Scattering Multifractal Analysis

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1 Introduction

Many observed phenomena, ranging from turbulent flows in fluid mechanics to network traffic or stock prices, are by nature irregular, singular almost everywhere, with long range dependence. Multifractal processes, introduced in [36] and formalized by Mandelbrot [23] are a large class of mathematical models which have a some form of self-similarity.

The Fourier power spectrum, which depends upon second order moments, is not sufficient to characterize such multifractal processes, beyond Gaussian processes. Multiple moments of wavelet coefficients appeared as a powerful technique to model non-Gaussian properties, in particular to compute the so-called spectrum of singularities [12, 1, 4]. However, the calculation of high order or negative moments leads to statistical estimators with high variance, which limits these type of techniques to long data sequences.

Wavelet scattering operators [22] provide new representations of stationary processes. Scattering coefficients of order m are expected values of non-linear functionals of the process obtained by iterating m times on complex wavelet transforms and modulus non-linearities. First order scattering coefficients are first order moments of wavelet coefficients, whereas second order scattering coefficients depend upon high order moments and specify non-gaussian behavior of stationary processes. A scattering representation is computed with a non-expansive operator and can be estimated with a low variance from a single process realization.

First and second order scattering coefficients provide state of the art classification results over stationary image and audio random textures [15, 2, ?]. These textures are realizations of complex non-Gaussian processes, including multifractal processes. This paper partly explains the efficiency of these texture classification algorithms by analyzing the properties of first and second order scattering coefficients of self-similar and multifractal processes. It shows that a scattering transform provides an alternative mathematical approach to characterize multifractal properties.

Section 2 introduces scattering representations for multifractal analysis. Section3 concentrates on self-similar processes with stationary increments, and particularly on fractional Brownian motions and Lévy processes. Section 4 studies scattering representations of multifractal cascades with stochastic self-similarities. Scattering representations of random multiplicative cascades are studied in Section 4.2. The main theorem of Section 4.4 computes a multifractal intermittency parameter from normalized second order scattering coefficients. Scattering transforms of turbulent flows and of financial time series are studied in Sections 5.1 and 5.2 respectively, showing that it reveals important properties of these processes.

Notations: We denote $\{X(t)\}_t \overset{d}{=} \{Y(t)\}_t$ the equality of all finite-dimensional distributions. The dyadic scaling of $X(t)$ is written $L_j X(t) = X(2^{-j})$. The auto-covariance of stationary process X is denoted $R_X(\tau) = \mathrm{Cov}(X(t), X(t+\tau))$.

We denote $B(j) \simeq F(j)$, $j \to \infty$ (resp $j \to -\infty$) if there exists $C_1, C_2 > 0$ and $J \in \mathbb{Z}$ such that $C_1 \leq \frac{B(j)}{F(j)} \leq C_2$ for all $j > J$ (resp for all $j < J$).

2 Scattering Transform

A large body of work has been deveoted to the study of multifractal properties with wavelet transform. Section 2.1 reviews the properties of wavelet tranform moments and Section 2.2 introduce multiscale scattering transform, computed with iterated wavelet transforms and modulus non-linearities.

2.1 Multifractal Wavelet Analysis

The distribution and scaling properties of multifractal processes have been thoroughly studied [3, 33, 13] through the scaling properties of wavelet transform moments. This section reviews the main properties of dyadic wavelet transforms for multifractal analysis. We consider multifractal processes having stationary increments, which means that the distribution of $\{X(t-\tau) - X(\tau)\}_t$ does not depend upon τ.

A wavelet $\psi(t)$ is a function of zero average $\int \psi(t)\,dt = 0$, which decays $|\psi(t)| = O((1+|t|^2)^{-1})$. A dyadic wavelet transform is calculated by scaling this wavelet

$$\forall j \in \mathbb{Z} \quad , \quad \psi_j(t) = 2^{-j}\psi(2^{-j}t) \ .$$

The wavelet tranform of a random process $X(t)$ at a scale 2^j is defined for all $t \in \mathbb{R}$ by

$$X \star \psi_j(t) = \int X(u)\psi_j(t-u)du \ . \tag{1}$$

Since $\int \psi(t)\,dt = 0$, if X has stationary increments then one can verify that $X \star \psi_j(t)$ is a stationary process [33]. The multiscale wavelet transform of $X(t)$ is:

$$WX = \{X \star \psi_j\}_{j\in\mathbb{Z}} \ . \tag{2}$$

We consider dyadic complex wavelets $\psi(t)$ which satisfy the Littlewood-Paley condition:

$$\forall \omega \in \mathbb{R}^+ \ , \quad \sum_{j=-\infty}^{\infty} |\hat{\psi}(2^j\omega)|^2 + \sum_{j=-\infty}^{\infty} |\hat{\psi}(-2^j\omega)|^2 = 2 \ . \tag{3}$$

If X is stationary and $\mathbf{E}(|X|^2) < \infty$, by computing each $\mathbf{E}(|X \star \psi_j|^2)$ as a function of the power spectrum of X and inserting (3), one can prove that:

$$\mathbf{E}(\|WX\|^2) = \sum_j \mathbf{E}(|X \star \psi_j|^2) = \mathbf{E}(|X|^2) - |\mathbf{E}(X)|^2 \ . \tag{4}$$

Multifractals have been studied by analyzing the scaling properties of wavelet moments, through exponent factors $\zeta(q)$ defined by:

$$E(|X \star \psi_j(t)|^q) \simeq 2^{j\zeta(q)} \ .$$

Under appropriate assumptions, the function $\zeta(q)$ can be related to the singularity spectrum of X through a Legendre transform [12, 29]. A major numerical difficulty is to reliably estimate moments for $q \geq 2$ and $q < 0$ because of the large variance of estimators.

2.2 Wavelet Scattering Transform

This section reviews the properties of scattering transform [22], computed by cascading wavelet transforms and a modulus non-linearity. As opposed to the wavelet moment approach, scattering coefficients are calculated by cascading contractive operators, which avoids amplifying the process variability, and hence leads to lower variance estimators. It have been shown to efficiently discriminate non-Gaussian audio processes [2, 14] and image textures [15].

The wavelet transform is calculated with a complex wavelet whose real and imaginary parts are orthogonal, for exaple even and odds. One may choose analytic wavelets for which $\hat{\psi}(\omega) = 0$ for $\omega < 0$. For example, the analytic part of third order Battle-Lemaire wavelets [21] and satisfies (3). First order scattering coefficients are first order wavelet moments:

$$\forall j_1 \in \mathbb{Z} \quad , \quad SX(j_1) = \mathbf{E}(|X \star \psi_{j_1}|) \ .$$

Second order scattering coefficients are computed by calculating the wavelet transform of each $|X \star \psi_{j_1}|$ and computing the expected values of its modulus:

$$\forall (j_1, j_2) \in \mathbb{Z}^2 \quad , \quad SX(j_1, j_2) = \mathbf{E}(||X \star \psi_{j_1}| \star \psi_{j_2}|) \ .$$

Similarly, m^{th} order scattering coefficients are calculated by applying m wavelet tranforms and modulus, which yields:

$$\forall (j_1, ..., j_m) \in \mathbb{Z}^m \quad , \quad SX(j_1, ..., j_m) = \mathbf{E}(|\ |X \star \psi_{j_1}| \star ...| \star \psi_{j_m}|) \ . \tag{5}$$

Observe that scattering coefficients are well defined as long as the process has a finite first order moment $\mathbf{E}(|X|) < \infty$. One can indeed verify by induction on m that

$$|SX(j_1, ..., j_m)| \leq \mathbf{E}(|X|)\ \|\psi\|_1^m \ .$$

Scattering coefficients of order m are non-linear functions of X, which mostly depend upon normalized moments of X up to the order 2^m [22]. If $X(t)$ is not stationary but has stationary increments then $X \star \psi_{j_1}(t)$ is stationary. It results that $SX(j_1, ..., j_m) \in \mathbb{R}^+$ is well defined for all $m \geq 1$ if $\mathbf{E}(|X(t)|) < \infty$.

In this paper we concentrate on first and second order scattering coefficients, which carry important multifractal properties, and compute normalized second order scattering coefficients defined by:

$$\widetilde{S}X(j_1, j_2) = \frac{SX(j_1, j_2)}{SX(j_1)} = \frac{\mathbf{E}(||X \star \psi_{j_1}| \star \psi_{j_2}|)}{\mathbf{E}(|X \star \psi_{j_1}|)} \ .$$

The following proposition relates these coefficients to the ratio between first and second order wavelet moments.

Proposition 2.1 *If $\mathbf{E}(|X(t)|^2 < \infty$ then for any $j_1 \in \mathbb{Z}$:*

$$\frac{\mathbf{E}(|X \star \psi_{j_1}|^2)}{\mathbf{E}(|X \star \psi_{j_1}|)^2} = 1 + \sum_{j_2=-\infty}^{+\infty} |\widetilde{S}X(j_1, j_2)|^2 + e_3(j_1) \tag{6}$$

with

$$e_3(j_1) = \sum_{j_2,j_3=-\infty}^{\infty} \frac{\mathbf{E}(|||X \star \psi_{j_1}| \star \psi_{j_2}| \star \psi_{j_3}|^2)}{\mathbf{E}(|X \star \psi_{j_1}|)^2} \ . \quad (7)$$

Proof: Applying the mean-square energy conservation (4) to $X \star \psi_j$ proves that

$$\mathbf{E}(|X \star \psi_j|^2) = |\mathbf{E}(|X \star \psi_j|)|^2 + \sum_{j_2=-\infty}^{+\infty} \mathbf{E}(||X \star \psi_j| \star \psi_{j_2}|^2) \ . \quad (8)$$

Applying again (4) to $||X \star \psi_j| \star \psi_{j_2}|$ proves that

$$\mathbf{E}(||X \star \psi_j| \star \psi_{j_2}|^2) = \mathbf{E}(||X \star \psi_j| \star \psi_{j_2}|)^2 + \sum_{j_3=-\infty}^{+\infty} \mathbf{E}(|||X \star \psi_j| \star \psi_{j_2}| \star \psi_{j_3}|^2).$$

Inserting this equation in (8) proves (6). □

The amplitude of scattering coefficients $\widetilde{S}X(j_1, j_2)$ can be neglected for $j_1 > j_2$. The decay of $\widetilde{S}X(j_1, j_2)$ for $j_2 - j_1 \leq 0$ depends essentially on the wavelet properties as opposed to the properties of X. Indeed $|X \star \psi_{j_1}|$ inherits the regularity of $|\psi_{j_1}|$. If $|\psi|$ is $\mathbf{C}^p$ and ψ has p vanishing moments then $\widetilde{S}X(j_1, j_2) = \mathbf{E}(||X \star \psi_{j_1}| \star \psi_{j_2}|)/\mathbf{E}(|X \star \psi_{j_1}|)$ decreases typically like $2^{p(j_2 - j_1)}$. In the following, we thus concentrate on $j_2 > j_1$ where normalized scattering coefficients reveal the properties of X.

For multifractals, the ratio between first and second order wavelet moments define the intermittency factor $\zeta(2) - 2\zeta(1)$:

$$\frac{\mathbf{E}(|X \star \psi_j|^2)}{\mathbf{E}(|X \star \psi_j|)^2} \simeq 2^{j(\zeta(2) - 2\zeta(1))} \ . \quad (9)$$

Corollary 2.2 *For multifractals with stationary increments*

$$\sum_{j_2=j+1}^{+\infty} |\widetilde{S}X(j, j_2)|^2 = O(2^{j(\zeta(2) - 2\zeta(1))}) \ . \quad (10)$$

Monofractals are particular examples where $\zeta(q) = C\,q$ is linear. It results that $\sum_{j_2=j+1}^{+\infty} |\widetilde{S}X(j, j_2)|^2 = O(1)$. Section 3 concentrates on self-similar monofractals. When there is intermittency, the summation (17) does not provide a good estimation of $\zeta(2) - 2\zeta(1)$ because the energy of higher order terms $e_3(j_1)$ in (7) is not negligible and in fact is dominating. For multifractal cascades, Section 4 proves that one can still compute this factor by estimating higher order scattering coefficients from normalized second order coefficients.

3 Self-Similar Processes

The properties of self-similar processes are analyzed with a scattering tranfsorm. Scattering coefficients of fractional Brownian motions and Levy stable processes are studied in more details.

3.1 Scattering Self-Similarity

Self-similar processes of exponent H are stochastic processes $X(t)$ that are invariant in distribution under a scaling of space or time:

$$\forall\, s > 0\ ,\ \{X(st)\}_t \overset{d}{=} \{s^H X(t)\}_t\ . \tag{11}$$

We consider self-similar processes having stationary increments. Fractional Brownian motions and α-stable Lévy processes are examples of Gaussian and non-Gaussian self-similar processes with stationary increments.

If X is self-similar, then applying (11) with a change of variable $u' = 2^{-j}u$ in (1) proves that

$$\forall j \in \mathbb{Z}\quad,\quad \{X \star \psi_j(t)\}_t \overset{d}{=} 2^{jH}\,\{X \star \psi(2^{-j}t)\}_t\ .$$

Because of the stationarity, the moments do not depend on t and for all $q \in \mathbb{Z}$

$$\mathbf{E}(|X \star \psi_j|^q) = 2^{jHq}\,\mathbf{E}\{|X \star \psi|^q\}\ \text{ so }\ \zeta(q) = qH\ . \tag{12}$$

The self-similarity thus implies a scaling of wavelet coefficients moments, up to a multiplicative factor which depends upon the distribution of the process. The following proposition proves that normalized second order scattering coefficients can be written as a univariate function which characterizes important fractal properties.

Proposition 3.1 *If X is a self-similar process with stationary increments then for all $j_1 \in \mathbb{Z}$*

$$SX(j_1) = 2^{j_1 H}\,\mathbf{E}(|X \star \psi|), \tag{13}$$

and for all $(j_1, j_2) \in \mathbb{Z}^2$

$$\widetilde{S}X(j_1,j_2) = S\widetilde{X}(j_2 - j_1)\ \ \textit{with}\ \ \widetilde{X}(t) = \frac{|X \star \psi(t)|}{\mathbf{E}(|X \star \psi|)}\ . \tag{14}$$

Proof: Since $\psi_{j_1} = 2^{-j_1} L_{j_1}\psi$, a change of variables yields $L_{j_1}|X \star \psi| = |L_{j_1}X \star \psi_{j_1}|$, and hence

$$|X \star \psi_{j_1}| = L_{j_1}|L_{-j_1}X \star \psi| \overset{d}{=} 2^{j_1 H}\,L_{j_1}|X \star \psi|\ . \tag{15}$$

If $Y(t)$ is stationary then $\mathbf{E}(L_j Y(t)) = \mathbf{E}(Y(t))$, which proves (13).

By cascading (15) we get

$$\forall\,(j_1, j_2)\ ,\ ||X \star \psi_{j_1}| \star \psi_{j_2}| \overset{d}{=} 2^{j_1 H}\,L_{j_1}||X \star \psi| \star \psi_{j_2 - j_1}|\ , \tag{16}$$

so $SX(j_1,j_2) = 2^{j_1 H}\,\mathbf{E}(||X \star \psi| \star \psi_{j_2-j_1}|)$. Together with (13) it proves (14). $\square$.

Property (14) proves that if X is self-similar then a normalized scattering coefficient $\widetilde{S}X(j_1,j_2)$ is a function of $j_2 - j_1$. With an abuse of notation, we shall thus write $\widetilde{S}X(j_1,j_2) = \widetilde{S}X(j_2 - j_1)$, and $\widetilde{S}X(j) = S\widetilde{X}(j)$ with $\widetilde{X} = |X \star \psi|/\mathbf{E}(|X \star \psi|)$. Since $2\zeta(1) - \zeta(2) = 0$ if $E(|X \star \psi|^2) < \infty$ then Corollary 2.2 implies that

$$\sum_{j=1}^{+\infty} |\widetilde{S}X(j)|^2 < \infty\ . \tag{17}$$

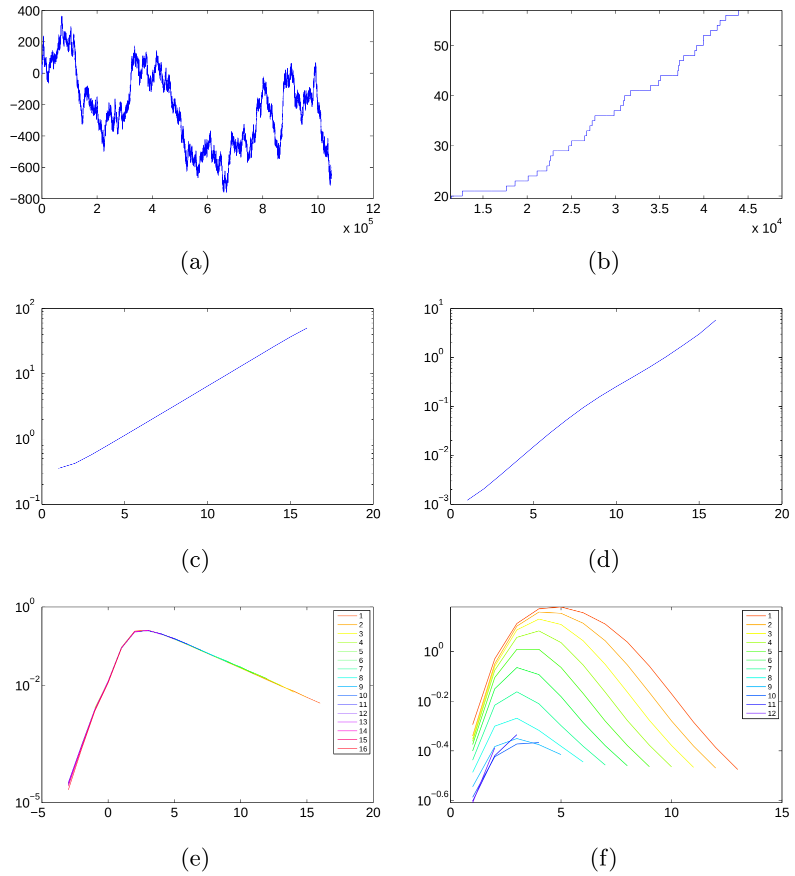


Figure 1: Left column: (a) Realization of Brownian motion $X(t)$ with $H = 1/2$, (c) $\log SX(j_1)$ as a function of j_1, (e) $\log \widetilde{S}X(j_1, j_2)$ as a function of $j_2 - j_1$ for several values of j_1. Right column: (b) Realization of a Poisson process $Y(t)$. (d) $\log SY(j_1)$ as a function of j_1 and (f) $\log \widetilde{S}Y(j_1, j_2)$ as a function of $j_2 - j_1$ for several values of j_1.

Figure 1 shows the first and second order scattering coefficients of two processes with stationary increments. Figure1(a,c,e) shows a realization of a Brownian Motion X, its first and normalized second order scattering coefficients. As shown by Proposition 3.1 the normalized second order coefficients satisfy $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_2 - j_1)$. The wavelet used in computations is a Battle-Lemarie spline wavelet of degree 3. Since $|\psi|$ is $\mathbf{C}^p$ with $p = 3$ and ψ has $p + 1$ vanishing moments, $\widetilde{S}X(j) = O(2^{pj})$ for $j \le 0$, as verified in figure 1(e). It reveals no information on X. For $j \ge 0$, one can observe that $\widetilde{S}X(j) \sim 2^{-j/2}$. Next section proves this decay in the more general context of fractional Brownian motions. Figure 1(b,d,e) gives a realization of $Y(t) = \int_0^t dP(u)$, where dP is a Poisson point process. The processes $X(t)$ and $Y(t)$ have stationary increments with same power spectrum, but their second order scattering coefficients behave very differently. Since $Y(t)$ is not self-similar $\widetilde{S}Y(j_1, j_2)$ depends upon both j_1 and j_2 as opposed to $j_2 - j_1$.

First order scattering coefficients $SX(j_1) = \mathbf{E}(|X \star \psi_{j_1}|)$ are wavelet coefficients first order moments. Their decay as j_1 increases depend upon the global regularity of X. This regularity may be modified with a fractional derivative or a fractional integration. Let $\hat{f}(\omega)$ be the Fourier transform of $f \in \mathbf{L}^2(\mathbb{R}^d)$. For $\alpha \in \mathbb{R}$, we define $D^\alpha f$ as a function or tempered distribution whose Fourier transform is

$$\widehat{D^\alpha f}(\omega) = (i\omega)^\alpha \hat{f}(\omega) \ .$$

For $\alpha > 0$ it is a fractional derivative and for $\alpha < 0$ it is a fractional integration.

Proposition 3.2 *If X is a self-similar process with stationary increments then for all $\alpha \in \mathbb{R}$ and all $j_1 \in \mathbb{Z}$*

$$SD^\alpha X(j_1) = 2^{j_1(H-\alpha)}\, \mathbf{E}(|X \star D^\alpha \psi|) \ , \tag{18}$$

and for all $(j_1, j_2) \in \mathbb{Z}^2$

$$\widetilde{S}D^\alpha X(j_1, j_2) = S\widetilde{X}_\alpha(j_2 - j_1) \ \ \textit{with} \ \ \widetilde{X}_\alpha = \frac{|X \star D^\alpha \psi|}{\mathbf{E}(|X \star D^\alpha \psi|)} \ . \tag{19}$$

Proof: Observe that $L_j D^\alpha = 2^{-\alpha j_1} D^\alpha L_{j_1}$ and since $\psi_{j_1} = 2^{-j_1} L_{j_1} \psi$ we get

$$|D_\alpha X \star \psi_{j_1}| = 2^{-\alpha j_1} |X \star L_{j_1} D_\alpha \psi| = 2^{-\alpha j_1} L_{j_1} |L_{-j_1} X \star D_\alpha \psi| \ . \tag{20}$$

Since $\{L_{-j_1} X(t)\} \overset{d}{=} 2^{j_1 H} \{X(t)\}_t$, taking the expected value proves (18).

The proof of (19) follows the same argument as the proof of (14), with the identity (20). $\square$.

If $\alpha > 0$ and ψ is in the Sobolev space H^α then $D^\alpha \psi \in \mathbf{L}^2(\mathbb{R}^d)$ is a wavelet which resembles to ψ. This is also true if $\alpha < 0$ and ψ has $\alpha + 1$ vanishing moments. It then results that X and $D^\alpha X$ have nearly the same normalized second order scattering coefficients. Fractional Brownian motions give a simple illustration of this property.

3.2 Fractional Brownian Motions

We compute the normalized scattering representation of Fractional Brownian Motions, which are the only self-similar Gaussian processes.

A fractional Brownian motion of Hurst exponent $0 < H < 1$ is defined as a zero mean Gaussian process $\{X(t)\}$, satisfying

$$\forall\, t, s > 0 \;, \mathbf{E}(X(t)X(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\mathbf{E}(X(1)^2) \;.$$

It is self-similar and satisfies

$$\forall\, s > 0 \;,\; \{X(st)\}_t \stackrel{d}{=} s^H \{X(t)\}_t \;.$$

Proposition 3.1 proves in (13) that

$$SX(j_1) = 2^{Hj_1}\mathbf{E}(|X \star \psi|) \;.$$

This is verified by Figure 2(a) which displays $\log_2 SX(j_1)$ for several fractional brownian motions with $H = 0.2, 0.4, 0.6, 0.8$. It increases linearly as a function of j_1, with a slope H.

Figure 2(b) display second order coefficients $\log_2 \widetilde{S}(j_1, j_2)$ as a function of $j_2 - j_1$. As opposed to first order coefficients, they nearly do not depend upon H, and all curves superimpose. Proposition 3.1 shows that scattering coefficients $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_2 - j_1)$ only depend on $j_2 - j_1$. They have a fast exponential decay to zero when $j_2 - j_1 \leq 0$ decreases, which mostly does not depend on X but on the wavelet regularity and its vanishing moments. We shall thus concentrate on the properties of scattering coefficients for $j_2 \geq j_1$ on the rest of the paper.

Modulo a proper initialization at $t = 0$, if X is a fractional Brownian motion of exponent H then $D^\alpha X$ is a fractional Brownian motion of exponent $H - \alpha$. We thus expect from (19) in Proposition 3.2 that $\log_2 \widetilde{S}X(j_2 - j_1)$ nearly does not depend upon H when $j_2 - j_1 > 0$. One can observe that $\log_2 \widetilde{S}X(j)$ has a slope of $-1/2$ when j increases, which is proved by the following theorem. It does not depend upon H and it is in fact a characteristic of wide-band Gaussian processes. We suppose that ψ is a $\mathbf{C^1}$ analytic function, with at least two vanishing moments and that ψ and its derivative is $O((1 + |u|^2)^{-1})$.

Theorem 3.3 *Let $X(t)$ be a Fractional Brownian Motion with Hurst exponent $0 < H < 1$. There exists a constant $C > 0$ such that*

$$\lim_{j\to\infty} 2^{j/2}\widetilde{S}X(j) = C \;. \tag{21}$$

Proof: Proposition 3.1 proves in (14) that $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_2 - j_1)$, with $\widetilde{S}X(j) = \mathbf{E}(|\widetilde{X} \star \psi_j|)$ and $\widetilde{X}(t) = |X \star \psi(t)|/\mathbf{E}(|X \star \psi|)$. Let $B(t)$ be a Brownian motion and $dB(t)$ be the Wiener measure. The two processes $X \star \psi(t)$ and $D^{H-1}dB \star \psi(t)$ are Gaussian stationary processes having same power spectrum so

$$\{|X \star \psi(t)|\}_t \stackrel{d}{=} \{|D^{H-1}dB \star \psi(t)|\}_t \stackrel{d}{=} \{|dB \star D^{H-1}\psi(t)|\}_t \;.$$

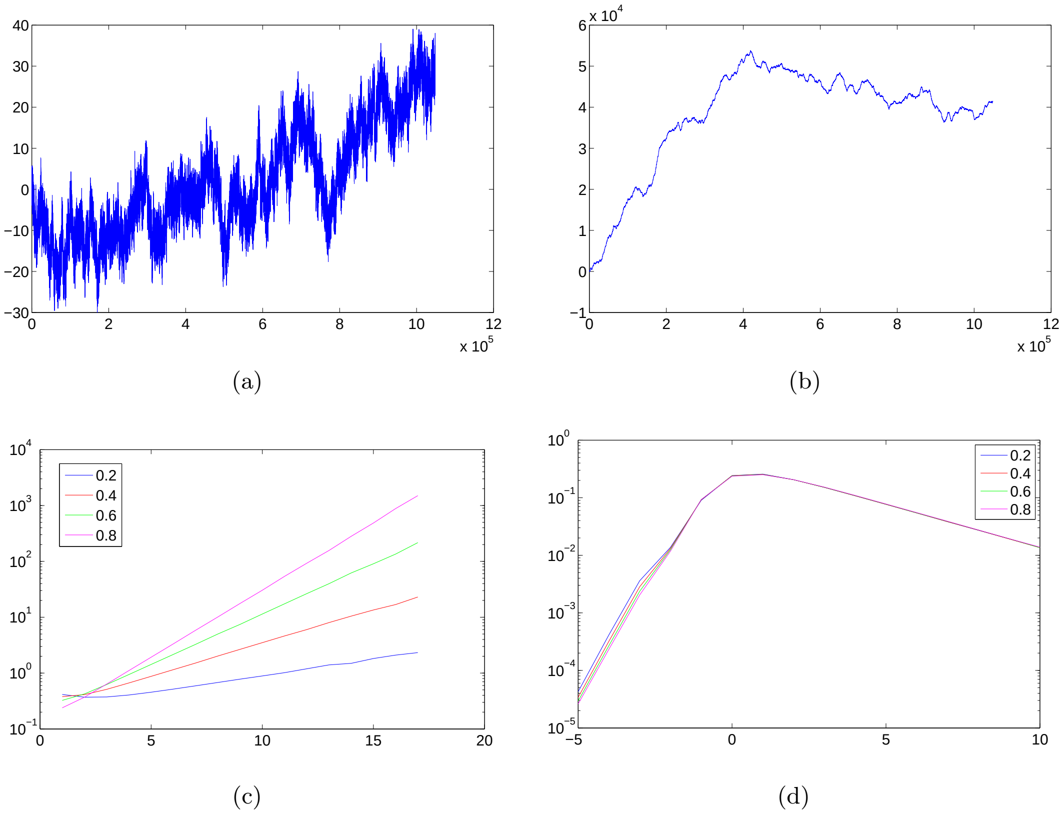


Figure 2: (a) Realization of a FBM with $H = 0.2$, (b) Realization of a FBM with $H = 0.8$ (c) $\log_2 SX(j_1)$ as a function of j_1 for fractional Brownians of exponents $H = 0.2,\ 0.4,\ 0.6,\ 0.8$. (d) $\log_2 \widetilde{S}X(j_1, j_2) = \log_2 S\widetilde{X}(j_2 - j_1)$ as a function of $j_2 - j_1$ for the same processes.

It results that

$$\widetilde{S}X(j) = \frac{\mathbf{E}(||dB \star D^{H-1}\psi| \star \psi_j|)}{\mathbf{E}(|X \star \psi|)} \tag{22}$$

Since ψ has two vanishing moments and is $\mathbf{C}^1$ with all derivatives which are $O((1 + |u|^3)^{-1})$, one can verify that $|D^{H-1}\psi(u)| = O((1+|u|^2)^{-1})$. It results that $|dB \star D^{H-1}\psi|$ is stationary process whose autocorrelation has some decay. As the scale 2^j increases, the convolution with ψ_j performs a progressively wider averaging. By applying a central-limit theorem for dependant random variables, the following lemma applied to $\varphi = D^{H-1}\psi$ proves that $2^{j/1}|dB \star D^{H-1}\psi| \star \psi_j$ converges to a Gaussian processes and that its first moment converges to a constant when j goes to ∞. The theorem result (21) stating that $2^{j/2}\widetilde{S}X(j)$ converges to a constant is thus derived from (22).

Lemma 3.4 *If* $\varphi(u) = O((1 + |u|^2)^{-1})$ *then*

$$2^{j/2}|dB \star \varphi| \star \psi_j(t) \xrightarrow[j\to\infty]{l} \mathcal{N}(0, \sigma^2\mathrm{Id}) \ , \tag{23}$$

with $\sigma^2 = \|\psi\|_2^2 \int R_Y(\tau) d\tau$ *and*

$$\lim_{j\to\infty} \mathbf{E}(|2^{j/2}|dB\star\varphi|\star\psi_j|) = \sigma\sqrt{\frac{\pi}{2}} \ . \tag{24}$$

□

Fractional Brownian motions have moments that scale linearly $\zeta(q) = qH$ and hence $\zeta(2) - 2\zeta(1) = 0$. Since $\widetilde{S}X(j) \sim 2^{-j/2}$ for $j \geq 1$, as expected from Corollary 2.2, $\sum_j \widetilde{S}X(j)^2 < \infty$. Numerically we compute $\sum_j \widetilde{S}X(j)^2 \approx 0.22$. If ψ is analytic, then $|X\star\psi_j|(t)$ is a Rayleigh random variable for all j. In this case,

$$1.273\ldots = \frac{4}{\pi} = \frac{\mathbf{E}(|X\star\psi|^2)}{\mathbf{E}(|X\star\psi|)^2} \approx 1 + \sum_{j\geq 1}\widetilde{S}(j)^2 \approx 1.22 \ ,$$

which means that the energy of higher order terms e_3 in (7) is small compared to the lower order terms.

3.3 α-stable Lévy Processes

In this section, we compute numerically the scattering coefficients of α-stable Lévy processes and give qualitative arguments explaining their behavior.

The Lévy-Khintchine formula [19] characterizes infinitely divisible distributions from their characteristic exponents. Self-similar Lévy processes have stationary increments with heavy tailed distributions. Their realizations contain rare, large, events which critically influence their moments. In particular, an α-stable Lévy process $X(t)$ only has moments of order strictly smaller than α.

If $\alpha > 1$ then $X(t)$ has stationary increments and $\mathbf{E}(|X(t)|) < \infty$. Its scattering coefficients are thus well defined. If $\alpha < 2$ then the second order moments are infinite so one can not analyze these processes with second order moments. An α-stable Lévy processes satisfies the self-similarity relation

$$\{X(st)\}_t \stackrel{d}{=} s^{\alpha^{-1}}\{X(t)\}_t \ , \tag{25}$$

so Propositon 3.1 proves that

$$SX(j_1) = 2^{j_1\alpha^{-1}}\,\mathbf{E}(|X\star\psi|)\,. \tag{26}$$

This is verified in Figure 3 which shows that $\log_2 SX(j_1)$ has a slope of α^{-1} as a function of j_1. First order coefficients thus do not differentiate a Lévy stable processes from fractional Brownian motions of Hurst exponent $H = \alpha^{-1}$.

The self-similarity implies that normalized second order coefficients satisfy $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_2 - j_1)$. However, they have a very different behavior than second order scattering coefficients of fractional Brownian motion. Figure 3 shows that $\log \widetilde{S}X(j)$ has a slope of $\alpha^{-1} - 1$ and hence that

$$\forall j \geq 1 \ , \ \widetilde{S}X(j) \simeq 2^{j(\alpha^{-1}-1)} \ . \tag{27}$$

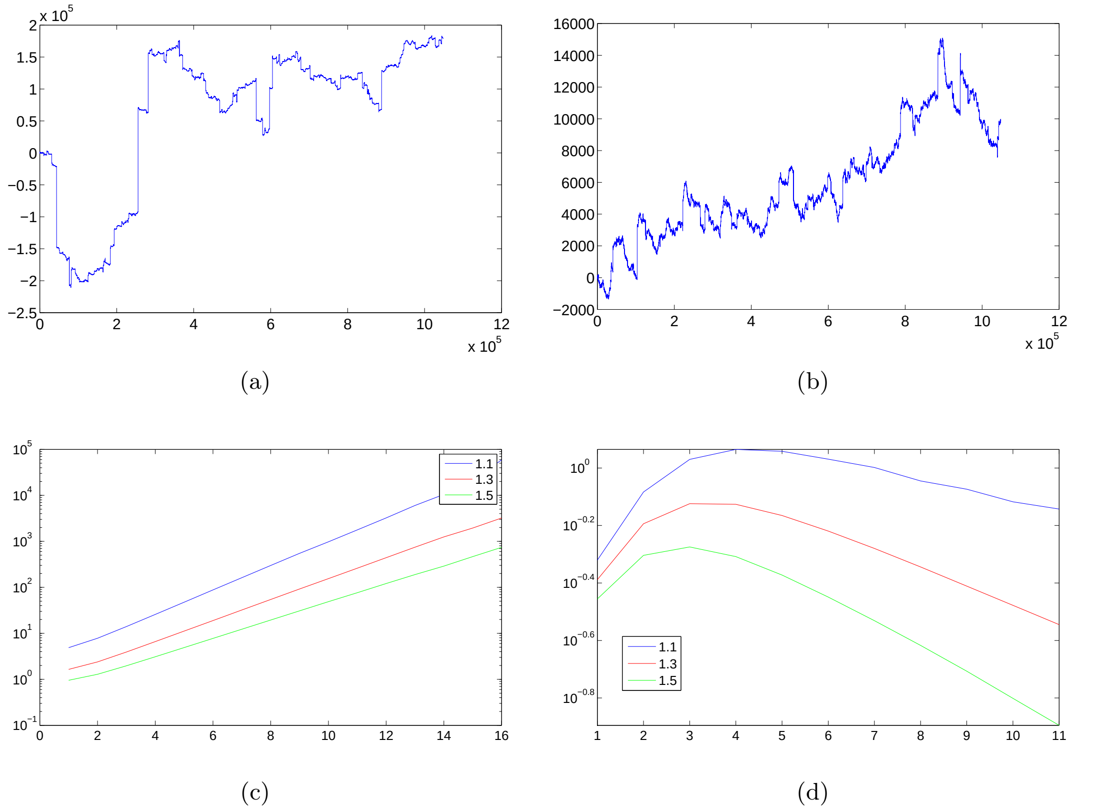


Figure 3: (a): Realization of a Lévy process with $\alpha = 1.1$, (b): Realization of a Lévy process with $\alpha = 1.5$, (c) $\log SX_\alpha(j_1)$ as a function of j_1 for α-stable Lévy processes with $\alpha = 1.1,\ 1.2,\ 1.3$. The slopes are α^{-1}. (d) $\log \widetilde{S}X_\alpha(j_2 - j_1)$ as a function of $j_2 - j_1$ number of realizations. The slopes are $\alpha^{-1} - 1$.

This property can be explained as follows. Proposition 3.2 proves in (14) that

$$\widetilde{S}X(j) = \frac{\mathbf{E}(||X \star \psi(t)| \star \psi_j|)}{\mathbf{E}(|X \star \psi|)} \ . \tag{28}$$

The stationary process $|X \star \psi(t)|$ computes the amplitude of local variations of the process X. It is dominated by a sparse sum of large amplitude bumps of the form $a\,|\psi(t-u)|$, where a and u are the random amplitudes and positions of rare jumps in $dX(t)$, distributed according to the Lévy measure. It results that

$$\mathbf{E}(||X \star \psi| \star \psi_j|) \simeq \mathbf{E}(|dX \star |\psi| \star \psi_j|). \tag{29}$$

If $2^j \gg 1$ then $|\psi| \star \psi_j \approx \|\psi\|_1\, \psi_j$, and $\mathbf{E}(|dX \star \psi_j|) \simeq 2^{j(\alpha^{-1}-1)}$ because the Lévy jump process $dX(t)$ satisfies the self-similarity property

$$\{dX(st)\}_t \stackrel{d}{=} s^{\alpha^{-1}-1}\{dX(t)\}_t \ .$$

Inserting (29) in (28) gives the scaling property (27).

For $\alpha = 2$, the Lévy process X is a Brownian motion and we recover that $\widetilde{S}X(j) \simeq 2^{-j/2}$ as proved in Theorem 3.3. For $\alpha < 2$ the scaling law is different because its value essentially depends upon the rare jumps of large amplitudes, which do not exist in Brownian motions. Observe that $\sum_{j=1}^{+\infty} \widetilde{S}X(j)^2 < \infty$ although $\mathbf{E}(|X \star \psi_j|^2) = \infty$, which shows that the variance blow-up is due to the higher order term $e_3(j)$ in (6).

4 Scattering of Multifractal Cascades

We study the scattering representation of multifractal processes which satisfy stochastic scale invariance property. Section 4.2 studies the particularly important case of log-infinitely divisible multiplicative processes. Section 4.4 shows under general conditions that one can compute a measure of intermittency from normalized scattering coefficients.

4.1 Stochastic Self-Similar Processes

We consider processes with stationary increments which satisfy the following stochastic self-similarity:

$$\forall\, 1 \geq s > 0 \ , \ \{X(st)\}_{t \leq 2^J} \stackrel{d}{=} A_s \cdot \{X(t)\}_{t \leq 2^J} \ , \tag{30}$$

where A_s is a log-infinitely divisible random variable independent of $X(t)$ and the so-called *integral scale* 2^J is chosen (for simplicity) as a power of 2. The Multifractal Random Measures (MRM) introduced by [30, 6] are important examples of such processes. Let us point out that MRM's can be seen as stationary increments versions of the multiplicative cascades initially introduced by Yaglom [36] and Mandelbrot [23, 24], and further studied by Kahane and Peyriere [16]. Strictly speaking, these multiplicative cascades do not satisfy (30). However they do satisfy an extremely similar equation when sampled on a dyadic-grid, and consequently, all the results that we obtained on MRM's are easily generalized to multiplicative cascades. For the sake of conciseness, we did not include them here.

In this section and the following ones, we consider a wavelet ψ of compact support equal to $[-1/2, 1/2]$. Since X has stationary increments and satisfies (30), with a change of variables we verify that

$$\forall j \leq J \quad , \quad \{X \star \psi_j(t)\}_t \stackrel{d}{=} A_{2^j}\, \{X \star \psi(2^{-j}t)\}_t \ ,$$

and hence for all $q \in \mathbb{Z}$ and $j \leq J$

$$\mathbf{E}(|X \star \psi_j|^q) = \mathbf{E}(|A_{2^j}|^q)\, \mathbf{E}\{|X \star \psi|^q\} \simeq C_q 2^{j\zeta(q)} \ , \tag{31}$$

where $\zeta(q)$ is a priori a non-linear function of q. Since the self-similarity is upper bounded by an integral scale, the convexity of moments [12] implies that $\zeta(q)$ is a concave function of q. Similarly to Proposition 3.1, the following proposition shows that normalized scattering coefficients capture stochastic self-similarity with a univariate function.

Proposition 4.1 *If X is randomly self-similar in the sense of (30) with stationary increments then for all $j_1 \leq J$*

$$SX(j_1) = \mathbf{E}(|A_{2^{j_1}}|)\,\mathbf{E}(|X \star \psi|). \tag{32}$$

If $2^{j_1} + 2^{j_2} \leq J$ then

$$\widetilde{S}X(j_1, j_2) = S\widetilde{X}(j_2 - j_1) \quad \textit{with} \quad \widetilde{X}(t) = \frac{|X \star \psi(t)|}{\mathbf{E}(|X \star \psi|)}\ . \tag{33}$$

Proof: Property (13) is particular case of (31) for $q = 1$. If $2^{j_1} + 2^{j_2} \leq J$, with the same derivations as for (16), we derive from (30) that

$$||X \star \psi_{j_1}| \star \psi_{j_2}| \stackrel{d}{=} A_{2^{j_1}}\, L_{j_1} ||X \star \psi| \star \psi_{j_2 - j_1}|\ , \tag{34}$$

so $SX(j_1, j_2) = \mathbf{E}(A_{2^{j_1}})\,\mathbf{E}(||X \star \psi| \star \psi_{j_2-j_1}|)$. Together with (32) it proves (33). $\Box$.

Figure 4 shows the normalized scattering of a multiplicative cascade process described in Section 4.2, with an integral scale $2^J = 2^{17}$. As proved in (33), if $j_1 < j_2 < J$ then $\widetilde{S}X(j_1, j_2)$ only depends on $j_2 - j_1$. This stops to be valid when 2^{j_2} reaches the integral scale 2^J.

Propositions 3.1 and 4.1 show that normalized scattering coefficients can be used to detect the presence of self-similarity, both deterministic and stochastic, since in that case necessarily $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_2 - j_1)$. This necessary condition is an alternative to the scaling of the q-order moments, $\mathbf{E}(|X \star \psi_j|^q) \simeq C_q 2^{j\zeta(q)}$, which is difficult to verify empirically for $q \geq 2$ or $q < 0$.

Multifractals typically become decorrelated beyond their integral scale. For $j_2 > J$, the decorrelation at large scales induces a normalized second order scattering which converges to that of the Gaussian white noise, with an asymptotic behavior $C 2^{-j/2}$ as seen in Section 3.2. Consequently, the resulting normalized second order scattering is

$$\widetilde{S}X_J(j_1, j_2) \approx \begin{cases} \widetilde{S}X(j_2 - j_1) & \text{if } j_1 < J \text{ and } j_2 < J \\ C\,2^{(J - j_2)/2} & \text{if } j_1 < J \text{ and } j_2 \geq J \\ C\,2^{(j_1 - j_2)/2} & \text{if } J \leq j_1 < j_2 \end{cases}\ . \tag{35}$$

4.2 Log-infinitely divisible Multifractal Random Processes

Multiplicative cascades as introduced by Mandelbrot in [23, 24] are built as an iterative process starting at scale 2^J with the Lebesgue measure on the interval $[0, 2^J]$. The iteration consists in cutting this interval at the middle point and multiplying the mass on each interval by a log-infinitely divisible variable (iid versions are used on each interval). One then gets a random measure "living" at scale 2^{J-1} (i.e., it is uniform on intervals of length 2^{J-1} : $[0, 2^{J-1}]$ and $[2^{J-1}, 2^J]$). The iteration is then applied recursively on each sub-interval. At the nth iteration, one gets a random measure living at scale 2^{J-n} (i.e., which is uniform on each interval of the form $[k2^{J-n}, (k+1)2^{J-n}]$). The object of interest is the weak limit ($n \to +\infty$) of this random measure. At a given point t it

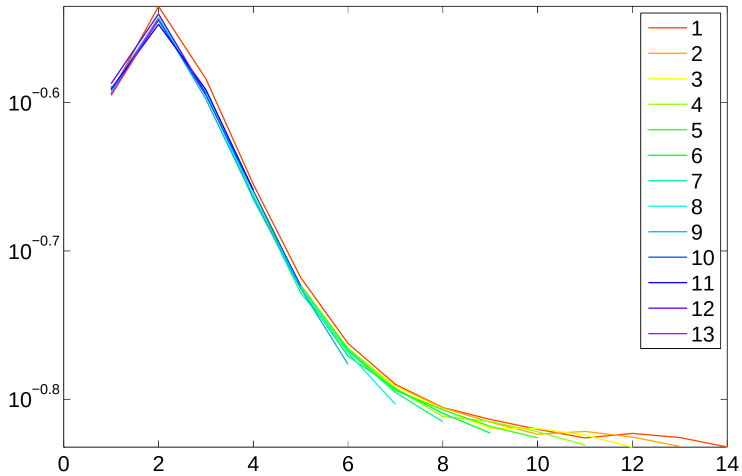


Figure 4: Second order scattering for an MRM random cascade with $\lambda^2 = 0.04$. We plot the curves $\widetilde{S}_j(l) = \widetilde{S}dX(j, j+l)$ as a function of l, and different colors stand for different values of j.

can be written as an infinite product $\prod_{n=1}^{+\infty} W_n$ where W_n are iid log-infinitely divisible variables.

Multifractal Random Measures can be seen as stationary increments versions of these multiplicative cascades. It is basically obtained by infinitely decomposing each iteration step, i.e., no longer going directly from scale $s = 2^{J-n}$ to scale $s = 2^{J-n-1}$ but going from an arbitrary scale s to a scale rs where r is infinitely close to 1. Thus, they are built using an infinitely divisible random noise dP distributed in the half-plane (t, s) $(s > 0)$. Using the previous notations, the noise around (t, s) can be seen as the equivalent of the infinitely divisible variable $\log W_{\log s}(t)$. More precisely, if $\omega_l^{2^J}(t) = \int_{\mathcal{A}_l^{2^J}(t)} dP$ where $\mathcal{A}_l^{2^J}(t)$ is the cone in the (t, s) half-plane pointing to point $(0, s)$ and truncated for $s < l$, the MRM is defined as the weak limit

$$dM(t) = \lim_{l \to 0} e^{\omega_l^{2^J}(t)} dt. \tag{36}$$

For a rigorous definition of $\omega_l^{2^J}$ and of MRM, we refer the reader to [6]. One can prove that dM is a stochastic self-similar process in the sense of (30) with A_s a log-infinitely divisible variable. It is multifractal in the sense that (31) holds for some non-linear function $\zeta(q)$ which is uniquely defined by infinitely divisible law chosen for dP.

Let us point out that the self-similarity properties of dM are mainly direct consequences of the following self-similarity properties of $\omega_l^{2^J}$

- A "global" self-similarity property which is true for all T and all $s > 0$:

$$\{\omega_{sl}^{sT}(st)\}_t \overset{law}{=} \{\omega_l^T(t)\}_t. \tag{37}$$

- The stochastic self-similarity property which is true for all T and $s < 1$:

$$\{\omega_{sl}^T(su)\}_{u<T} \stackrel{law}{=} \{\Omega_s + \omega_l^T(u)\}_{u<T} \tag{38}$$

where Ω_s is an infinitely divisible random variable independent of $\omega_l^T(u)$ such that $E(e^{q\Omega_s}) = e^{-(q-\zeta(q))\ln(s)}$.

Some more precise results that we will be used in the following are gathered in Appendix C.

Let us first state the expression for the first order scattering coefficient of the random process $dM(t)$

Proposition 4.2 (First order scattering of an MRM) *The wavelet transform of the $dM(t)$ process writes, for all $t \in [0, T-1]$,*

$$dM \star \psi_j \stackrel{law}{=} e^{\Omega_{2^{j-J}}} \epsilon(t2^{-j}) \tag{39}$$

where Ω_s is the log-infinitely random variable defined in Eq. (38) *and $\epsilon(t)$ is a stationary random process independent of $\Omega_{2^{j-J}}$ The expression for the first order scattering coefficient follows*

$$SdM(j) = E(|dM \star \psi_j|) = K, \tag{40}$$

where $K = E(|\epsilon(t)|)$.

Proof: Let $dM_l = e^{\omega_l^{2^J}(t)}dt$. From (37), one has:

$$dM_l \star \psi_j \stackrel{law}{=} 2^{-j} \int e^{\omega_{2^{j-J}l}^{2^j}(u2^{j-J})} \psi(2^{-j}(u-t))du \ ,$$

and thus, by setting $s = 2^{j-J}$, from Eq. (38) and (37):

$$\begin{aligned}
dM_l \star \psi_j &\stackrel{law}{=} 2^{-j} e^{\Omega_s} \int \psi(2^{-j}(u-t)) e^{\omega_l^{2^j}(u)} du \\
&\stackrel{law}{=} 2^{-j} e^{\Omega_s} \int \psi(2^{-j}(u-t)) e^{\omega_{l2^{-j}}^1 (2^{-j}u)} du \\
&\stackrel{law}{=} e^{\Omega_s} \int \psi(u-t) e^{\omega_{l2^{-j}}^1 (u)} du \ .
\end{aligned}$$

Taking the limit $l \to 0$ at fixed j ends the proof for (39) with

$$\epsilon(t) = \lim_{l \to 0} \int \psi(u-t) e^{\omega_l^1(u)} du. \tag{41}$$

Let us remark that, since we supposed that ψ is compact support of size 1, the process $\epsilon(t)$ is a 1-dependent process, i.e., $\forall \tau > 1$, $\epsilon(t+\tau)$ is independent from $\{\epsilon(t')\}_{t' \le t}$. (40) is a direct consequence of (39) and of the fact that $E(e^{\Omega_s}) = 1$. □

The normalized second order scattering coefficient is given by the following theorem :

Theorem 4.3 *Let us suppose that $\zeta(2) > 1$. Then the normalized second order scattering $\tilde{S}dM(j_1, j_2)$ depends only on $j_1 - j_2$ and there exists a constant $\tilde{K}$ such that, in the limit $j_1 \to -\infty$ (j_2 fix)*

$$\lim_{j_1 \to -\infty} SdM(j_1, j_2) = \tilde{K}. \tag{42}$$

Proof: As for the first order, using first (37) and then (38) with $s = 2^{j_2 - J}$ we obtain :

$$\begin{aligned} |\psi_{j_2} \star |\psi_{j_1} \star dM_l||(t) &\overset{law}{=} |\psi_{j_2} \star |\psi_{j_1} \star dM_l||(0) \\ &\overset{law}{=} 2^{-j_2} e^{\Omega_{2^{j_2-J}}} \left| \int \psi(-u 2^{-j_2}) 2^{-j_1} \left| \int \psi(\frac{u-v}{2^{j_1}}) e^{\omega_l^{2^{j_2}}(v)} dv \right| du \right| \end{aligned}$$

Making the changes of variables $u' = u2^{-j_2}$ and $v' = v2^{-j_1}$ and using (37), leads to

$$|\psi_{j_2} \star |\psi_{j_1} \star dM_l||(t) \overset{law}{=} e^{\Omega_{2^{j_2-J}}} \left| \int \psi(-u) \left| \int \psi(2^{j_2-j_1} u - v) e^{\omega_{2^{-j_1} l}^{2^{j_2-j_1}}(v)} dv \right| du \right|$$

Since j_2 is fixed, with no loss of generality, in the following we can set $j_2 = 0$. Using (37), one gets

$$|\psi \star |\psi_{j_1} \star dM_l(0)|| \overset{law}{=} e^{\Omega_{2^{-J}}} \left| \int \psi(-u) \left| \psi_{j_1} \star e^{\omega_l^1}(u) \right| du \right| \tag{43}$$

We are now using the Lemma C.2 proved in Appendix C with $\alpha = \frac{1-2\nu}{1+F(2)}$ ($\nu < 1/2$). We get :

$$E(|\eta_{j_1}|) = O(2^{j_1 \nu}), \tag{44}$$

and

$$\begin{aligned} \psi_{j_1} \star e^{\omega_l^1}(u) &= 2^{-j_1} e^{\omega^1_{2^{j_1}\alpha}(u)} \int \psi(\frac{u-v}{2^{j_1}}) e^{\tilde{\omega}_l^{2^{j_1}\alpha}(v)} dv + \eta_{j_1,l}(u) \\ &\overset{law}{=} e^{\omega^1_{2^{j_1}\alpha}(u)} \int \psi(u 2^{-j_1} - v) e^{\tilde{\omega}^1_{l 2^{-j_1}\alpha}(v 2^{j_1(1-\alpha)})} dv + \eta_{j_1,l}(u) \\ BUGBUG &\overset{law}{=} e^{\Omega_{2^{j_1(1-\alpha)}}} e^{\omega^1_{2^{j_1}\alpha}(u)} \int \psi(u 2^{-j_1} - v) e^{\tilde{\omega}^1_{l 2^{-j_1}}(v)} dv + \eta_{j_1,l}(u) \\ &\underset{l \to 0}{\to} e^{\Omega_{2^{j_1(1-\alpha)}}} e^{\omega^1_{2^{j_1}\alpha}(u)} \tilde{\epsilon}(2^{-j_1} u) + \eta_{j_1}(u) \end{aligned}$$

where $\tilde{\epsilon}(t) = \lim_{l \to 0} \int \psi(t - v) e^{\tilde{\omega}^1_{l 2^{-j_1}}(v)} dv$ is a 1-dependent noise. If follows that :

$$\lim_{l \to 0} \psi \star |\psi_{j_1} \star dM_l(0)| \overset{law}{=} e^{\Omega_{2^{-J}}} e^{\Omega_{2^{j_1(1-\alpha)}}} \int \psi(-u) e^{\omega^1_{2^{j_1}\alpha}(u)} |\tilde{\epsilon}(2^{-j_1} u)| du + \int \psi(-u) \eta_{j_1}(u) du$$

Let $\tilde{K} = E(|\tilde{\epsilon}(t)|)$ and let us define the centered process : $\bar{\epsilon}(t) = |\tilde{\epsilon}(t)| - \tilde{K}$.
Thus we can write

$$\lim_{l \to 0} \psi \star |\psi_{j_1} \star dM_l|(0) \overset{law}{=} e^{\Omega_{2^{-J}}} (I + II + III), \tag{45}$$

where

$$I = \tilde{K}e^{\Omega_{2^{j_1(1-\alpha)}}} \int \psi(-u)e^{\omega^1_{2^{j_1}\alpha}(u)}du \quad (46)$$

$$II = e^{\Omega_{2^{j_1(1-\alpha)}}} \int \psi(-u)e^{\omega^1_{2^{j_1}\alpha}}\bar{\epsilon}(2^{-j_1}u)du \quad (47)$$

$$III = \int \psi(-u)\eta_{j_1}(u)du \quad (48)$$

Then,

$$SdM(j_1, 0) = E(|I + II + III|)$$

Since $\int \psi(u)e^{\omega^1_{2^{j_1}\alpha}}du$ converges in law, when $j_1 \to -\infty$, towards $\epsilon(t)$, (where $\epsilon(t)$ is an independent copy of the process defined in (41)), we have, in the limit $j_1 \to -\infty$:

$$E(|I|) \to K\tilde{K} \quad (49)$$

Thus

$$|SdM(j_1, 0) - \tilde{K}K| \leq |E(|I + II|) - E(|I|)| + E(|III|)$$

From the Lemma, we know that $\lim_{j_1 \to -\infty} E(|III|) = 0$. Moreover

$$|E(|I + II|) - E(|I|)| \leq E(|II|) \leq \sqrt{E(|II^2|)}.$$

From the expression of II and the fact that $\bar{\epsilon}$ is a 1-dependent process, we have:

$$E(|II|^2) \leq ||\psi||^2_\infty \int_0^1 \int_0^1 E\left(e^{\omega^1_{2^{j_1}\alpha}(u)+\omega^1_{2^{j_1}\alpha}(v)}\right) E\left(\bar{\epsilon}(2^{-j_1}u)\bar{\epsilon}(2^{-j_1}v)\right) dudv \quad (50)$$

$$\leq ||\psi||_\infty E(\bar{\epsilon}^2)2^{j_1}E(e^{2\omega^1_{2^{j_1}\alpha}}) \quad (51)$$

which goes to 0 when $j_1 \to -\infty$. Thus $SdM(j_1, 0)$ converges to $\tilde{K}K$ which proves (42). □

Figure 5 displays the first and second order scattering of log-Normal MRM cascades for different value of the intermittency $\lambda^2 = \zeta''(0)$, confirming the results predicted by both Proposition 4.2 and Theorem 4.3 : first order coefficients are constants $(= K)$ and normalized second order coefficients $\widetilde{S}dM(j)$ converge towards a constant $(= \tilde{K})$ as $j \to \infty$.

Another important example of a stochastic self-similar process can be simply obtained from a MRM dM and a Brownian motion $B(t)$ by composing them : $Y(t) = B(M(t))$. It is quite straightforward to prove that $Y(t)$ is stochastically self-similar and satisfies 31. It is referred to as the Multifractal Random Walk and has been initially introduced in [6]. The proofs are extremely similar to the ones for the MRM. We just state here the equivalent of Proposition 4.2 and Theorem 4.3 without the proofs.

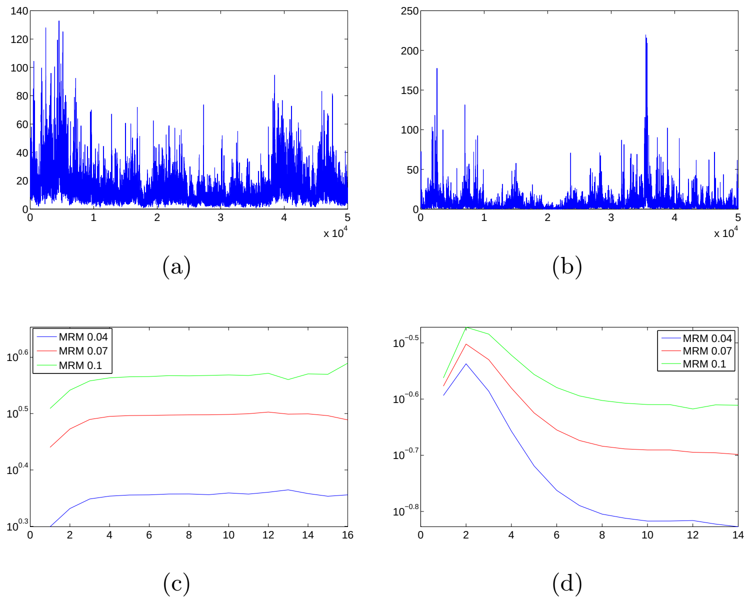


Figure 5: (a) Realization X of an MRM multifractal cascade with $\lambda^2 = 0.04$, (b) Realization of MRM with $\lambda^2 = 0.1$, (c) First order scattering $\log_2 \widetilde{S}dM(j)$ for $\lambda^2 = 0.04, 0.07, 0.1$. (b) Normalized second order scattering $\log_2 \widetilde{S}dM(j)$.

Theorem 4.4 *Let Y a MRW associated to the function $\zeta(q)$. There exists a constant K' such that*

$$SdY(j) = E(|dY \star \psi_j|) = 2^{j\frac{1-\zeta(2)}{2}} K'. \quad (52)$$

Let us suppose that $\zeta(2) > 1$. Then, there exists a constant $\tilde{K}'$ such that the normalized second order scattering $\tilde{S}dY(j_1, j_2)$ satisfies

$$\tilde{S}dY(j_1, j_2) = \tilde{K}' + O\left(2^{j_1 \nu}\right), \quad \forall\, \nu < 1/2. \quad (53)$$

4.3 From Scattering To Intermittency

This section shows that normalized scattering coefficients can be used to compute the intermittency of multifractal random cascades, measured with $\zeta(2) - 2\zeta(1)$.

The intermittency is an important aspect of a multifractal process, describing how the distribution of its increments changes with the scale, and is measured from the curvature of $\zeta(q)$. If X is a multifractal process with integral scale 2^J, Proposition 2.1 relates the intermittency, measured as $2\zeta(1) - \zeta(2)$, with renormalized scattering coefficients:

$$2^{j(\zeta(2)-2\zeta(1))} \simeq \frac{\mathbf{E}(|X \star \psi_j|^2)}{\mathbf{E}(|X \star \psi_j|)^2} = 1 + \sum_{j_2=-\infty}^{+\infty} |\widetilde{S}X(j, j_2)|^2 + e_3(j)\ , \quad (54)$$

with

$$e_3(j) = \frac{1}{SX(j)^2} \sum_{j', j''} \mathbf{E}(\,|\,||X \star \psi_j| \star \psi_{j'}| \star \psi_{j''}|^2)\ .$$

If X is an intermittent multifractal process, and since $\zeta(q)$ is a concave function [13], $\zeta(2) - 2\zeta(1) < 0$ and hence (54) grows exponentially as $j \to -\infty$. If X becomes decorrelated beyond its integral scale 2^J, $\widetilde{S}X(j, j_2) \simeq 2^{(J-j_2)/2}$ for $j_2 > J$, as seen in (35), and $\widetilde{S}X(j, j_2) \simeq 2^{(j_2-j)p}$ for $j_2 \leq j$ if $|\psi| \in C^p$, and hence their contribution in the sum (54) does not affect the exponential growth. On the other hand, since $|\widetilde{S}X(j)| \leq \|\psi\|_1$, $\forall j$, it results that this exponential growth must necessarily come from the term $e_3(j)$.

In order to measure the intermittency, one must account for this term, which in general depends upon high order scattering coefficients. However, the following theorem, proved in Appendix B, shows that when X is a multiplicative cascade, one can characterize the intermittence, measured as $\zeta(2) - 2\zeta(1)$, using normalized scattering coefficients.

Theorem 4.5 *Let $X(t)$ be a log-infinitely divisible Multiplicative cascade with stationary increments, satisfying (30) with an integral scale 2^J and $\mathbf{E}(|X \star \psi_j|^q) = C_q 2^{j\zeta(q)}$. If $2\zeta(1) - \zeta(2) > 0$, then $\rho = 2^{2\zeta(1)-\zeta(2)}$ is the only root in $\mathbb{R}^+$ of the equation*

$$\sum_j \widetilde{S}X(j)^2 L(j)\, x^j = 1 \ , \tag{55}$$

where $L(j) = \frac{C_1^2}{C_2} \mathbf{E}(||X \star \psi_{-j}| \star \psi|^2) \mathbf{E}(||X \star \psi_{-j}| \star \psi|)^{-2}$ satisfies

$$\lim_{j \to \infty} L(j) = 1 \ . \tag{56}$$

This theorem proves that the intermittency factor $2\zeta(1) - \zeta(2)$ can be computed from normalized scattering coefficients $\widetilde{S}X(k)$ and the ratios $L(k)$, which converge to 1 as $k \to \infty$. These ratios correspond to the correction that needs to be applied to scattering coefficients, which are first order moments, in order to measure second order moments. As opposed to the increments $X \star \psi_j$, where the gap between first and second order moments widens and is precisely given by the intermittency factor $2^{j(2\zeta(1)-\zeta(2))}$, the processes $|X \star \psi_j| \star \psi$ are less intermittent, and as a result their second moments can be well approximated by their first order moments, which are precisely the second order scattering coefficients of X.

One can extend Theorem 4.5 to higher order scattering coefficients up to a certain order m. In that case, the correction terms will correspond to the processes $||X \star \psi_j| \star \psi_{j_2}| \ldots \star \psi_{j_m}| \star \psi$, which are even less intermittent than $|X \star \psi_j| \star \psi$. The limit $m = \infty$ corresponds to the scattering conservation of energy, proved in the case of discrete processes [**?**] and conjectured in the continuous case:

$$\mathbf{E}(|X \star \psi_j|^2) = SX(j)^2 + \sum_{j_2=-\infty}^{+\infty} SX(j, j_2)^2 + \sum_{m=3}^{+\infty} \sum_{(j_2, \ldots, j_m) \in \mathbb{Z}^{m-1}} SX(j, j_2, ..., j_m)^2 \ . \tag{57}$$

4.4 Estimation of Intermittency

The previous section showed that scattering coefficients provide estimations of intermittency factors for log infinitely divisible multifractal cascades. This result is generalized by relating normalized scattering coefficients to the intermittency factor $\zeta(2) - 2\zeta(1)$, which specifies the relative scaling of first and second order moments

$$\frac{\mathbf{E}(|X \star \psi_j|^2)}{\mathbf{E}(|X \star \psi_j|)^2} \simeq 2^{j(\zeta(2)-2\zeta(1))} \ . \tag{58}$$

Theorem 4.5 characterizes the intermittency factor $2\zeta(1) - \zeta(2)$ from normalized scattering coefficients and the ratios $L(j) = \frac{C_1^2}{C_2}\mathbf{E}(||X \star \psi_{-j}| \star \psi|^2)\mathbf{E}(||X \star \psi_{-j}| \star \psi|)^{-2}$. Since $\lim_{k\to\infty} L(k) = 1$, one can obtain an estimator using only normalized scattering, by approximating $L(k) = 1$. Moreover, since $\widetilde{S}X(j) \simeq 2^{jp}$, $j < 0$ if $\psi \in C^p$, we can also neglect the terms $j < j_0$ for some $j_0 \leq 0$. This estimator thus computes $\zeta(2) - 2\zeta(1)$ from the smallest positive root of the equation

$$\sum_{k\geq j_0} \widetilde{S}X(k)^2 x^l = 1 \ , \tag{59}$$

which requires to estimate $\widetilde{S}X$.

We begin by introducing an estimator of scattering coefficients from a single realization of a self-similar process $X(t)$ and compare the resulting estimator of $2\zeta(1) - \zeta(2)$ to other estimators for multifractal cascades.

We suppose that ψ has a compact support normalized to $[-1/2, 1/2]$. A realization of $X(t)$ is measured at a resolution normalized to 1 over a domain of size 2^J. We can thus compute wavelet coefficients $X \star \psi_j(n)$ for for $2^J > 2^j > 1$ and $2^j \leq n \leq 2^J - 2^j$. The expected values $SX(j_1) = \mathbf{E}(|X \star \psi_{j_1}|)$ and $SX(j_1, j_2) = \mathbf{E}(||X \star \psi_{j_1}| \star \psi_{j_2}|)$ are computed with time averaging unbiased estimators:

$$S_J X(j_1) = \frac{1}{2^J - 2^{j_1}} \sum_{n=2^{j_1-1}}^{2^J - 2^{j_1-1}} |X \star \psi_{j_1}(n)| \tag{60}$$

and

$$S_J X(j_1, j_2) = \frac{1}{2^J - 2^{j_1} - 2^{j_2}} \sum_{n=2^{j_1-1}+2^{j_2-1}}^{2^J - 2^{j_1-1} - 2^{j_2-1}} ||X \star \psi_{j_1}| \star \psi_{j_2}(n)| \ . \tag{61}$$

An estimator of $\widetilde{S}X(j_1, j_2) = SX(j_1, j_2)/SX(j_1)$ is given by

$$\widetilde{S}_J X(j_1, j_2) = \frac{S_J X(j_1, j_2)}{S_J X(j_1)} \ . \tag{62}$$

If $X(t)$ is self-similar or stochastically self-similar, then Propositions 3.1 and 4.1 prove that that $\widetilde{S}X(j_1, j_2) = \widetilde{S}X(j_1 + k, j_2 + k)$ for any $k \in \mathbb{Z}$ if $2^{j_1} + 2^{j_2} < 2^J$. An estimator

Table 1: Estimation of $2\zeta(1) - \zeta(2) = \lambda^2$ for an MRW multifractal cascade for different values of λ^2. The table gives the mean and the standard deviation of each estimator computed with the scattering equation (64), the moment regression (65) and the log covariance regression (66).

λ^2	Regression moments	Regression Log-Cov	Scattering
0.05	$0.05 \pm 8 \cdot 10^{-3}$	$0.05 \pm 6 \cdot 10^{-4}$	0.05 ± 10^{-3}
0.1	0.09 ± 10^{-2}	$0.1 \pm 2 \cdot 10^{-3}$	$0.1 \pm 2 \cdot 10^{-3}$
0.15	$0.14 \pm 2 \cdot 10^{-2}$	$0.15 \pm 2 \cdot 10^{-3}$	0.15 ± 10^{-3}
0.2	$0.23 \pm 2 \cdot 10^{-2}$	$0.2 \pm 3 \cdot 10^{-3}$	$0.24 \pm 3 \cdot 10^{-3}$

of $\widetilde{S}X(j)$ is obtained averaging the estimators of $\widetilde{S}X(j_1, j_1+k)$ for $1 \le j_1 < J$, in order to minimize the resulting variance [14]:

$$\widetilde{S}_J X(j) = (1 - 2^{-J}) \sum_{k=1}^{J} 2^{-k} \widetilde{S}_J X(k, k+j) \ . \tag{63}$$

Indeed the relative variance amplitude of each $\widetilde{S}_J X(k, k+j)$ is typically proportional to 2^k, because they are computed from empirical averages over number of samples proportional to 2^{-k}, as shown by (60) and (61).

Normalized scattering coefficients are estimated up to a maximum scale $2^K < 2^J$. Large scale coefficients do not affect much the value of the smallest root of equation (59) we thus simply set $\widetilde{S}_j(k) = \widetilde{S}_J(K)$ for $k \ge K$. We also set $j_0 = 0$ in numerical experiments. Equation (59) then becomes

$$\sum_{k=0}^{K} \widetilde{S}_J X(k)^2 x^k + \widetilde{S}_J X(K)^2 \frac{x^{K+1}}{1-x} = 1 \ ,$$

which amounts to finding the smallest root ρ of the equation

$$(1-x) \sum_{k=0}^{K} \widetilde{S} X(k)^2 x^k + \widetilde{S}_J X(K)^2 x^{K+1} + x = 1 \ , \tag{64}$$

and estimate $2\zeta(1) - \zeta(2)$ by $\max(0, -\log_2(\rho))$. Observe that ρ depends mostly upon the values of $\widetilde{S}_J X(k)$ for small values of k which are smaller variance estimators.

Numerical experiments are performed for MRW multifractal cascades studied in Section 4.2, in which case $\zeta(q) = \left(\frac{1}{2} + \lambda^2\right) q - \frac{\lambda^2}{2} q^2$, so $\zeta(2) - 2\zeta(1) = -\lambda^2$. Table 1 reports the results of the intermittency estimation for MRW for several values of λ^2. We simulate cascades using $N = 2^{16}$ points. We estimate the expected scattering representation by averaging over 32 realizations, which is then used to estimate the intermittency. We repeat this experience over 8 runs in order to compute the standard deviation of the estimators. The estimate based on the scattering coefficients is compared with a linear regression on the estimated first and second order moments

$$2\log_2 \mathbf{E}(|X \star \psi_j|^2) - \log_2 \mathbf{E}(|X \star \psi_j|)^2 \approx j(\zeta(2) - 2\zeta(1)) + C \ . \tag{65}$$

The wavelet moments $\mathbf{E}(|X \star \psi_j|^2)$ and $\mathbf{E}(|X \star \psi_j|)$ are estimated with empirical averages of $|X \star \psi_j|$ and $|X \star \psi_j|^2$. We also include a log-covariance estimator from [5] which estimates

$$\mathrm{Cov}\left(\log|X \star \psi_j(t)|, \log|X \star \psi_j(t+l)|\right) \simeq -\lambda^2 \ln\left(\frac{l}{2^J}\right) + o\left(\frac{j}{l}\right) \ . \tag{66}$$

Table 1 shows that the estimation of $\lambda^2 = 2\zeta(1) - \zeta(2)$ based on the scattering equation (64) outperforms the regression on the moments, and has a variance comparable to the covariance of the logarithm. For large values of λ^2 the approximation $L(k) = 1$ introduces non-negligible errors, which can be compensated using higher ordre scattering coefficients.

5 Applications

5.1 Scattering of Turbulence Energy Dissipation

Random cascade models and multifractal analysis were originally introduced in the the context of phenomenology of fully developed turbulence [27, 11, 32, 24]. Turbulent regimes that appear in a wide variety of experimental situations, are characterized by random fluctuations over a wide range of time and space scales. The main physical picture behind this complexity was introduced by Richardson and Kolmogorov [34, 17]: the fluid receives kinetic energy at large scales and dissipates this energy at small scales where fluctuations are well known to be of intermittent nature. The overall range of scales between injection and dissipation is called the inertial range and only depends on the Reynolds number. Making a theory of this "energy cascade" across the inertial range remains one of the most famous challenges in classical physics. According to the previous picture and as proposed in the pioneering papers of Kolmogorov and Obhukov [18, 31], the local dissipation field $\epsilon(\vec{x}, t)$ is described by a multiplicative cascade and its multiscaling properties are the main signature of intermittency. There has been a lot of experimental studies that have been devoted to the estimation of the exponent spectrum associated with these multifractal properties. Most of them agree with a log-normal cascade of intermittency exponent $\lambda^2 \simeq 0.2$ (see e.g. [11, 27]). The pertinence of such a (log-normal) random cascade model can be verified with scattering coefficients estimated from experimental data. The data we used has been recorded by the group of B. Castaing in Grenoble in a low temperature gazeous Helium jet which Taylor scale based Reynolds number is $R_\lambda = 703$ [10]. If one supposes the flow homogeneous and isotropic, the local energy dissipation rate at a given time and location is given by:

$$\epsilon(\vec{x}, t) = 15\nu \left(\frac{\partial v_{||}}{\partial x}\right)^2 \ ,$$

where $v_{||}(\vec{x}, t)$ is the stream-wise component of velocity and ν stands for the kinematic viscosity constant. If one assumes the validity of the Taylor frozen-flow hypothesis [11], a surrogate of the dissipation field can be obtained from the temporal evolution of longitudinal velocity field as:

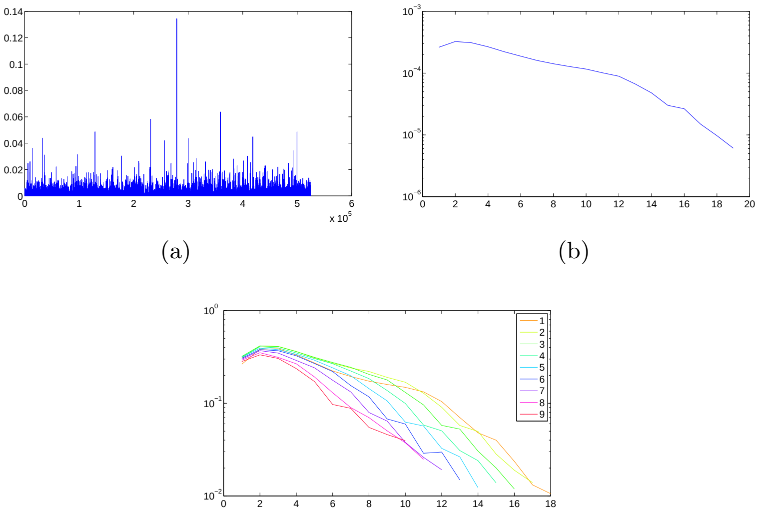
$$\epsilon(t) \simeq \left(\frac{\partial v_{||}}{\partial t} \right)^2_; . \quad (67)$$

Figure 6-(a) shows a sample of the field $\epsilon(t)$ as a function of time estimated from the experimental velocity records. The Kolmogorov (dissipative) scale η is observed at approximately 2^2 sample points, whereas the integral scale T is approximately 2^{11} sample points. The associated first order scattering coefficients are displayed in panel (b) of Figure 6. We first observe that at large scales, below the integral scale T, $S\epsilon(j) \simeq 2^{-jH}$, with $H \simeq 0.5$. As explained in section 3, this corresponds to a Gaussian white noise, as one expects for uncorrelated fluctuations at large scales. In the inertial range, the scaling law of the exponents is smaller. Let us notice that according to a multiplicative cascade model for ϵ, this exponent should be zero, i.e., $S\epsilon(j) \simeq C$ for $\eta \ll 2^j \ll T$. The fact that a standard conservative cascade model is not suited to reproduce the data is even more apparent in Panel (c) of Figure 6. In this Panel are reported the estimated scattering coefficients $\widetilde{S}\epsilon(j_1, j_2)$ as a function of $j_2 - j_1$ for different values of j_1. Unlike what is expected for a self-similar cascade (as e.g. in Fig. 4), the second order scattering coefficients do not depend only on $j_2 - j_1$. Moreover one does not observe any convex curve with an asymptotic constant behavior at large $j_2 - j_1$. This striking departure from stochastic self-similarity is likely to originate from the fact that Taylor hypothesis does not rigorously hold. A single probe provides measures of velocity temporal variations at a fixed space location that involve both Lagrangian and Eulerian fluctuations. This point has been discussed in details by Castaing in ref. [9] in order to explain the behavior of the correlation functions of velocity amplitude variations. Unlike standard scaling analysis, the analysis relying on second order scattering allows one to detect if the self-similarity hypothesis, even in the stochastic sense of random cascades, is sound.

5.2 Analysis of Financial Data

Since Mandelbrot's pioneering work on the fluctuations of cotton price in early sixties, it is well known that market price variations are poorly described by the standard geometric Brownian motion [25]: Extreme events are more probable than in a Gaussian world and variance fluctuations are well known to be of intermittent and correlated nature. Many recent studies have shown that multifractal processes and more precisely, random cascade models provide a class of random processes that reproduce faithfully most of empirical properties observed in financial time series (see e.g. [26, 28, 20, 7] or [8] for a review). In this framework the logarithm of the price of a given asset $X(t)$ is generally modeled as :

$$X(t) = B(\Theta(t)), \quad (68)$$



(a) (b)

(c)

Figure 6: (a) Realization of dissipation $\epsilon(t) = \left(\frac{\partial v}{\partial t}\right)^2$ in a turbulent flow. (b) First order scattering coefficients $\log S\epsilon(j)$ as a function of j, estimated from 4 realizations of 2^{19} samples each. (c) Second order Scattering coefficients $\log \widetilde{S}\epsilon(j_1, j_2)$ estimated from the same data. We plot curves $\log \widetilde{S}\epsilon(j_1, j_1 + l)$ as a function of l for different values of j_1.

where B is Brownian motion which is independent of the positive non-decreasing process $\Theta(t)$ which is a multifractal process satisfying the self-similarity property (30). The process $\Theta(t)$ is generally referred to as the *trading time* [25] and somewhat describes the intrinsic time (versus the physical time) of the market. Thus $d\Theta(t)$ can be seen as the instantaneous variance of the Brownian motion at time t. The multifractal structure of $X(t)$ is entirely deduced from the multifractal structure of $\Theta(t)$.

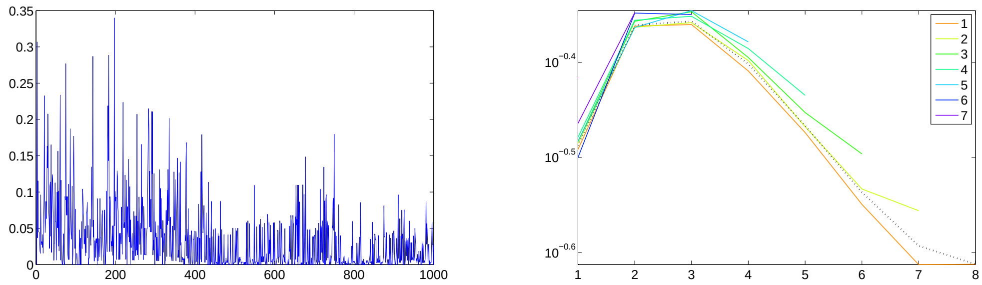
In this section, we compute the normalized scattering coefficients to analyze the process $\Theta(t)$[1].

Analysis of high-frequency ("tick-by-tick") Euro-Bund data

Euro-Bund is one of the most actively traded financial asset in the world. It corresponds to a future contract on an interest rate of the euro-zone and it is traded on the Eurex electronic market (in Germany). The typical number of trades is around 40.000 per day and in this study we have used 800 trading days going from 2009 May to 2012 September. Each trade occurs at a given price, whose logarithm is modeled by $X(t)$ using (68).

Every single day, the sum of the quadratic variations of $X(t)$ are computed on a rolling interval of 30 seconds (after preprocessing the microstructure noise using [35] technique). This can be considered as an estimation of the 30 second variance $\int_t^{t+30s} d\Theta$. It is well known that intraday financial data are subject to very strong seasonal intraday

[1]We will use a proxy of $\Theta(t)$ since only $X(t)$ is directly visible on the market .



(a) (b)

Figure 7: (a) One trading day of the German BUND (b) estimated $\widetilde{S}F(j_1, j_2)$ for different values of j_1.

effects (e.g., the variance is systematically stronger at opening and closing time than at lunch time). In order to remove them, we used a standard technique consisting in normalizing the variance by the intraday seasonal variance (computed by averaging every day the 5min variance at a particular time of the day).

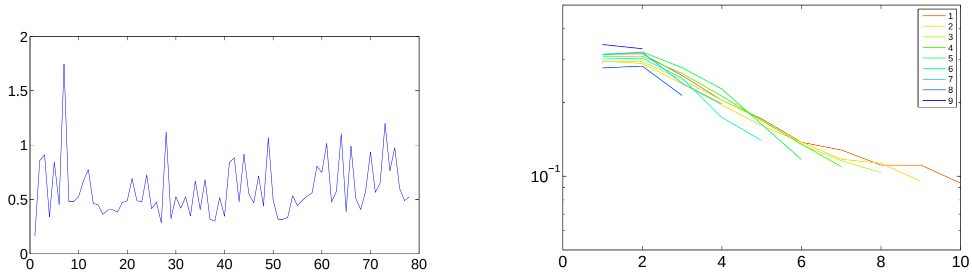
Figure 7(a) shows the resulting "deseasonalized" 30s variance for a particular day. The scattering coefficients have been computed independently for each single day and then averaged. Fig. 7(b) shows $\widetilde{S}F(j_1, j_2)$ for different values of j_1. We observe that this function does not depend on j_1 and varies very little. It confirms the stochastic self-similarity of the variance process and that the normalized scattering function behaves like the one of an MRM.

Analysis of 5 minutes S&P 100 index

The same analysis is performed on S&P 100 index sampled every 5 minutes from April 8th 1997 to December 17th 2001. It opens 6.5 hours from 9:30am to 4:00pm. The S&P 100 Index is a stock market index of United States stocks maintained by Standard & Poor's. It is a subset of the S&P 500 index. We perform the same preprocessing of the data as we did on the Euro-Bund data except that

- since it is sampled at a lower frequency there is no need to remove the microstructure noise,
- we used high and low values on each 5mn interval to compute an estimation on the 5mn variance $\int_t^{t+5mn} d\Theta$
- all the days are concatenated and the overnight period has been preprocessed using the usual deseasonalizing algorithm.

Figure 8(a) shows the deseasonalized 5mn variance during a trading day. Fig. 8(b) shows $\widetilde{S}F(j_1, j_2)$ for different values of j_1. We observe that this function does not depend on j_1 and varies very little. Again it confirms the stochastic self-similarity of the variance



(a) (b)

Figure 8: (a) One trading day of the S&P 100 index (b) estimated $\widetilde{S}F(j_1, j_2)$ for different values of j_1.

process and that the normalized scattering behaves like the one of an MRM (see Section 4.2).

A Proof of lemma 3.4

Let $Y_j(t) = 2^{j/2}|dB \star \varphi| \star \psi_j(t)$. To prove that $\mathbf{E}(|Y_j|)$ converges to a constant, we shall prove that the distribution of Y_j is asymptotically gaussian:

$$Y_j(t) \xrightarrow[j\to\infty]{l} A = A_1 + iA_2 \tag{69}$$

where A_1 and A_2 are two zero-mean independant Gaussian distributions of variance $\sigma^2/2$ with

$$\sigma^2 = \|\psi\|_2^2 \int R_{|dB\star\varphi|}(\tau) d\tau$$

which is the first result of Lemma 3.4. It implies that

$$|Y_j| \xrightarrow[j\to\infty]{l} |A| \ ,$$

where $|A|$ has a Rayleigh distribution of parameter σ. We shall also prove that

$$\lim_{j\to\infty} \mathbf{E}(|Y_j|^2) = E(|A|^2) = \sigma^2 \ . \tag{70}$$

This will allow us to conclude that

$$\lim_{j\to\infty} \mathbf{E}(|Y_j|) = \mathbf{E}(|A|) = \sigma\sqrt{\frac{\pi}{2}} \ . \tag{71}$$

and hence finish the proof of Lemma 3.4, by applying the following theorem on uniform integrability of sequences of random variables $X_j = |Y_j|$.

Theorem A.1 *([?], thm 6.1-6.2) Let $\{X_j\}j \in \mathbb{N}$ be a sequence of random variables. If $X_j \xrightarrow{d} X_\infty$ and*

$$\sup_j \mathbf{E}(|X_j|^{1+\delta}) < \infty \quad \textit{for} \quad \delta > 0 \ ,$$

then

$$\lim_{j\to\infty} \mathbf{E}(X_j) = \mathbf{E}(X_\infty) \ .$$

The convergence (69) of Y_j relies on the use of a central-limit theorem for real dependant random variables. The extension to the two-dimensional complex random variables Y_j is done by considering arbitrary linear combinations of its real and imaginary parts. The Cramer-Wold theorem proves that if $Y_j = Re(Y_j) + i\,Im(Y_j)$ satisfies

$$\forall (\alpha,\beta) \in \mathbb{R}^2 \quad , \quad \alpha Re(Y_j) + \beta Im(Y_j) \underset{j\to\infty}{\overset{l}{\longrightarrow}} \alpha A_1 + \beta A_2 \tag{72}$$

then $Y_j \underset{j\to\infty}{\overset{l}{\longrightarrow}} A_1 + iA_2$, so (69) is satisfied.

The random variables A_1 and A_2 are zero-mean independant Gaussian random variables of variance σ^2 if and only if $\alpha A_1 + \beta A_2 = \mathcal{N}(0, (\alpha^2+\beta^2)\sigma^2)$ for all $(\alpha,\beta) \in \mathbb{R}^2$.

Since $Y_j = 2^{j/2}|dB \star \varphi| \star \psi_j(t)$,

$$\bar{Y}_j = \alpha Re(Y_j) + \beta Im(Y_j) = 2^{j/2}|dB \star \varphi| \star \bar{\psi}_j(t)$$

with $\bar{\psi} = \alpha Re(\psi) + \beta Im(\psi)$. Since ψ is analytic, its real and imaginary parts are respectively even and odd. They are thus orthogonal and $\|Re(\psi)\|_2 = \|Im(\psi)\|_2$. It results that $\|\bar{\psi}\|^2 = (\alpha^2+\beta^2)\|\psi\|^2/2$. Proving the Cramer-Wolf hypothesis (72) amounts to proving that

$$\bar{Y}_j(t) \underset{j\to\infty}{\overset{l}{\longrightarrow}} \mathcal{N}(0, \bar{\sigma}^2) \tag{73}$$

with $\bar{\sigma}^2 = \|\bar{\psi}\|_2^2 \int R_{|dB\star\varphi|}(\tau)d\tau$. The second order moment condition (70) is proved by showing that

$$\lim_{j\to\infty} \mathbf{E}(|\bar{Y}_j|^2) = \bar{\sigma}^2 \tag{74}$$

given that $|Y_j|^2 = |Re(Y_j)|^2 + |Im(Y_j)|^2$.

We now concentrate on the proof of (73) and (74) for $\bar{Y}_j = 2^{j/2}|dB \star \varphi| \star \bar{\psi}_j(t)$. Let us write $\varphi_\Delta = \varphi\,\mathbf{1}_{[-\Delta/2,\Delta/2]}$. We shall limit ϕ to a compact support by defining $\{\Delta_j\}_{j\geq 0}$ with $\lim_{j\to\infty} \Delta_j = \infty$ and decompose

$$|dB \star \varphi(t)| = |dB \star \varphi_{\Delta_j} + dB \star (\varphi - \varphi_{\Delta_j})| \ .$$

As a result

$$|dB \star \varphi(t)| = |dB \star \varphi_{\Delta_j}| + Z_j(t)$$

with $\mathbf{E}(|Z_j|) \leq \mathbf{E}(|dB \star (\varphi - \varphi_{\Delta_j})|)$. Since dB is the Wiener measure, if $\theta \in \mathbf{L}^2(\mathbb{R}^d)$ then

$$\mathbf{E}(|dB \star \theta|) \leq \mathbf{E}(|dB \star \theta|^2)^{1/2} = \|\theta\|_2 \ , \tag{75}$$

so $\mathbf{E}(|Z_j|) \leq \|\varphi - \varphi_{\Delta_j}\|_2$. It results that

$$|dB \star \varphi| \star \bar{\psi}_j(t) = |dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(t) + Z_j \star \bar{\psi}_j(t) \ , \tag{76}$$

and

$$\mathbf{E}(|Z_j \star \bar{\psi}_j|) \leq \mathbf{E}(|Z_j|)\|\bar{\psi}_j\|_1 \leq \|\varphi - \varphi_{\Delta_j}\|_2 \, \|\bar{\psi}\|_1 \ .$$

Since $\lim_{j\to\infty} \Delta_j = \infty$, $\lim_{j\to\infty} \|\varphi - \varphi_{\Delta_j}\|_2 = 0$ so $Z_j \star \bar{\psi}_j$ converges to 0 in probability when j increases. So the limits of $|dB \star \varphi| \star \bar{\psi}_j(t)$ and $|dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(t)$ are equal.

We now prove (73) by applying Berk central limit theorem for dependent random variables [**?**] to $\bar{Y}_j = |dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(t)$, in order to show that it converges to a normal distribution. The proof will also verify (74).

Theorem A.2 (Berk Central-Limit) *For any $j \in \mathbb{N}$, let $\{S_{i,j}\}_{i=1,\ldots,n_j}$ be a sequence of zero mean random variables such that for any $i \leq n_j$ $S_{i,j}$ is independant of $S_{i+r,j}$ for $r \geq m_j$. If the following properties are satistied*

(i) $\exists \delta > 0$, $\lim_{j\to\infty} n_j^{-1} m_j^{2+2/\delta} = 0$

(ii) $\exists M > 0$, $\forall i, j > 0$, $\mathbf{E}(|S_{i,j}|^{2+\delta}) \leq M$

(iii) $\exists K > 0$, $\forall i, j, l > k > 0$, $Var(\sum_{i=k+1}^{k+l} S_{i,j}) \leq l\, K$

(iv) $\lim_{j\to\infty} n_j^{-1} Var(\sum_{i=1}^{n} S_{i,j}) = \sigma^2 > 0$

then

$$n_j^{-1/2} \sum_{i=1}^{n} S_{i,j} \xrightarrow[j\to\infty]{l} \mathcal{N}(0, \sigma^2) \ . \tag{77}$$

Since $|dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(t)$ is stationary, its distribution can be evaluated at $t = 0$

$$|dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(0) = \int |dB \star \varphi_{\Delta_j}|(u) \bar{\psi}_j(-u) du.$$

The central-limit theorem is applied by dividing this integrale into disjoint integrals

$$S_{i,j} = 2^j \int_{2^j b_{i,j}}^{2^j b_{i+1,j}} |dB \star \varphi_{\Delta_j}(u)| \, \bar{\psi}_j(-u) du \ , \tag{78}$$

where for each $j \in \mathbb{Z}$, $\{b_{i,j}\}_{1 \leq i \leq n_j}$ is an increasing sequence of points in $\mathbb{R} \cup \{\pm\infty\}$ such that

$$\forall i \ , \quad \int_{b_{i,j}}^{b_{i+1,j}} |\bar{\psi}(-u)| du = 2^{-j} \|\bar{\psi}\|_1 \ . \tag{79}$$

Since $\bar{\psi}$ is $\mathbf{C^1}$ and bounded, we verify that $n_j \simeq 2^j$. Summing these random variables gives

$$2^{-j/2} \sum_{i=1}^{n_j} S_{i,j} = 2^{j/2} \, |dB \star \varphi_{\Delta_j}| \star \bar{\psi}_j(0) \ . \tag{80}$$

We now show that the $S_{i,j}$ satisfy the hypothesis of the Beck central-limit theorem so that we can apply the convergence result (77) which implies (73).

Let us first prove that $S_{i,j}$ is independant of $S_{i+r,j}$ for $r \geq m_j$ which satisfies (i). Since $\bar\psi$ is bounded, it results that $\inf_{i,j} 2^j |b_{i,j} - b_{i+1,j}| = \eta > 0$. Since φ_{Δ_j} has a support of size Δ_j and dB is a Wiener Noise, it follows that $|dB \star \varphi_{\Delta_j}|(u)$ is indendant of $|dB \star \varphi_{\Delta_j}|(u')$ for $|u-u'| > \Delta_j$ and hence that $S_{i,j}$ is independant of $S_{i+r,j}$ for $r \geq m_j = \Delta_j/\eta$.

To verify (i) let us set $\delta = 1$. Since $n_j \simeq 2^j$, if we choose $\Delta_j = 2^{j/5}$ then

$$\lim_{j\to\infty} \frac{m_j^4}{n_j} \leq \eta^{-4} \lim_{j\to\infty} 2^{j(4/5-1)} = 0 \ . \tag{81}$$

We now verify condition (ii) with $\delta = 1$. Since $\bar\psi_j(u)$ has a zero integral, one can replace $|dX \star \varphi_{\Delta_j}(u)|$ by $Q_j(u) = |dX \star \varphi_{\Delta_j}|(u) - \mathbf{E}(|dX \star \varphi_{\Delta_j}|)$ in the definition (78) of $S_{i,j}$. It results that

$$\begin{aligned} \mathbf{E}(|S_{i,j}|^3) &\leq \iiint \mathbf{E}(Q_j(u)Q_j(u')Q_j(u''))\, 2^{3j} |\bar\psi_j(-u)|\, |\bar\psi_j(-u')|\, |\bar\psi_j(-u'')|\, du\, du'\, du'' \\ &\leq \mathbf{E}(|dB \star \varphi_{\Delta_j}|^3) \|\bar\psi\|_1^3 = 2^{5/2}\pi^{-1/2} \|\varphi_{\Delta_j}\|_2^3 \|\bar\psi\|_1^3 \leq 2^{5/2}\pi^{-1/2} \|\varphi\|_2^3 \|\bar\psi\|_1^3 \end{aligned} \tag{82}$$

Let us now verify condition (iii). The sum $A_{k,l,j} = \sum_{i=k}^{k+l} S_{i,j}$ is by definition

$$A_{k,l,j} = 2^j \int_{2^j b_{k,j}}^{2^j b_{k+l,j}} |dB \star \varphi_{\Delta_j}(u)|\, \bar\psi_j(-u) du = \int_{\mathbb{R}} |dB \star \varphi_{\Delta_j}(u)|\, h_{k,l,j}(u)\, du$$

with $h_{k,l,j}(u) = 2^j \bar\psi_j(-u) 1_{[2^j b_{k,j}, 2^j b_{k+l,j}]}(u)$. It results that

$$\mathrm{Var}(A_{k,l,j}) \leq \|R_{|dB\star\varphi_{\Delta_j}|}\|_1\, \|h_{k,l,j}\|_2^2 \ . \tag{83}$$

But, with a change of variable and applying (79) we get

$$\|h_{k,l,j}\|_2^2 = \int_{2^j b_{k,j}}^{2^j b_{k+l,j}} |\bar\psi(2^{-j}u)|^2\, du \leq \|\bar\psi\|_\infty \int_{b_{k,j}}^{b_{k+l,j}} 2^j |\bar\psi(u)|\, du \leq \|\bar\psi\|_\infty\, \|\bar\psi\|_1\, l \ .$$

We are now going to bound $\|R_{|dB\star\varphi_{\Delta_j}|}\|_1$ by using the decay $\varphi(u) = O((1+|u|^{-2}))$.

$$R_{|dB\star\varphi_{\Delta_j}|}(\Delta) = \mathbf{E}(|dB \star \varphi_{\Delta_j}(\Delta)|\ |dB \star \varphi_{\Delta_j}(0)|) - \mathbf{E}(|dB \star \varphi_{\Delta_j}|)^2 \ .$$

If $|\Delta| > |\Delta_j|$ then since the support of $\varphi_{\Delta_j}(u)$ and $\varphi_{\Delta_j}(u-\Delta)$ does not overlapp, $|dB \star \varphi_{\Delta_j}(\Delta)|$ and $|dB \star \varphi_{\Delta_j}(0)|$ are independant random variables so $R_{|dB\star\varphi_{\Delta_j}|}(\Delta) = 0$. Otherwise, we decompose

$$|dB \star \varphi_{\Delta_j}(u)| = |dB \star \varphi_\Delta(u) + dB \star (\varphi_{\Delta_j} - \varphi_\Delta(u)| \ .$$

Since $|dB \star \varphi_\Delta(0)|$ and $|dB \star \varphi_\Delta(\Delta)|$ are independant random variables,

$$|R_{|dB\star\varphi_{\Delta_j}|}(\Delta)| \leq |\mathbf{E}(|dB\star\varphi_\Delta|)^2 - \mathbf{E}(|dB\star\varphi_{\Delta_j}|)^2| + 2\mathbf{E}(|dB\star\varphi_\Delta|)\mathbf{E}(|dB\star(\varphi_{\Delta_j}-\varphi_\Delta)|) + \mathbf{E}(|dB\star(\varphi_{\Delta_j}-\varphi_\Delta)|)^2 \ .$$

Since $\mathbf{E}(|dB\star\theta|) \leq \mathbf{E}(|dB\star\theta|^2)^{1/2} \leq \|\theta\|_2$, by applying this to $\theta = \varphi_\Delta$ and $\theta = \varphi_{\Delta_j} - \varphi_\Delta$ one can verify that

$$|R_{|dB\star\varphi_{\Delta_j}|}(\Delta)| \leq 6\|\varphi\|_2\, \|\varphi - \varphi_\Delta\|_2 \ . \tag{84}$$

Since $\varphi(u) = O((1+|u|)^{-2})$ it results that $\|\varphi - \varphi_\Delta\|_2 = O((1+|\Delta|)^{-3/2})$ so $\|R_{|dB\star\varphi_{\Delta_j}|}\|_1$ is bounded independantly of j. Inserting this in (83) proves the theorem hypothesis (iii).

Let us now verify the hypothesis (iv). It results from (80) that

$$2^{-j}\mathrm{Var}(\sum_i S_{i,j}) = 2^j \mathrm{Var}(|dX\star\varphi_{\Delta_j}|\star\bar\psi_j) = 2^j \int \hat R_{|dX\star\varphi_{\Delta_j}|}(\omega)|\widehat{\bar\psi}(2^j\omega)|^2 d\omega \ .$$

We proved (84) that $R_{|dB\star\varphi_{\Delta_j}|} \in \mathbf{L}^1$ but the same inequality is valid for $R_{|dB\star\varphi_\Delta|}$ which proves that it is also in $\mathbf{L}^1$. It results that $\widehat R_{|dX\star\varphi|}$ is continuous. Since φ_{Δ_j} converges to φ in $\mathbf{L}^2\cap\mathbf{L}^1$ as $j\to\infty$, $\widehat R_{|dX\star\varphi^j|}(0)$ converges to $\widehat R_{|dX\star\varphi|}(0)$. Since $2^j|\widehat{\bar\psi}(2^j\omega)|^2$ converges to $\|\bar\psi\|_2^2\delta(\omega)$ when j goes to ∞

$$\lim_{j\to\infty} 2^{-j}\mathrm{Var}(\sum_i S_{i,j}) = \widehat R_{|dX\star\varphi|}(0)\|\bar\psi\|_2^2 = \bar\sigma^2,$$

which proves condition (iv). It also proves (74).

We can thus apply Theorem A.2 which proves that $2^{j/2}|dB\star\varphi|\star\bar\psi_j(t)$ converges in distribution to $\mathcal{N}(0\,\bar\sigma^2)$ and hence (73). It finishes the Lemma proof.

B Proof of Theorem 4.5

If X is a stationary process, let us write $\chi(X) = \mathbf{E}(|X|^2)\mathbf{E}(|X|)^{-2}$. Since by hypothesis $\mathbf{E}(|X\star\psi_j|^q) = C_q 2^{j\zeta(q)}$, and the wavelet decomposition is unitary, it results that

$$\begin{aligned}\forall\, j\ ,\ \chi(X\star\psi_j) &= 1 + \sum_{j'} \frac{\mathbf{E}(||X\star\psi_j|\star\psi_{j'}|^2)}{\mathbf{E}(|X\star\psi_j|)^2}\\ &= 1 + \sum_{j'} \widetilde S X(j,j')^2 \chi(|X\star\psi_j|\star\psi_{j'})\ .\end{aligned} \tag{85}$$

We will show that the contribution of the terms $j' > 0$ is bounded:

$$\sup_{j\leq j_0}\sum_{j'\geq 0} \widetilde S X(j,j')^2\chi(|X\star\psi_j|\star\psi_{j'}) = O(1)\ . \tag{86}$$

By plugging (86) into (85), since by definition $\chi(X\star\psi_j) = \frac{C_2}{C_1^2}2^{j(\zeta(2)-2\zeta(1))}$, it results that

$$\forall j\leq j_0\ ,\ 1 = \frac{C_1^2}{C_2}2^{-j(\zeta(2)-2\zeta(1))}(\sum_{j'\leq 0}\widetilde S X(j,j')^2\chi(|X\star\psi_j|\star\psi_{j'}) + O(1))\ . \tag{87}$$

Now, we use the stochastic self-similarity (30)

$$\begin{aligned} |X \star \psi_j| \star \psi_{j'}|(t) &\stackrel{l}{=} |D_{j'}X \star \psi_{j-j'}| \star \psi| \\ &= A_{2^{j'}}||X \star \psi_{j-j'}| \star \psi| \ . \end{aligned}$$

Since $\mathbf{E}(|X \star \psi_j|^q) = C_q 2^{j\zeta(q)}$, it results that $\mathbf{E}(A^q_{2^{j'}}) = 2^{j'\zeta(q)}$ and hence

$$\begin{aligned} \chi(||X \star \psi_j| \star \psi_{j'}|) &= \frac{\mathbf{E}(|X \star \psi_j| \star \psi_{j'}|^2)}{\mathbf{E}(|X \star \psi_j| \star \psi_{j'}|)^2} \\ &= 2^{(\zeta(2)-2\zeta(1))j'} \chi(|X \star \psi_{j-j'}| \star \psi|) \ . \end{aligned} \tag{88}$$

By substituting in (87) and using the fact that $\widetilde{S}X(j,j') = \widetilde{S}X(j'-j)$, we obtain

$$\forall j \leq j_0 \ , \ 1 = \sum_{j'<0} \widetilde{S}X(j'-j)^2 2^{(\zeta(2)-2\zeta(1))(j'-j)} L(j'-j) + O(2^{-j(\zeta(2)-2\zeta(1))}) \ . \tag{89}$$

Since $\zeta(2) - 2\zeta(1) < 0$, by letting $j \to -\infty$, we obtain (55).

Let us now prove (86). For that purpose, we write $X_j(t) = e^{\omega^{2^J}_{2^j}(t)}dt$ and $X^j(t) = \lim_{l\to 0} e^{\omega^{2^j}_l(t)}dt$. Using the same decomposition as in (45), we have

$$\frac{|X \star \psi_j| \star \psi_{j'}}{\mathbf{E}(|X \star \psi_j|)} = \frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)}{\mathbf{E}(|X \star \psi_j|)} X_{j\alpha} \star \psi_{j'} + \epsilon \star \psi_{j'} + \eta \star \psi_{j'} \ , \tag{90}$$

where $\alpha = \frac{1-2\nu}{3-\zeta(2)}$ is chosen with $\nu < 1/2$ and

$$\epsilon = \frac{1}{\mathbf{E}(|X \star \psi_j|)} X_{j\alpha} \left(|X^{j\alpha} \star \psi_j| - \mathbf{E}(|X^{j\alpha} \star \psi_j|)\right)$$

satisfy

$$\mathbf{E}(|\eta|^2) = O(2^{j\nu_1}) \ , \ \mathbf{E}(|\epsilon \star \psi_{j'}|^2) = O(2^{(j-j')\nu_2}) \ ,$$

with $\nu_1, \nu_2 > 0$, thanks to (100) and to (51). We thus have

$$\begin{aligned} \sum_{j'>0} \widetilde{S}X(j,j')^2 \chi(|X \star \psi_j| \star \psi_{j'}) &= \sum_{j'>0} \frac{\mathbf{E}(|X \star \psi_j| \star \psi_{j'}|^2)}{\mathbf{E}(|X \star \psi_j|)^2} \\ &\leq C \sum_{j'>0} \frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)^2}{\mathbf{E}(|X \star \psi_j|)^2} \mathbf{E}(|X \star \psi_{j'}|^2) + \mathbf{E}(|\epsilon \star \psi_{j'}|^2) + \mathbf{E}(|\eta \star \psi_{j'}|^2) \\ &\leq C \frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)^2}{\mathbf{E}(|X \star \psi_j|)^2} \sum_{j'>0} \mathbf{E}(|X \star \psi_{j'}|^2) + 2^{j\nu_2} \sum_{j'>0} O(2^{-j'}\nu_2) + \mathbf{E}(|\eta|^2) \\ &\leq C \frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)^2}{\mathbf{E}(|X \star \psi_j|)^2} \sum_{j'>0} \mathbf{E}(|X \star \psi_{j'}|^2) + O(2^{j \min(\nu_1,\nu_2)}) \ . \end{aligned} \tag{91}$$

Since X decorrelates beyond its integral scale 2^J, its second moments satisfy

$$\mathbf{E}(|X \star \psi_{j'}|^2) \simeq \begin{cases} 2^{j'\zeta(2)} & \text{if } j' < J \ , \\ 2^{-j'} & \text{if } j \geq J \ . \end{cases} \tag{92}$$

From (91) and (92), and since

$$\frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)^2}{\mathbf{E}(|X \star \psi_j|)^2} \underset{j \to -\infty}{\longrightarrow} 1 \ ,$$

we obtain (86).

Finally, let us prove (56). Using the decomposition (90) when $j' = 0$, we just showed that

$$\frac{|\,|X \star \psi_j| \star \psi|(t)}{\mathbf{E}(|X^{j\alpha} \star \psi_j|)} \overset{d}{=} |X_{j\alpha} \star \psi| + \epsilon_j \ ,$$

where $\mathbf{E}(|\epsilon_j|^2) = O(2^{j\nu})$ and $\nu > 0$. One can verify that if Y, Y' are random variables such that $\mathbf{E}(|Y - Y'|^2) \leq \epsilon$, then

$$|\chi(Y) - \chi(Y')| \leq C \frac{\epsilon}{\mathbf{E}(|Y|)^2}(1 + \chi(Y')) \ .$$

It results that

$$\chi(|X \star \psi_j| \star \psi|) = \frac{\mathbf{E}(|X \star \psi_j| \star \psi|^2)}{\mathbf{E}(|X \star \psi_j| \star \psi|)^2} = \chi(X_{j\alpha} \star \psi)\left(1 + O(2^{j\nu})\frac{\mathbf{E}(|X^{j\alpha} \star \psi_j|)^2}{\mathbf{E}(||X \star \psi_j| \star \psi|)^2}\right) \ . \tag{93}$$

Since $X_{j\alpha}$ converges weakly to X as $j \to -\infty$ and $\lim_{j \to -\infty} \widetilde{S}X(j, 0) = \tilde{K}$ from theorem 4.3, it results that

$$\lim_{j \to -\infty} L(j) = \frac{C_1^2}{C_2}\chi(X \star \psi) = 1 \ .$$

□

C Various results on the MRM measure

Lemma C.1 *The process $\omega_l^T(t)$ used for the construction of the MRM dM is an infinitely-divisible process whose two-points characteristic function reads:*

$$E\left(e^{p_1 \omega_l^T(t_1) + p_2 \omega_l^T(t_2)}\right) = e^{[F(p_1)+F(p_2)]\rho_l^T(0) + [F(p_1+p_2) - F(p_1) - F(p_2)]\rho_l^T(t_2 - t_1)} \tag{94}$$

where $F(-ip)$ is the cumulant generating function characterizing the infinitely divisible law as provided by the Levy-Khintchine formula where the drift term is chosen such that

$$F(1) = 0 \ , \tag{95}$$

and where the function $\rho_l^T(\tau)$ is defined by:

$$\rho_l^T(\tau) = \begin{cases} \ln(T/l) + 1 - |\tau|/l \ , & \textit{if } |\tau| \leq l \ , \\ \ln(T/|\tau|) \ , & \textit{if } l \leq |\tau| < T \ , \\ 0 \ , & \textit{otherwise} \ . \end{cases} \tag{96}$$

Moreover, the function $\zeta(p)$ which satisfies (31) *(with $X = dM$ where dM is the associated MRM) is given by*

$$\zeta(p) = p - F(p).$$

The proof of this Lemma is given in [6].
The next Lemma uses an alternative MRM measure considered in Ref. [6] defined by

$$d\tilde{M}(t) = \lim_{l \to 0} e^{\tilde{\omega}_l^T(t)} dt$$

where $\tilde{\omega}_l^T$ is defined exactly as the process ω_l^T but only differs by its ρ function which is replaced by : $\tilde{\rho}_l^T(\tau) = \rho_l^T(\tau) + \frac{\tau}{T} - 1$, for $\tau \leq T$. One can then easily show that $\tilde{\omega}_l^T$ is linked with ω_l^T by the following cascade property :

$$\forall l \leq a \leq T, \ \omega_l^T(u) \overset{a.s.}{=} \tilde{\omega}_l^a(u) + \omega_a^T(u) \tag{97}$$

where $\tilde{\omega}_l^a$ and ω_a^T are independent copies of the processes defined previously. Moreover, $\tilde{\omega}_l^T$ satisfies both (37) and (37).

We are now ready to state the last Lemma we will need.

Lemma C.2 *Let ω_l^T the infinitely divisible process associated to the MRM dM and ψ be a wavelet of support in $[0,1]$ such that $||\psi||_\infty < \infty$. For all α such that $0 < \alpha < 1$, one has:*

$$\forall l < 2^j \ , \ (\psi_j \star e^{\omega_l^T})(t) = e^{\omega_{2^{j\alpha}}^T(t)} \left(\psi_j \star e^{\tilde{\omega}_l^{2^{j\alpha}}} \right) + \eta_{l,j}(t) \ , \tag{98}$$

where the process $\eta_{l,j}(t)$ has a limit process $\lim_{l\to 0} \eta_{l,j}(t) = \eta_j(t)$ which satisfies, in the limit $j \to -\infty$,

$$E(|\eta_j(t)|) = O(2^{j\frac{1-\alpha(1+F(2))}{2}}) \tag{99}$$

and

$$E(|\eta_j(t)|^2) = O(2^{j(3-F(2)-\alpha)}) \ . \tag{100}$$

Without loss of generality we fix $t = 0$. Let us consider $0 < \alpha < 1$ and l and j small enough and such that:

$$l < 2^j < 2^{j\alpha} < T$$

Let us first remark that, for $u < 2^{j\alpha}$, one has from Eq. (94):

$$E\left(e^{p(\omega_{2^{j\alpha}}^T(u) + \omega_{2^{j\alpha}}^T(0))}\right) = 2^{-j\alpha F(2p)} T^{F(2p)} e^{F(2p)(1-u2^{-j\alpha})} \tag{101}$$

where $F(p) = \varphi(-ip) = p - \zeta(p)$. Hence, we have:

$$\begin{aligned} E\left(e^{2\omega^T_{2^{j\alpha}}(u)}\right) &= 2^{-j\alpha F(2)}T^{F(2)}e^{F(2)} \\ E\left(e^{\omega^T_{2^{j\alpha}}(u)+\omega^T_{2^{j\alpha}}(0)}\right) &= 2^{-j\alpha F(2)}T^{F(2)}e^{F(2)(1-u2^{-j\alpha})} . \end{aligned}$$

One defines $\eta_{l,j}$ as:

$$\eta_{l,j}(0) = 2^{-j}\int \psi(u2^{-j})\left(e^{\omega^T_l(u)} - e^{\tilde{\omega}^{2^{j\alpha}}_l(u)+\omega^T_{2^{j\alpha}}(0)}\right)du \tag{102}$$

Using dominated convergence, Eq. (97), $E(e^{\tilde{\omega}^{2^{j\alpha}}_l}) = 1$ and the fact that ψ is a bounded function of support $[0,1]$ one has:

$$\begin{aligned} E(\lim_{l\to 0}|\eta_{l,j}|) &= \lim_{l\to 0} E(|\eta_{l,j}|) \\ &\leq ||\psi||_\infty 2^{-j}\int_0^{2^j}\sqrt{E\left[\left(e^{\omega^T_{2^{j\alpha}}(u)} - e^{\omega^T_{2^{j\alpha}}(0)}\right)^2\right]}du \\ &= ||\psi||_\infty 2^{-j}\int_0^{2^j}\sqrt{E\left(e^{2\omega^T_{2^{j\alpha}}(0)} + e^{2\omega^T_{2^{j\alpha}}(u)} - 2e^{\omega^T_{2^{j\alpha}}(u)+\omega^T_{2^{j\alpha}}(0)}\right)}du \\ &= \sqrt{2}||\psi||_\infty 2^{-\frac{j\alpha F(2)}{2}}T^{\frac{F(2)}{2}}e^{\frac{F(2)}{2}}\int_0^1\left(1 - e^{-F(2)u2^{j(1-\alpha)}}\right)^{\frac{1}{2}}du \\ &\underset{j\to-\infty}{=} O(2^{j\frac{1-\alpha(1+F(2))}{2}}) \end{aligned}$$

which proves (99). In order to bound the second moment, we consider

$$\begin{aligned} & E(\lim_{l\to 0}|\eta_{l,j}|^2) = \lim_{l\to 0}E(|\eta_{l,j}|^2) \\ = \;& 2^{-2j}\iint_0^{2^j}\psi(2^{-j}u)\psi(2^{-j}u')\mathbf{E}(e^{\tilde{\omega}^{2^{j\alpha}}_l(u)+\tilde{\omega}^{2^{j\alpha}}_l(u')})\mathbf{E}\left((e^{\omega^T_{2^{j\alpha}}(u)} - e^{\omega^T_{2^{j\alpha}}(0)})(e^{\omega^T_{2^{j\alpha}}(u')} - e^{\omega^T_{2^{j\alpha}}(0)})\right)dudu' \\ \leq \;& 2^{-2j}2^{-j\alpha F(2)}(Te)^{F(2)}\iint_0^{2^j}|\psi(2^{-j}u)|\,|\psi(2^{-j}u')|\mathbf{E}(e^{\tilde{\omega}^{2^{j\alpha}}_l(u)+\tilde{\omega}^{2^{j\alpha}}_l(u')}) \,. \\ \cdot \;& \left|e^{-F(2)|u-u'|2^{-j\alpha}} + 1 - e^{-F(2)|u|2^{-j\alpha}} - e^{-F(2)|u'|2^{-j\alpha}}\right|dudu' \\ = \;& 2^{-j\alpha F(2)}(Te)^{F(2}\iint_0^1|\psi(u)||\psi(u')|\mathbf{E}(e^{\tilde{\omega}^{2^{j\alpha}}_l(2^ju)+\tilde{\omega}^{2^{j\alpha}}_l(2^ju')}) \,. \\ \cdot \;& \left|e^{-F(2)|u-u'|2^{j(1-\alpha)}} + 1 - e^{-F(2)|u|2^{j(1-\alpha)}} - e^{-F(2)|u'|2^{j(1-\alpha)}}\right|dudu' \\ \underset{j\to-\infty}{=} \;& O(2^{j(1-\alpha(1+F(2)))})\mathbf{E}(|e^{\tilde{\omega}^{2^{j\alpha}}_l(2^ju)}\star|\psi||^2) \\ = \;& O(2^{j(1-\alpha(1+F(2))+\zeta(2)+\alpha F(2)}) = O(2^{j(3-F(2)-\alpha)}) . \end{aligned}$$

$\square$.

Let us remark that one could obtain a smaller error with a smoother variant of the ω_l. Indeed, as shown by (Vargas, Robert, 2008) it is possible to choose the way ω_l is regularized at scale l. One can thus define a MRM process using ω_l with a covariance function that is C^2 at $\tau = 0$. In that case, in Eq. (101), the function $\rho_l(u)$ would be proportional to $2^{-2j\alpha}u^2$ and the error mean absolute value could be bounded by $2^{j(1-\alpha-F(2)/2)}$.

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