routines to synthetize these processes are available on our web pages:
`www.isima.fr/∼chainais`,
`www.stat.rice.edu/∼riedi`,
`perso.ens-lyon.fr/patrice.abry`.

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A Outline of proofs

This section outlines the proofs of our main theoretical results which characterize the scaling properties of an IDC-Motion and its associated IDC-Random Walk. The reader is referred to [9] for detailed proofs. While scaling behaviors are rather easy to describe, their mathematical proof calls for some technical assumptions.

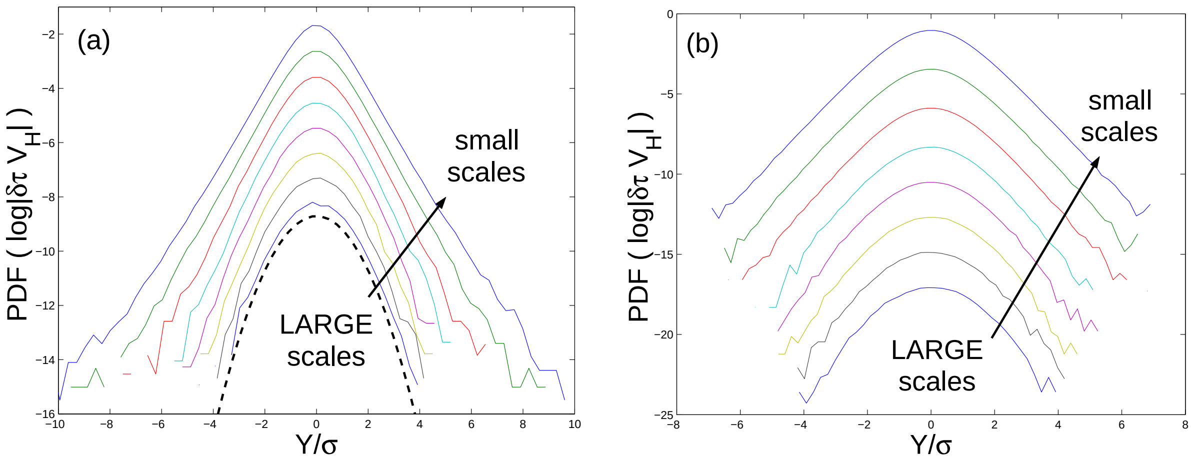

FIG. 9: **Evolution of probability density functions of the increments $\delta_\tau V_H$** from Gaussian at large scales to non Gaussian at smaller scales, respectively for **(a) a power law scaling cascade** and **(b) a warped cascade**.

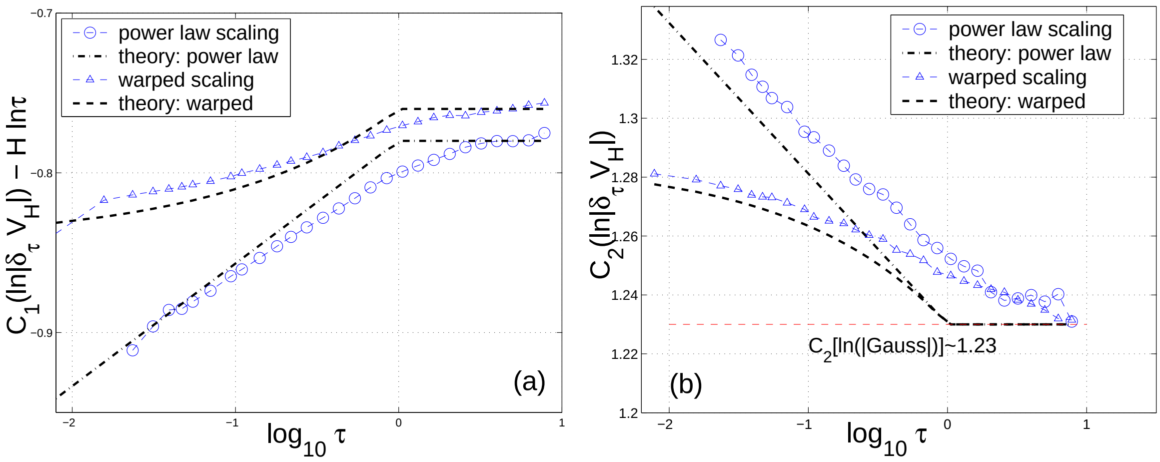

FIG. 10: **Cumulants of the log of increments of $\mathbf{V_H}$** from a warped cascade deviates from power laws: **(a)** $C_1 - H\ln\tau$ compared to $-H\varphi'(0)m(\mathcal{C}_\tau)$; **(b)** C_2 compared to $-H^2\varphi''(0)m(\mathcal{C}_\tau)$.

Let us start by making precise the rescaling property of IDCs. To this end we introduce for $r < b^n$,

$$A_r^{(n)}(t) = \frac{1}{b^n}\int_0^{tb^n} \frac{Q_r(s)}{Q_{b^n}(s)} ds. \tag{33}$$

This cascade has control measure $dm^{(n)}(t,r)$ where

$$g^{(n)}(r) := b^{2n} g(b^n r)\cdot \mathbf{1}_{[0,1]}. \tag{34}$$

Since $m^{(n)}(\mathcal{C}_{r/b^n}(s)) = m(\mathcal{C}_r(b^n s)\backslash \mathcal{C}_{b^n}(b^n s))$ we may understand $A^{(n)}$ as a rescaled zoom into the small scale details of A. In the power law scaling case ($dm(t,r) = dtdr/r^2$) we have $g^{(n)} = g$ and, thus, $A^{(n)}$ is equal in distribution to A.

Lemma 1

Let Q_r be an Infinitely Divisible Cascading Noise and A_r its Motion. Let $0 < r \le b < 1$. Then there exists a non-decreasing process $A_r^{(1)}$ independent of Q_b, such that

$$A_r(t) = b\int_0^t Q_b(s)d[A_r^{(1)}(\frac{s}{b})]. \tag{35}$$

In analogy, we may replace A by $A^{(n)}$ and $A^{(1)}$ by $A^{(n+1)}$.

Proof of Lemma 1.

For the duration of the proof, we introduce the "bandlimited cone"

$$\mathcal{C}_r^b(t) := \{(t', r') \in \mathcal{C}_r(t) \,:\, r' \le b\} = \mathcal{C}_r(t)\backslash \mathcal{C}_b(t). \tag{36}$$

and set

$$Q_r^b(s) := \exp\left[\rho(1)m(\mathcal{C}_r^b)\right]\exp\left[M(\mathcal{C}_r^b(s))\right]. \tag{37}$$

By convention, $Q_r^b(s) = 1$ if $r = b$. Note that $\mathbb{E}[Q_r^b(s)] = 1$. Note also that for any $r < b$ and any t we have $m(\mathcal{C}_r) = m(\mathcal{C}_r^b) + m(\mathcal{C}_b)$ and thus

$$Q_r(s) = Q_r^b(s)\cdot Q_b(s). \tag{38}$$

Now define $Q_r^{(1)}(s) = Q_r^b(bs)$ and set

$$A_r^{(1)}(t) = \int_0^t Q_r^{(1)}(s)ds = \int_0^t Q_r^b(bs)ds = \frac{1}{b}\int_0^{bt} Q_r^b(s)ds. \tag{39}$$

Note that $\mathbb{E}[A_r^{(1)}(t)] = t$. Also, (35) follows by elementary operation. Further, $A^{(1)} = \lim_{r\to 0} A_r^{(1)}$ and Q_b are independent since they are built using disjoint sets of the time-scale half-plane $\mathcal{P}^+$. Finally, $Q_r^b(b\cdot)$ forms an IDC-noise with control measure $m^{(1)}$ as claimed, which can be verified by defining $M^{(1)}(\mathcal{C}_{r/b}(s)) = M(\mathcal{C}_r^b(bs)$. Note that $m^{(1)}(\mathcal{C}_{r/b}(s)) = m(\mathcal{C}_r^b(bs)) = m(b\mathcal{C}_{r/b}(s))$. $\diamondsuit$

If the integrand Q_b in (35) were constant over the interval $[0,t]$ a scaling law of moments would immediately follow. A measure for the variation of the integrand which will prove useful is the following:

$$\Delta_b(t) := \frac{\mathbb{E}\sup_{0\le s\le t}|Q_b(s)^q - Q_b(0)^q|}{\mathbb{E}[Q_b(0)^q]}. \quad (40)$$

The next lemma quantifies by how much the scaling law deviates from the Binomial case where $Q_b(t)$ is indeed a constant for $t < b$.

Lemma 2
Fix $q > 0$. Let $0 < r \le b < 1$ and $0 \le t \le 1$. Then,

$$\mathbb{E}A_r(t)^q = b^q \cdot \mathbb{E}[Q_b(0)^q] \cdot \mathbb{E}[A_r^{(1)}(t/b)^q] \cdot (1+\varepsilon). \quad (41)$$

The error term ε is bounded as: $|\varepsilon| \le \Delta_b(t)$.

Proof of Lemma 2.
We will be using the fact [17] that

$$\left|\left(\int_I x(s)d\mu(s)\right)^q - C\right| \le \sup_{s\in I}|x(s)^q\mu(I)^q - C| \quad (42)$$

Applying it for the measure μ induced by $A_r^{(1)}(\cdot/b)$ and using (9) and (35) we obtain

$$\begin{aligned}
&\left|A_r(t)^q - b^q Q_b(0)^q A_r^{(1)}(t/b)^q\right| \\
&= \left|\left(b\int_0^t Q_b(s)d[A_r^{(1)}(s/b)]\right)^q - b^q\cdot Q_b(0)^q \cdot A_r^{(1)}(t/b)^q\right| \\
&\le b^q \sup_{0\le s\le t}\left|Q_b(s)^q A_r^{(1)}(t/b)^q - Q_b(0)^q A_r^{(1)}(t/b)^q\right| \\
&= b^q\cdot A_r^{(1)}(t/b)^q \cdot \sup_{0\le s\le t}|Q_b(s)^q - Q_b(0)^q|. \quad (43)
\end{aligned}$$

$\diamondsuit$

The error term (41) in lemma 2 can be bounded for certain IDCs, such as the ones featured in the next lemma. To formulate it, some notation is required. For an IDC Motion A_r with control measure $dm(t,r) = g(r)dtdr$ we set for convenience

$$\overline{g}(b) := \int_b^1 g(r)dr \quad (44)$$

as well as for $b \in (0,1)$ and $\nu > 0$

$$C_{b,\nu}[g] := \sup_{0<t\le b}\frac{1}{t^\nu}\cdot\Delta_b(t) \in [0,\infty] \quad (45)$$

Lemma 3
Fix $q > 0$. Let $0 < t \le b < 1$.

CPC Case: If Q_b is a Compound Poisson Cascade with weights W which possess finite q-th moments, then $C_{b,1}[g]$ is finite. In other words, for all $t < b$:

$$\frac{1}{t}\Delta_b(t) \le C_{b,1}[g] < \infty. \quad (46)$$

Log-normal Case: For any log-normal IDC with

$$\rho(q) := -q\mu - q^2\sigma^2, \quad (47)$$

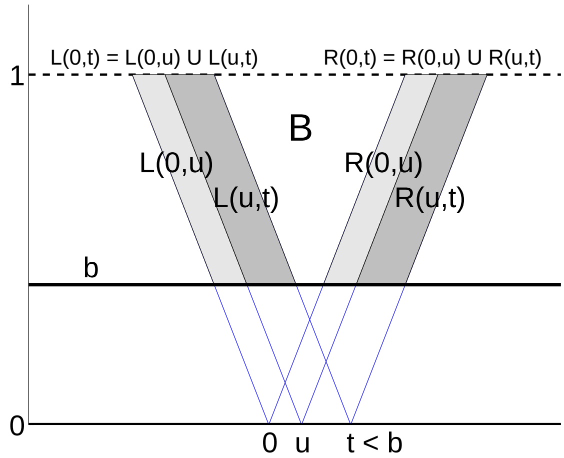

FIG. 11: **Definition of $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{R}$.**

$C_{b,1/2}[g]$ is finite. More precisely, given $q > 0$, $b \le 1$ and g, there exist real numbers J, c_1 and c_2 depending only on q, b, μ, σ^2 and on $\overline{g}(b)$ such that:

$$\frac{1}{\sqrt{t}}\Delta_b(t) \;\le\; (J\cdot c_1\sqrt{t} + c_2)\cdot\max(1, e^{\rho(q)\overline{g}(b)}). \quad (48)$$

In both cases, if in addition $g^{(n)}$ as defined in Theorem 1 converges, then the bounds $C_{b,1/2}[g^{(n)}]$ remain uniformly bounded as $n\to\infty$.

Proof of Lemma 3.
First, we simplify the expressions and separate independent from dependent parts of $M(\mathcal{C}_b(u))$ and $M(\mathcal{C}_b(0))$. Thus, we write easily

$$\Delta_b(t) \;=\; \frac{\mathbb{E}\sup_{0\le u\le t}|e^{qM(\mathcal{C}_b(u))} - e^{qM(\mathcal{C}_b(0))}|}{\mathbb{E}[e^{qM(\mathcal{C}_b(0))}]} \quad (49)$$

and introduce the following parallelepiped as subsets of the time-scale strip:

$$\begin{aligned}
\mathcal{L}(u,v) &= \{(s,r) : b\le r\le 1, -r+u\le s< -r+v\}, \\
\mathcal{R}(u,v) &= \{(s,r) : b\le r\le 1, r+u\le s< r+v\}, \quad (50)\\
\mathcal{B} &= \mathcal{C}_b(t)\cap\mathcal{C}_b(0) = \{(s,r) : b\le r\le 1, -r+t\le s\le r\}.
\end{aligned}$$

Checking the constraints on the variable s one verifies quickly the following decomposition of a cone $\mathcal{C}_b(u)$ into *disjoint* sets which is valid for $u\in[0,t]$ and for $t\le b$ (see FIG. 11):

$$\mathcal{C}_b(u) \;=\; \mathcal{L}(u,t)\cup\mathcal{B}\cup\mathcal{R}(0,u). \quad (51)$$

As a particular case we have $\mathcal{C}_b(0) = \mathcal{L}(0,t)\cup\mathcal{B}$. Noting that $\mathcal{L}(u,v)\cup\mathcal{L}(v,w) = \mathcal{L}(u,w)$ with disjoint union whenever $u\le v\le w$, we find $M(\mathcal{C}_b(u)) - M(\mathcal{C}_b(0)) = M(\mathcal{R}(0,u)) - M(\mathcal{L}(0,u))$ and may write $\Delta_b(t)$ as:

$$\frac{\mathbb{E}[e^{qM(\mathcal{B})}]}{\mathbb{E}[e^{qM(\mathcal{C}_b(0))}]}\cdot\mathbb{E}\left[e^{qM(\mathcal{L}(0,t))}\sup_{0\le u\le t}\left|\frac{e^{qM(\mathcal{R}(0,u))}}{e^{qM(\mathcal{L}(0,u))}} - 1\right|\right] \quad (52)$$

Here, we used that the term $e^{qM(\mathcal{B})}$ is statistically independent of the others in the enumerator. We note that

$$\begin{aligned}
\frac{\mathbb{E}[e^{qM(\mathcal{B})}]}{\mathbb{E}[e^{qM(\mathcal{C}_b(0))}]} &= e^{-\rho(q)(m(\mathcal{B})-m(\mathcal{C}_b(0)))} \\
&= e^{\rho(q)m(\mathcal{L}(0,t))} \\
&\le \max(1, \exp[\rho(q)m(\mathcal{L}(0,b))]). \quad (53)
\end{aligned}$$

using the fact that $m(\mathcal{L}(0,t))$ is monotonous in t. Note that $m(\mathcal{L}(0,b)) = \overline{g}(b)$. It remains to bound the second term in (52).

Now the idea is to show that with t very small and thus u small, the control measures $m(\mathcal{R}(0,u))$ and $m(\mathcal{L}(0,u))$ are very small, thus the corresponding random variables are small with high probability and thus $e^{qM(\mathcal{R}(0,u))}$ and $e^{qM(\mathcal{L}(0,u))}$ are both close to 1. Thus their quotient is close to one and the contribution to the last term in (52) is small with large probability.

As a matter of fact, that quotient is exactly equal to 1 with large probability in the CPC case. The log-normal case is somewhat more intricate but relies only on standard bounds [9].

$\diamondsuit$

Theorem

Let $q > 0$. Let $\rho(\cdot)$ defining as above some infinitely divisible law. Let A_r be an IDC Motion with control measure $g(r)dtdr$. Assume that there are constants $b \in (0,1)$ and $\nu > 0$ such that $C_{b,\nu}[g^{(n)}]$ are finite and remain bounded as $n \to \infty$; assume that A_r as well as $A_r^{(n)}$ for large n converge in $\mathcal{L}_q$. Then there exist constants $\overline{C}_q$ and $\underline{C}_q$ and $\overline{C}'_q$, $\underline{C}'_q$ such that for any $t < b$

$$\underline{C}_q t^q e^{-\varphi(q)m(\mathcal{C}_t)} \leq \mathbb{E}A(t)^q \leq \overline{C}_q t^q e^{-\varphi(q)m(\mathcal{C}_t)}, \tag{54}$$

$$\underline{C}'_q t^{qH} e^{-\varphi(qH)m(\mathcal{C}_t)} \leq \mathbb{E}[|V_H(t)|^q] \leq \overline{C}'_q t^{qH} e^{-\varphi(qH)m(\mathcal{C}_t)}. \tag{55}$$

The assumptions of the Theorem are verified for compound Poisson distributions as well as for the Normal distributions, assuming that the functions $g^{(n)}$ converge (see above lemmas as well as [9]).

The proof of Theorem 1 relies on iterating (41) n times keeping b fixed. Thus, we will apply it successively with t/b^k to the cascades $A_r^{(k)}$ introduced in (33), for $k = 0,\ldots,n-1$. Note that $A_r^{(k)}$ possesses the control measure $g^{(k)}(r)dtdr$ which leads to

$$\mathbb{E}A_r(t)^q = (b^q \cdot \mathbb{E}[Q_b(0)^q])^n \cdot \mathbb{E}[A_r^{(n)}(t/b^n)^q] \cdot \prod_{k=0}^{n-1}(1+\varepsilon_k). \tag{56}$$

The error terms can be bounded using $C_b[g^{(k)}]$. $\diamondsuit$

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