Estimation in models driven by fractional Brownian motion

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$(X_s)_{s\in\mathbb{R}^+}$ is a solution of the following equation

$$X(t) = c + \int_0^t \sigma(X(u))db_H(u) + \int_0^t \mu(X(u))du.$$

Where $b_H(u)$ is the fractional Brownian motion with $H > 1/2$.

Our main interest in this work is to provide estimators for the function $\sigma(\cdot)$.
We observe instead of $X(t)$ a mollified version :

$X_\varepsilon(t) = \varphi_\varepsilon * X(t)$ for φ a density with bounded support.

For simple models we can estimated jointly H and σ. In more involve models we can estimate only σ by using a functional method.

We study first the estimation problem for the following four models :

$$X(t) = \sigma\, db_H(t) + \mu\, dt, \tag{1}$$

$$X(t) = \sigma\, db_H(t) + \mu\, X(t)\, dt, \tag{2}$$

$$X(t) = \sigma\, X(t)\, db_H(t) + \mu\, X(t)\, dt, \tag{3}$$

$$X(t) = \sigma\, X(t)\, db_H(t) + \mu\, dt, \tag{4}$$

with $X(0) = c$.

The solution of these equations are respectively :

(1) $X(t) = \sigma\, b_H(t) + \mu\, t + c$,

(2) $X(t) = \sigma\, b_H(t) + \exp(\mu\, t)\,[\sigma\, \mu\, \left(\int_0^t b_H(s)\, \exp(-\mu\, s)\, ds\right) + c]$,

(3) $X(t) = c\, \exp(\mu\, t + \sigma\, b_H(t))$,

(4) $X(t) = \exp(\sigma\, b_H(t))\left(c + \mu\, \int_0^t \exp(-\sigma\, b_H(s))\, ds\right)$.

We consider the problem of estimating simultaneously H and $\sigma > 0$.

Suppose we observe instead of $X(t)$ a smoothed by convolution process $X_\varepsilon(t) = \varphi_\varepsilon * X(t)$, where $\varphi_\varepsilon = 1/\varepsilon\varphi(\cdot/\varepsilon)$, with φ as before and where we have extended $X(\cdot)$ by means of $X(t) = c$, if $t < 0$.

From now on, we shall denote for each $t \geq 0$ and $\varepsilon > 0$,

$$Z_\varepsilon^X(t) := \begin{cases} \dfrac{\varepsilon^{(2-H)}\ddot{X}_\varepsilon(t)}{\sigma_{2H}}, & \text{for the first two models} \\[2ex] \dfrac{\varepsilon^{(2-H)}\ddot{X}_\varepsilon(t)}{\sigma_{2H}X_\varepsilon(t)}, & \text{for the other two.} \end{cases}$$

And let us consider

$$A_k(\varepsilon) := \frac{1}{\sigma^k ||N||_k^k}\left(\int_0^1 \left|Z_\varepsilon^X(u)\right|^k du\right) - 1.$$

and

$$S_g(\varepsilon) := \varepsilon^{-\frac{1}{2}} ds \int\limits_0^1 g(Z_\varepsilon(u))du.$$

It holds, for $k \geq 1$,

$$A_k(\varepsilon)_{\varepsilon \to 0} \to 0.$$

Furthermore

$$\frac{1}{\sqrt{\varepsilon}} A_k(\varepsilon) = S_{g_k}(\varepsilon) + o_{a.s.}(1), \text{ where}$$

$$g_k(x) := \frac{|x|^k}{||N||_k^k} - 1.$$

We can propose estimators of H and σ, by observing $X_\varepsilon(u)$ at several scales of the parameter ε, $h_i = \varepsilon\, c_i$, $c_i > 0$, $i = 1, \cdots, \ell$. In this aim, let us define

$$M_k(\varepsilon) := \begin{cases} \displaystyle\int_0^1 \left|\ddot{X}_\varepsilon(u)\right|^k du, & \text{for the first two models} \\ \displaystyle\int_0^1 \left|\frac{\ddot{X}_\varepsilon(u)}{X_\varepsilon(u)}\right|^k du, & \text{for the other two.} \end{cases}$$

Using assertion 1) above, we get

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{k(2-H)} M_k(\varepsilon)}{\sigma_{2H}^k \sigma^k \,||N||_k^k} du \to 1,$$

from which we obtain

$$\log(M_k(\varepsilon)) = k(H-2)\log(\varepsilon) + \log(\sigma_{2H}^k \sigma^k ||N||_k^k) + o_{a.s.}(1).$$

The following regression model can be written, for each scale h_i :

$$Y_i = a_k X_i + b_k + \xi_i, \ \ i = 1, \cdots, \ell,$$

where $a_k := k(H-2)$, $b_k := \log(\sigma_{2H}^k \sigma^k ||N||_k^k)$ and for $i = 1, \cdots, \ell$, $Y_i := \log(M_k(h_i))$, $X_i := \log(h_i)$. Hence, the least squares estimators $\widehat{H}_k$ of H and $\widehat{B}_k$ of b_k are defined as

$$k(\widehat{H}_k - 2) := \sum_{i=1}^{\ell} z_i \log(M_k(h_i)) \quad \text{and}$$

$$\widehat{B}_k := \frac{1}{\ell} \sum_{i=1}^{\ell} \log(M_k(h_i)) - k(\widehat{H}_k - 2) \frac{1}{\ell} \sum_{i=1}^{\ell} \log(h_i),$$

$$z_i := \frac{y_i}{\sum_{i=1}^{\ell} y_i^2} \text{ and } y_i := \log(c_i) - \frac{1}{\ell}\sum_{i=1}^{\ell}\log(c_i).$$

Then we propose

$$\widehat{\sigma_{2H}^k} := \sigma_{2\widehat{H}_k}^k,$$

as estimator of σ_{2H}^k and

$$\widehat{\sigma^k} := \frac{\exp(\widehat{B}_k)}{\widehat{\sigma_{2H}^k}||N||_k^k},$$

as estimator of σ^k. Finally, we propose $\widehat{\sigma}_k$ as estimator of σ defined by

$$\widehat{\sigma}_k := \left(\widehat{\sigma^k}\right)^{\frac{1}{k}}.$$

Theorem For $k \geq 1$ and $1/2 < H < 1$,

1) $\widehat{H}_k$ is a strongly consistent estimator of H and

$$\frac{1}{\sqrt{\varepsilon}}(\widehat{H}_k - H) \underset{\varepsilon \to 0}{\Rightarrow} \mathcal{N}(0, \sigma^2_{g_k,l}(\mathbf{c}, \sqrt{\mathbf{c}}\,(\mathbf{z}/k))),$$

where g_k is defined by (5) and

$$g_k(x) = \sum_{n=1}^{\infty} \hat{g}_{2n,k} H_{2n}(x) \text{ with } \hat{g}_{2n,k} \quad = \quad \frac{1}{(2n)!} \prod_{i=0}^{n-1} (k - 2i).$$

2) $\widehat{\sigma}_k$ is a weakly consistent estimator of σ and

$$\frac{1}{\sqrt{\varepsilon}\log(\varepsilon)}(\widehat{\sigma}_k - \sigma) \underset{\varepsilon \to 0}{\Rightarrow} \mathcal{N}(0, \sigma^2 \sigma^2_{g_k,l}(\mathbf{c}, \sqrt{\mathbf{c}}\,(\mathbf{z}/k))).$$

$$\frac{1}{\sqrt{\varepsilon}}(\widehat{H}_k - H) \underset{\varepsilon \to 0}{\Rightarrow} \mathcal{N}(0, \sigma^2_{g_k,l}(\mathbf{c}, \sqrt{\mathbf{c}}\,(\mathbf{z}/k))),.$$

Remark : In the proof of the theorem one shows that the asymptotic behavior of $\left(\frac{(\hat{H}_k - H)}{\sqrt{\varepsilon}}, \frac{(\widehat{\sigma}_k - \sigma)}{\sqrt{\varepsilon}\log(\varepsilon)}\right)$ is a degenerated Gaussian law. Hence, we could not provide simultaneous confidence intervals for H and σ.

This is why we shall consider another type of estimation.

We get back to more general models of the form

$$X(t) = c + \int_0^t \sigma(X(u))db_H(u) + \int_0^t \mu(X(u))du.$$

and our goal is to provide functional estimation of $\sigma(\cdot)$. Indeed, in a precedent work we considered the case where $\mu(\cdot) \equiv 0$ and we proved that, if h is a continuous function and $1/2 < H < 3/4$:

$$\lim_{\varepsilon\to 0}\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{(1-H)}}{\sigma_{2H}}\int_0^1 h(X_\varepsilon(u))\left|\dot{X}_\varepsilon(u)\right|du=\int_0^1 h(X(u))\sigma(X(u))du,$$

Moreover a rate of convergence was also established

$$\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{(1-H)}}{\sigma_{2H}}\int_0^1 h(X_\varepsilon(u))\left|\dot{X}_\varepsilon(u)\right|du-\int_0^1 h(X(u))\sigma(X(u))du\right],$$

converges stably towards

$$\sigma_{g_2}\int_0^1 h(X(u))\sigma(X(u))d\widehat{W}(u).$$

Here, $\widehat{W}(\cdot)$ is a standard Brownian motion independent of $b_H(\cdot)$, $g_2(x)=\sqrt{\frac{\pi}{2}}|x|-1.$

Trying weaken the restricted condition $1/2 < H < 3/4$, we use the second derivative $\ddot{X}_\varepsilon(\cdot)$. We get the following two results, if $1/2 < H < 1$, it holds

$$\frac{1}{E[|N|^k|]}\int_0^1 h(X_\varepsilon(u))\left|\frac{\varepsilon^{(2-H)}}{\sigma_{2H}}\ddot{X}_\varepsilon(u)\right|^k du \to \int_0^1 h(X(u))\,[\sigma(X(u))]^k\,du$$

$$\frac{1}{\sqrt{\varepsilon}}[\frac{1}{E[|N|^k]}\int_0^1 h(X_\varepsilon(u))\,|\frac{\varepsilon^{(2-H)}}{\sigma_{2H}}\ddot{X}_\varepsilon(u)|^k du$$

$$-\int_0^1 h(X(u))\,[\sigma(X(u))]^k du]$$

$$\Rightarrow \sigma_{g_k}\int_0^1 h(X(u))\,[\sigma(X(u))]^k\,d\widehat{W}(u).$$

Here, $g_k(x) = \frac{1}{E[|N|^k]}|x|^k - 1$ and $\widehat{W}(\cdot)$ is still a standard Brownian motion independent of $b_H(\cdot)$.

Now we get back to the model with $\mu(\cdot)$ not necessarily identically null and we are going to prove similar results as above for this example.

Let $X(t)$ be the solution of the equation

$$dX(t) = \sigma(X(t))\, db_H(t) + \mu(X(t))\, dt.$$

We denote P the probability measure induced by the fBm over the σ-algebra $\mathcal{G}$. If G is a measurable and bounded real function defined on the space $C([0,1],\mathbb{R})$ of continuous real functions, we have

$$E[G(X)]_P = E[G(K(b_H))\Lambda]_P, \tag{5}$$

where Λ will be defined later on and $K(\cdot)$ is solution of the O.D.E. . To get this equality we apply the Girsanov theorem of Decreusefond-Üstünel. Namely let $Y(t) = K(b_H(t))$ and let define

$\tilde{b}_H(t) := b_H(t) - \int_0^t \frac{\mu(Y(s))}{\sigma(Y(s))} ds$. By using the Itô's formula we get

$$dY(t) = \sigma(Y(t))\, d\tilde{b}_H(t) + \mu(Y(t))\, dt.$$

Furthermore, there exists a probability measure $\tilde{P}$ absolutely continuous w.r.t. P, such that with this probability $\tilde{P}$ the process $\tilde{b}_H(\cdot)$ is a fBm with parameter $0 < H < 1$. Hence, we have

$$E[G(Y)]_{\tilde{P}} = E[G(K(b_H))\Lambda]_P,$$

where Λ is the Radon-Nikodim derivative of $\tilde{P}$ w.r.t P. Since the two processes $X(\cdot)$ and $Y(\cdot)$ have the same distribution over P and $\tilde{P}$ respectively, we get

$$E[G(X)]_P = E[G(Y)]_{\tilde{P}} = E[G(K(b_H))\Lambda]_P,$$

and equality follows.

Let us define the set of trajectories

$$\Delta := \{x \in C([0,1],\mathbb{R}) : \lim_{\varepsilon\to 0} \frac{1}{E[|N|^k]} \int_0^1 h(x_\varepsilon(u)) \left| \frac{\varepsilon^{(2-H)}\ddot{x}_\varepsilon(u)}{\sigma_{2H}} \right|^k du = \int_0^1 h(x(u)) \, [\sigma(x(u))]^k \, du\}.$$

If we choose G as 1_Δ, using the equality one obtains

$$E[1_\Delta(X)]_P = E[1_\Delta(K(b_H))\Lambda]_P = E[\Lambda]_P = 1,$$

where above we have used the convergence prove before and $P(K(b_H) \in \Delta) = 1$, thus our result follows.

To prove the weak convergence we use only that the stable convergence of measure still holds under absolutely continuity change of measures.

If we get back to the four models defined in the beginning the above result provides estimators for σ when H is supposed known. Indeed, if we set for $k \geq 1$,

$$\tilde{\sigma}_k := \frac{||Z_\varepsilon^X(\cdot)||_k}{||N||_k}$$

then we obtain the following theorem.

Theorem : For $k \geq 1$, $\tilde{\sigma}_k$ is a strongly consistent estimator of σ and

$$\frac{1}{\sqrt{\varepsilon}}(\tilde{\sigma}_k - \sigma) \underset{\varepsilon \to 0}{\Rightarrow} \mathcal{N}(0, \frac{\sigma^2}{k^2}\sigma_{g_k}^2),$$

where $g_k = \frac{|x|^k}{||N||^k} - 1.$

Remark 1 : Note that the rate of convergence is $1/\sqrt{\varepsilon}$ instead of $1/(\sqrt{\varepsilon}\log(\varepsilon))$ as before. This is due to the fact that here H is supposed known.

Remark 2 : The variance $\sigma^2_{g_k}/k^2$ is minimal for $k = 2$ and then the best estimator for σ in the sense of minimal variance is obtained for $k = 2$.

Test of hypothesis

Let us consider the three stochastic differential equations, for $t \geq 0$,

$$dX_\varepsilon(t) = \sigma_\varepsilon(X_\varepsilon(t))db_H(t) + \mu(X_\varepsilon(t))dt \text{ with } X_\varepsilon(0) = c,$$

$X_\varepsilon(t) = c$, for $t < 0$ and we consider testing the hypothesis

$$H_0 : \sigma_\varepsilon(x) = \sigma \ (\text{resp. } \sigma_\varepsilon(x) = \sigma\, x),$$

against the sequence of alternatives

$$H_\varepsilon : \sigma_\varepsilon(x) = \sigma_\varepsilon := \sigma + \sqrt{\varepsilon}\,(d + F(\sqrt{\varepsilon}))\ \text{(resp. } \sigma_\varepsilon(x) = \sigma_\varepsilon\, x\text{)},$$

where σ, d are positive constants, $F(\cdot)$ is a positive function such that $F(\sqrt{\varepsilon}) \to 0$ and $\mu(x) = \mu$ or $\mu(x) = \mu\, x$ (resp. $\mu(x) = \mu\, x$). We are interested in observing the following functionals

$$F_\varepsilon := \frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{(2-H)}}{\sigma_{2H}}\int_0^1 \left|\ddot{Y}_\varepsilon(u)\right| du - \sigma\right]$$

(resp.

$$F_\varepsilon := \frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{(2-H)}}{\sigma_{2H}}\int_0^1 \left|\ddot{Y}_\varepsilon(u)\right| du - \sigma\int_0^1 |Y_\varepsilon(u)|\, du\right]).$$

We get if $1/2 < H < 1$, then

$$F_\varepsilon \Rightarrow \sigma_{g_1}\, \sigma\, N \,+\, d$$

(resp. F_ε converges stably towards

$$\sigma_{g_1}\, \sigma \int_0^1 |X(u)|\, d\widehat{W}(u) + d \int_0^1 |X(u)|\, du,)$$

Remark 1 : There is an asymptotic bias d (resp. a random asymptotic bias $d \int_0^1 |X(u)| du$), and the larger the bias the easier it is to discriminate between the two hypotheses.