stochastic processes and their applications

Stochastic Processes and their Applications 86 (2000) 177–182

www.elsevier.com/locate/spa

Stochastic integral representation and properties of the wavelet coefficients of linear fractional stable motion

Lieve Delbeke*, Patrice Abry

Royal Meteorological Institute of Belgium, Department of Meteorological Research and Development, Ringlaan 3, 1180 Brussels, Belgium

Received 3 July 1997; received in revised form 15 July 1999

Abstract

Let $0 < \alpha \leqslant 2$ and let $T \subseteq \mathbb{R}$. Let $\{X(t), t \in T\}$ be a linear fractional α-stable $(0 < \alpha \leqslant 2)$ motion with scaling index H $(0 < H < 1)$ and with symmetric α-stable random measure. Suppose that ψ is a bounded real function with compact support $[a,b]$ and at least one null moment. Let the sequence of the discrete wavelet coefficients of the process X be

$$\left\{ D_{j,k} = \int_{\mathbb{R}} X(t)\psi_{j,k}(t)\,\mathrm{d}t,\ j,k \in \mathbb{Z} \right\}.$$

We use a stochastic integral representation of the process X to describe the wavelet coefficients as α-stable integrals when $H - 1/\alpha > -1$. This stochastic representation is used to prove that the stochastic process of wavelet coefficients $\{D_{j,k},\ k \in \mathbb{Z}\}$, with fixed scale index $j \in \mathbb{Z}$, is strictly stationary. Furthermore, a property of self-similarity of the wavelet coefficients of X is proved. This property has been the motivation of several wavelet-based estimators for the scaling index H. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Linear fractional stable motion; Wavelet analysis; Stable integral; Self-similarity

1. Introduction

Let $0 < \alpha \leqslant 2$ and let $T \subseteq \mathbb{R}$. Let $X = \{X(t), t \in T\}$ be a measurable version of linear fractional α-stable $(0 < \alpha \leqslant 2)$ motion (or also linear fractional Lévy motion or LFSM) with scaling index $H \in (0,1)$ and symmetric α-stable $(S\alpha S)$ random measure. Linear fractional α-stable motion is a class of self-similar processes with stationary increments. It is the most commonly used class of self-similar processes that extends the fractional Brownian motion. It indeed offers an efficient tool to model data exhibiting scaling properties but whose probability density functions depart from the Gaussian law.

LFSM is mathematically defined as follows: let $\alpha \in (0,2]$ and let M be an α-stable random measure on $\mathbb{R}$ with Lebesgue control measure m. The linear fractional stable

* Corresponding author. Tel.: +32-2-373-05-42; fax: +32-2-373-05-48.
E-mail addresses: believe@oma.be (L. Delbeke), pabry@physique.ens-lyon.fr (P. Abry)

0304-4149/00/$ - see front matter
PII: S0304-4149(99)00092-7

motion is then defined by

$$X(t) = \int_{\mathbb{R}} f_{\alpha,H}(t,x) M(\mathrm{d}x), \tag{1}$$

where

$$f_{\alpha,H}(t,x) = ((t-x)_+)^{H-1/\alpha} - ((-x)_+)^{H-1/\alpha}, \tag{2}$$

and where $0 < H < 1$. The notation $(x)_+$ means that $(x)_+ = x$ if $x \geqslant 0$ and 0 otherwise. In this paper, we suppose that the α-stable random measure M is symmetric and we use a measurable version of the process. We need the measurability of the process X when we represent the wavelet coefficients of X as stable integrals. If $0 < \alpha \leqslant 2$, we know from Samorodnitsky and Taqqu (1994) that there exists a measurable version of the defined process (1) if $T \subseteq \mathbb{R}$. Let ψ be a bounded real function with compact support $[a,b]$ and at least one null moment. A function with these properties is an example of a wavelet function (see Cohen, 1992). The wavelet functions of this paper have to satisfy the mentioned properties. Define the sequence of discrete wavelet coefficients $\{D_{j,k},\ j,k \in \mathbb{Z}\}$ of the process X as:

$$D_{j,k} = \int_{\mathbb{R}} X(t)\psi_{j,k}(t)\,\mathrm{d}t, \tag{3}$$

where $\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k)$ and $j,k \in \mathbb{Z}$.

In Section 2, we will represent the wavelet coefficients of X as stable integrals. Therefore, we need a Fubini-type result that permits us to interchange a deterministic and a stochastic integration. In Section 3, we use the stochastic representation of the wavelet coefficients to show the main results of this work: the stochastic process $\{D_{j,k},\ k \in \mathbb{Z}\}$ is strictly stationary for fixed scale index j. We also show that, if $j \in \mathbb{Z}$ and $k_1, \dots, k_m \in \mathbb{Z}$, then

$$(D_{j,k_1}, \dots, D_{j,k_m}) \stackrel{\mathrm{d}}{=} 2^{j(H+1/2)}(D_{0,k_1}, \dots, D_{0,k_m}) \tag{4}$$

or also, for any $j \in \mathbb{Z}$,

$$\{D_{j,k},\ k \in \mathbb{Z}\} \stackrel{\mathrm{d}}{=} \{2^{j(H+1/2)} D_{0,k},\ k \in \mathbb{Z}\}.$$

This property of self-similarity has been the motivation and starting point for the definition of several wavelet-based estimators for the scaling index H. Such estimators are discussed and studied in detail in Delbeke (1998). We will use the following notation:

$$L^\alpha(E, \varepsilon, \nu) = \left\{ f\colon f \text{ is measurable}, \int_E |f(x)|^\alpha \nu(\mathrm{d}x) < \infty \right\},$$

where $E \subset \mathbb{R}$, ε is the Borel σ-algebra on (E, m).

2. The distribution of a wavelet coefficient

In order to deduce the distribution of the wavelet coefficients of the process X, we use a measurable version of X. If $0 < \alpha < 2$ and $T \subseteq \mathbb{R}$, it is proved by Theorem 11.1.1 of Samorodnitsky and Taqqu (1994, p. 498) that X has a measurable version. When

$\alpha = 2$, the process X is clearly measurable since it has continuous sample paths (see Exercise 9.15 (ii) in Samorodnitsky and Taqqu, 1994). The proof of the next theorem is given in the appendix.

Theorem 1. *Let* $0 < \alpha \leqslant 2$ *and* $0 < H < 1$ *such that* $H - 1/\alpha > -1$. *Let* $X = \{X(t),\ t \in T \subset \mathbb{R}\}$ *be a measurable version of LFSM. Let* $v(\mathrm{d}t) = g(t)\,\mathrm{d}t$ *where* g *is a bounded positive function with compact support. Then*

$$\int_T |X(t)| v(\mathrm{d}t) < \infty \quad a.s. \tag{5}$$

We use the previous theorem to deduce the following result.

Theorem 2. *Let* $T \subset \mathbb{R}$. *Let* $X = \{X(t),\ t \in T\}$ *be a measurable version of LFSM. Suppose that* $v(\mathrm{d}t) = g(t)\,\mathrm{d}t$ *where* $g : [a,b] \to \mathbb{R}$ *is a bounded positive function. Then*

$$\int_a^b X(t) v(\mathrm{d}t) = \int_{\mathbb{R}} \left(\int_a^b f_{\alpha,H}(t,x) v(\mathrm{d}t) \right) M(\mathrm{d}x) \quad a.s. \tag{6}$$

Thus, in particular, $\int_a^b f_{\alpha,H}(t,x) v(\mathrm{d}t) \in L^\alpha(\mathbb{R}, \varepsilon, m)$.

Proof. This follows directly from lemma Samorodnitsky and Taqqu (1994, p. 511) and Theorem 1. □

To find properties of the wavelet coefficients of LFSM, we will use the following corollary.

Corollary 1. *Let* T *be a compact set in* $\mathbb{R}$ (*if* $0 < \alpha \leqslant 2$) *or* $T = \mathbb{R}$ (*if* $0 < \alpha < 2$). *Let* $X = \{X(t),\ t \in T\}$ *be a measurable version of LFSM and suppose that* $\psi : [a,b] \to \mathbb{R}$ *is a bounded function with at least one null moment. Let* $\psi_1 = \frac{1}{2}(|\psi| + \psi)$ *and* $\psi_2 = \frac{1}{2}(|\psi| - \psi)$ *and suppose that* ψ_1 *and* ψ_2 *are bounded. Then*

$$\int_a^b X(t) v(\mathrm{d}t) = \int_{\mathbb{R}} \left(\int_a^b f_{\alpha,H}(t,x) \psi(t)\,\mathrm{d}t \right) M(\mathrm{d}x) \quad a.s. \tag{7}$$

Thus, in particular, $\int_a^b f_{\alpha,H}(t,x) \psi(t)\,\mathrm{d}t \in L^\alpha(\mathbb{R}, \varepsilon, m)$.

Proof. The signed measure $v(\mathrm{d}t) = \psi(t)\,\mathrm{d}t$ can be written as the difference of the two positive measures $v_1(\mathrm{d}t) = \psi_1(t)\,\mathrm{d}t$ and $v_2(\mathrm{d}t) = \psi_2(t)\,\mathrm{d}t$. The result follows then from Theorem 2. □

3. Main results

By Proposition 3.4.2 of Samorodnitsky and Taqqu (1994), we know that the wavelet coefficients

$$D_{j,k} = \int_{\mathbb{R}} X(t) \psi_{j,k}(t)\,\mathrm{d}t, \quad j,k \in \mathbb{Z}, \tag{8}$$

are jointly stable and, for $\theta_{j_u,k_u} \in \mathbb{R}$, $j_1,\dots,j_n$, $k_1,\dots,k_m \in \mathbb{Z}$,

$$E\mathrm{e}^{\mathrm{i}\sum_{u=1}^{m}\sum_{v=1}^{n}\theta_{j_u,k_v}D_{j_u,k_v}} = \exp\left\{-\int_{\mathbb{R}}\sum_{u=1}^{m}\sum_{v=1}^{n}\theta_{j_u,k_v}\int_{\mathbb{R}} f_{\alpha,H}(t,x)\phi_{j_u,k_v}(t)\,\mathrm{d}t\right|^{\alpha}\mathrm{d}x.$$

Since the wavelet coefficients are defined on the same probability space as the random measure M, the sequence $\{D_{j,k},\ j,k\in\mathbb{Z}\}$ is a stochastic process. The wavelet coefficients of LFSM satisfy the following properties.

Theorem 3. (1) *For each fixed $j\in\mathbb{Z}$, the stochastic process $\{D_{j,k}, k\in\mathbb{Z}\}$ is strictly stationary*;

(2) *for each $j\in\mathbb{Z}$ and $k_1,\dots,k_m$, we have*

$$(D_{j,k_1},\dots,D_{j,k_m}) \stackrel{\mathrm{d}}{=} 2^{j(H+1/2)}(D_{0,k_1},\dots,D_{0,k_m}). \tag{9}$$

Proof. (i) The distribution of $D_{j,k}$ does not depend on k since

$$\begin{aligned}
D_{j,k} &= \int_{\mathbb{R}} X(t)\psi_{j,k}(t)\,\mathrm{d}t = 2^{j/2}\int_{\mathbb{R}} X(2^j(u+k))\psi(u)\,\mathrm{d}u\\
&= 2^{j/2}\int_{\mathbb{R}}(X(2^j(u+k)) - X(2^jk))\psi(u)\,\mathrm{d}u\\
&\stackrel{\mathrm{d}}{=} 2^{j/2}\int_{\mathbb{R}} X(2^ju)\psi(u)\,\mathrm{d}u = D_{j,0}.
\end{aligned}$$

Similarly, one can prove that, for $k_1,\dots,k_n$, $k\in\mathbb{Z}$ and $\theta_{k_i}\in\mathbb{R}$,

$$\sum_{i=1}^{n}\theta_{k_j}D_{k_i+k} \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n}\theta_{k_j}D_{j,k_i}.$$

(ii) We prove that $D_{j,k} \stackrel{\mathrm{d}}{=} 2^{j(H+1/2)}D_{0,k}$:

$$\begin{aligned}
D_{j,k} &= 2^{j/2}\int_{\mathbb{R}} X(2^j(u+k))\psi(u)\,\mathrm{d}u\\
&\stackrel{\mathrm{d}}{=} 2^{j(H+1/2)}\int_{\mathbb{R}} X(u+k)\psi(u)\,\mathrm{d}u\\
&= 2^{j(H+1/2)}D_{0,k}.
\end{aligned}$$

Similarly, one can prove that, for $k_1,\dots,k_m\in\mathbb{Z}$ and $\theta_{k_i}\in\mathbb{R}$,

$$\sum_{i=1}^{m}\theta_{k_i}D_{j,k_i} \stackrel{\mathrm{d}}{=} 2^{j(H+1/2)}\sum_{i=1}^{m}\theta_{k_i}D_{0,k_i}. \qquad \square$$

4. Conclusion

We have shown the following properties of the wavelet coefficients of linear fractional stable motion:

- the wavelet coefficients are stable integrals;

- the stochastic process of wavelet coefficients with fixed scale index is strictly stationary;
- the self-similarity of linear fractional stable motion leads to a scaling property of the wavelet coefficients of the process.

This property has given rise to several estimation methods for estimating the scaling index H. Studies and comparisons of various wavelet-based estimators for the scaling index H of the fractional Brownian motion can be found in Delbeke (1998). Moreover, in many situations, data exhibiting scaling properties also have non-Gaussian probability density functions.

5. For further reading

The following references are also of interest to the reader: Billingsley, 1995; Cohen et al., 1993; Maejima, 1994; Maejima, 1995.

Acknowledgements

The authors are grateful to an anonymous reviewer for his detailed comments on an earlier version.

Appendix

Proof of Theorem 1. Take $[a,b]=[0,1]$ and let $C>0$ such that $|\psi(t)|\leqslant C\ \forall t\in[a,b]$. Let $d=H-1/\alpha$.

The case $H>1/\alpha$: it follows immediately since g has a compact support and LFSM has continuous sample paths.

The case $H<1/\alpha$: if $\alpha<1$, we use Theorems 11.3.2 and 11.4.1 of Samorodnitsky and Taqqu (1994). One needs to prove that

$$\int_{\mathbb{R}}\left(\int_0^1 |f_{\alpha,H}(t,x)|\nu(\mathrm{d}t)\right)^{\alpha}\mathrm{d}x<\infty.$$

We have

$$\int_{\mathbb{R}}\left(\int_0^1 |f_{\alpha,H}(t,x)|\nu(\mathrm{d}t)\right)^{\alpha}\mathrm{d}x\leqslant C^{\alpha}\int_{-\infty}^1\left(\int_0^1 |f_{\alpha,H}(t,x)|\,\mathrm{d}t\right)^{\alpha}\mathrm{d}x.$$

Let

$$I_{\alpha,H}=\int_{-\infty}^1\left(\int_0^1 |f_{\alpha,H}(t,x)|\,\mathrm{d}t\right)^{\alpha}\mathrm{d}x.$$

Then

$$I_{\alpha,H}=\int_{-\infty}^0\left(\int_0^1 |(t-x)^d-(-x)^d|\,\mathrm{d}t\right)^{\alpha}\mathrm{d}x+\int_0^1\left(\int_x^1 (t-x)^d\,\mathrm{d}t\right)^{\alpha}\mathrm{d}x. \tag{10}$$

The first integral of (10) may diverge at (i) $x = -\infty$ or at (ii) $x = 0$. As $x \to -\infty$, the function

$$\int_0^1 |(t-x)^d - (-x)^d| \, dt$$

behaves like $d/2|x|^{d-1}$. Thus, the integrand converges at $x = -\infty$ since $(d-1)\alpha + 1 < 0$. As $|x| \to 0$ in the first integral (10), then the integrand behaves like $(-x)^{d\alpha}$, which is integrable since the condition to be satisfied, $d\alpha + 1 > 0$, requires that $H > 0$, which is also the case. Note that the second integral of (10) is finite since we suppose that $d + 1 > 0$.

If $\alpha > 1$, one needs to show that

$$\int_0^1 \left(\int_{\mathbb{R}} |f_{\alpha,H}(t,x)|^\alpha \, dx \right)^{1/\alpha} dt.$$

This follows immediately from

$$\int_{\mathbb{R}} |f_{\alpha,H}(t,x)|^\alpha \, dx = ct^{H\alpha}$$

because LFSM is well-defined and self-similar. For $\alpha = 1$, one needs to verify that

$$\int_0^1 dt \int_{\mathbb{R}} dx |f(t,x)|(1 + \log_+ A(t,x)) < \infty,$$

with $A(t,x)$ defined by

$$A(t,x) = \frac{|f(t,x)| \int_0^1 \int_{\mathbb{R}} |f(t,v)| g(u) \, dv \, du}{\int_{\mathbb{R}} |f(t,v)| \, dv \int_0^1 |f(u,x)| g(u) \, du}.$$

This can be done.

References

Billingsley, P., 1995. Probability and Measure. Wiley Series in Probability and Mathematical Statistics, New York.

Cohen, A., 1992. Wavelet bases, approximation theory and subdivision schemes. In: E.W. Cheney et al. (Eds.), Approximation Theory VII. Academic Press, Boston, pp. 63–85.

Cohen, A., Daubechies, I., Jawerth, B., Vial, P., 1993. Multiresolution analysis, wavelets and fast algorithms on an interval. C. R. Acad. Sci. Paris 316, 417–421.

Delbeke, L., 1998. Wavelet based estimators for the Hurst parameter of a self-similar process. Doctoral Dissertation, Department of Sciences, K.U.Leuven.

Maejima, M., 1994. Self-similar processes and limit theorems. Sugaku Expositions 2, 102–123.

Maejima, M., 1995. Self-similar Processes. Notes of Tutorial at K.U.Leuven.

Samorodnitsky, G., Taqqu, M.S., 1994. Stable Non-Gaussian Random Processes. Chapman & Hall, New York.