Revisiting Scaling, Multifractal, and Multiplicative Cascades with the Wavelet Leader Lens

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ABSTRACT

In the recent past years, scaling, random multiplicative cascades, multifractal stochastic processes became common paradigms used to analyse a large variety of different empirical times series characterised by scale invariance phenomena or properties. Scale invariance implies that no characteristic scale can be identified in data or equivalently that all scales are equally important. It also means that all scales are in relation ones with the others, hence the connection to multiplicative cascades, which, by construction, tie together a wide range of scales. Data with scale invariance are also often characterised by a high irregularity of their sample path. This variability is usually accounted for by Multifractal analysis. Hence, in applications, the three notions, scaling, multiplicative cascade and multifractal are often used ones for the others and even confusingly mixed up. These assimilations, that turned out to be fruitful in the early stages of the study of scaling, are now often responsible for misleading analysis and erroneous conclusions. Wavelet coefficients have long been used with relevance to analyse scaling. However, very recently, it has been shown that the analysis of multifractal properties can be significantly improved both conceptually and practically by the use of quantities referred to as *wavelet leaders*. The goals of this article are to introduce the wavelet leader based multifractal analysis, to detail its qualities and to show how it enables an insightful visit of the relationships between scaling, multifractal and multiplicative cascades.

Keywords: Wavelet, Wavelet Leader, Scaling, Multifractal Analysis, Multifractal Formalism, Multiplicative Cascade, Compound Poisson Cascade

1. MOTIVATION: SCALING, MULTIFRACTAL AND CASCADES ?

Scale Invariance. In the last twenty years, scale invariance phenomena have been observed or used as an analysis paradigm in a wide range of different applications and systems with very different nature (see e.g.,[1] for a review of application domains). Scale invariance means that no specific scale of time (or space, or else) that plays a characteristic role can be identified in the data under study; or equivalently, that all scales are equally important. It also implies that the common data analysis procedures based on the search for a characteristic scale are to be replaced by new ones aiming at analysing relationships between scales and mechanisms that relate them. This is why multiplicative cascades, originally introduced in the field of hydrodynamic turbulence by Yaglom, Obukhov and Mandelbrot (cf. e.g.,[2,3]), have been massively used as a relevant model to describe scaling. Scaling are also largely associated with the notion of high irregularity in the time evolution of the data. This strong variability is commonly described via a mathematical theory referred to as the *Multifractal Analysis* (cf. e.g.,[4,5]).

Following those intuitions, the key words *cascade, multifractal* and *scaling* and the corresponding notions are very often heuristically used interchangeably one for the other. Though this association originally led to fruitful intuitions and substantial progresses, it now causes potential confusions and induces misanalysis of empirical data when model identification, model testing or parameter estimation come in order. Let us try to depict scaling, multifractal and multiplicative cascade and their relationships more precisely.

Multiresolution Quantities. Let $\{X(t)\}_{t\in\mathbb{R}}$ denote the sample path of a 1D stochastic process $\{X(t)\}_{t\in\mathbb{R}}$, that we intend to study. Scaling in X are usually studied through multiresolution quantities, $T_X(a,t)$. A multiresolution quantity is loosely defined as the result of the comparison of X against a reference pattern dilated

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by factor $a > 0$ and translated to time position t. A classical example is provided by the increments of X, $T_X(a,t) = X(t+a\tau_0) - X(t)$ (τ_0 being an arbitrary constant). However, it has long been recognised that wavelet transforms constitute ideal tools for the study of scale invariance phenomena (see e.g.,[6–9] for review papers). Therefore, in this text, we concentrate on wavelet coefficients: $T_X(a,t) = \langle 1/a\psi_0((\cdot - t)/a)|X(\cdot)\rangle$ (ψ_0 being an arbitrary mother wavelet).

Scaling. It is commonly said that X posseses scaling properties when the time averages of (some function F) of $|T_X(a,t)|^q$ follow power law behaviours with respect to the scale a

$$\frac{1}{n_a}\sum_{k=1}^{n_a} F((|T_X(a,t_k)|^q) \simeq c_q |a|^{\zeta(q)}, \tag{1}$$

where n_a denotes the number of $T_X(a,t)$ available at scale a. These power laws are supposed to hold in a given range of scales, $a_m \le a \le a_M$, with $a_M/a_m \gg 1$, and for a given range of orders q. The notation $F(x)$ is used loosely to account for any (possibly non linear) transformation operating on the whole set of multiresolution coefficients $\{T_X(a,t),\, a \in \mathbb{R}^{*+}, t \in \mathbb{R}\}$. In applications, the most commonly encountered choice is the simple identity, $F(x) = x$, and the $\zeta(q)$ are usually referred to as the *scaling exponents*.
However, in specific contexts (cf. Section 3.2), the key word *scaling* may also refer to power law behaviours of the ensemble averages (or moments) of order q of $T_X(a,t)$,

$$\mathbb{E}|T_X(a,t_k)|^q = c_q |a|^{\xi(q)}. \tag{2}$$

Again, restrictions on the range of scales a and orders q are in order and under suitable conditions (stationarity in time and ergodicity) the time averages above can be thought of as ensemble averages (see e.g.,[10, 11]).

Multiplicative Cascades. Multiplicative cascades are built from iterative or recursive *split/multiply* procedures that hence produce interdependencies between the different scales of the resulting process. The term *multiplicative* refers to the fact that cascades are defined as the product of independent identically distributed positive random variables W, hence called *multipliers*. If the construction procedure ensures that the number of multipliers involved in the product evolves as the logarithm of the scale, then the resulting process has scaling and multifractal properties. A precise construction of a multiplicative process will be detailed in Section 3.2. Because the multipliers are the central quantities in these constructions, their moments of order q, $\mathbb{E}W^q$, play a key role in the description of multiplicative cascades. More precisely, some function G of $\mathbb{E}W^q$ will be of particular interest:

$$\phi(q) = G(q, \mathbb{E}W^q). \tag{3}$$

The precise form of G depends on the details of the cascade construction (cf. Section 3.2).

Multifractal Spectrum. Processes with scaling are usually characterised by sample paths that present a high variability. This irregular behaviour can fruitfully be analysed in terms of Hölder exponent (see definition 1 below). For a wide variety of processes, usually referred to as multifractal processes, the Hölder exponent is not constant along the sample paths but instead varies widely, apparently erratically, from one time position to another. The variability of this Hölder exponent can be described through the multifractal spectrum, $D(h)$, which consists of the Hausdorff dimension of the set of points with Hölder exponent h. Through the Legendre transform of $D(h)$, one can define another function of q:

$$\eta(q) = \inf_h (1 + qh - D(h)). \tag{4}$$

Relationships ? The separate introductions of the three notions *scaling, multiplicative cascades, Multifractal* naturally lead us to introduce different functions: the scaling exponents $\zeta(q)$ (and/or $\xi(q)$), the function of the multipliers $\phi(q)$, the multifractal spectrum $D(h)$ (or $\eta(q)$), respectively. These different functions of q are very often mixed up. The reasons are both practical and theoretical. Practically, these functions are all considered as *the same image* of the scale invariance property which is tracked in the analysed time series. Theoretically, standard multifractal formalisms proposed in the literature relate for instance $\eta(q)$ (or $D(h)$) to $\zeta(q)$. However, the major source of confusion lies in the fact that multiplicative cascades are also deeply tied to multifractal and scaling. Indeed, their sample paths are truly multifractal (in the sense that they are characterised by a non trivial multifractal spectrum), hence a theoretical connection between $\phi(q)$ and $\eta(q)$. They also exhibit scaling as in Eq. (1) (with non trivial $\zeta(q)$ functions, i.e., with functions that depart from a linear behaviour in q), hence a theoretical relation between $\phi(q)$ and $\zeta(q)$. For recently proposed multiplicative cascades, such as compound Poisson cascades, one can even obtain scaling properties as in Eq. (2), hence a relation between $\phi(q)$ and $\xi(q)$. For a very long time, multiplicative cascades remained the only theoretically controlled stochastic processes that could be used to model scaling and multifractal properties in empirical time series. Though new families of multifractal processes, such as random wavelet series were recently defined,[12, 13] multiplicative cascades remain the only versatile models used in applications so far. This prominent role led to significant successes at the early stages of the analysis of scaling. However, it is now responsible for confusing assimilation of the notions of scaling, multifractal and cascades and hence of the functions: $D(h)$, $\zeta(q)$, $\phi(q)$. Those incorrect associations prevent the development of more precise and refined analysis procedures that turn out to be necessary for model identification, parameter estimation or meaningful physical analysis.

Goals of this article. The use of wavelet enabled significant successes in the study of scaling. However, very recently, it has been shown that the analysis of multifractal properties can be significantly improved both conceptually and practically with the use of a quantity referred to as *wavelet leaders*.[14]
Therefore, the goals of the present article are twofold. First, it proposes to review the recent theoretical results related to the wavelet leader based multifractal formalism, to illustrate how and why it improves previously formulated multifractal formalisms. Second, it intends to show that this wavelet leader lens enables a meaningful, insightful and fruitful revisiting of the relationships between cascades, scaling and multifractal and to detail and clarify the correspondences and differences between the functions η, ζ, ξ, ϕ defined above.

Outline. Section 2 defines wavelet coefficients and wavelet leaders, it introduces multifractal analysis with the corresponding wavelet leader based multifractal formalism. It discusses its relation to previous formalisms. Section 3 makes use of the rich example of compound Poisson cascades to detail the clarifications, brought by the wavelet leader prism, to the relationships between Multiplicative Cascades, Scaling and Multifractal.

2. WAVELET BASED MULTIFRACTAL ANALYSIS

2.1. Wavelet Leaders

Wavelet Coefficients. Let $\psi_0(t)$ denote a reference pattern called the *mother-wavelet*. It is usually requested that ψ_0 possesses jointly strongly concentrated time and frequency supports: It therefore acts as an elementary atom of information. The reconstruction property implies that ψ_0 satisfy the so-called *admissibility condition*:

$$\int_{\mathbb{R}} \psi_0(t)dt \equiv 0. \tag{5}$$

The mother wavelet is also characterised by a strictly positive integer $N \geq 1$, called its *number of vanishing moments*, defined as:

$$\forall k = 0, 1, \dots, N-1, \quad \int_{\mathbb{R}} t^k \psi_0(t)dt \equiv 0, \quad \int_{\mathbb{R}} t^N \psi_0(t)dt \neq 0. \tag{6}$$

Let $\{\psi_{j,k}(t) = 2^{-j}\psi_0(2^{-j}t - k), j \in \mathbb{N}, k \in \mathbb{N}\}$ denote templates of ψ_0 dilated to scales 2^j and translated to time positions $2^j k$. Let $\{X(t), t \in \mathbb{R}\}$ denote the continuous time time series (or function) to be analysed. The discrete wavelet transform of X is defined through the computation of its coefficients $d_X(j,k)$ as:

$$d_X(j,k) = \int_{\mathbb{R}} X(t)\, 2^{-j}\, \psi_0(2^{-j}t - k)\, dt. \tag{7}$$

For a detailed introduction to wavelet decompositions, the reader is referred to e.g.,.[6]

Wavelet Leaders. In the remainder of this text, we further assume that the mother wavelet ψ_0 is chosen with compact time support and arbitrarily large regularity and that the collection $\{2^{-j/2}\psi_0(2^{-j}t-k), j \in \mathbb{N}, k \in \mathbb{N}\}$ forms an orthonormal basis of $\mathrm{L}^2(\mathbb{R})$. Therefore, any L^2 function X can be written as:

$$X(t) = \sum_{j,k \in \mathbb{Z}} d_X(j,k)\ \psi_0(2^{-j}t-k). \tag{8}$$

Moreover, let us define an alternative indexing for the dyadic intervals:

$$\lambda \equiv \lambda_{j,k} \ = \left[k2^j, (k+1)2^j\right), \tag{9}$$

so that d_λ and ψ_λ stand for $d_X(j,k)$ and $2^{-j}\psi_0(2^{-j}t-k)$, respectively. The wavelet ψ_λ is essentially localized near the interval λ, more precisely, when the wavelets are compactly supported $\exists C>0$ such that $\forall \lambda$, *supp* $(\psi_\lambda) \subset C\lambda$ (where $C\lambda$ denotes the interval homothetic to λ, centered at the same location and C times larger). Finally, let 3λ denote the union of the interval λ and its 2 adjacent dyadic intervals, two dyadic intervals are called adjacent if they share the same size and are next to each other: $3\lambda \equiv 3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$.

If $X \in L^\infty$, one has

$$|d_\lambda| \leq \int |X(t))||\psi_{j,k}(t)|dt \leq C \parallel \psi_0 \parallel_{L^1} \parallel X \parallel_{L^\infty},$$

and the quantities,

$$L_X(j,k) \equiv L_\lambda = \sup_{\lambda' \subset 3\lambda} |d_{\lambda'}| \tag{10}$$

are thus finite and referred to as a *wavelet leaders.*[14] Indeed, $L_X(j,k)$ consists of the (non linear) replacement of the usual wavelet coefficient $d_X(j,k)$ with a *neighbour maximum* chosen in a narrow time neighbourhood ($\lambda' \subset 3\lambda$), along all finer scales ($2^{j'} \leq 2^j$). Furthermore, we denote by $\lambda_{j,k}(t_0)$ the dyadic interval of size 2^j containing t_0 and

$$L_j(t_0) = \sup_{\lambda' \subset 3\lambda_{j,k}(t_0)} |d_{\lambda'}|.$$

Note that the supremum is taken not only on $\lambda_{j,k}(t_0)$ but also on the 2 adjacent dyadic intervals.

2.2. Multifractal Analysis

The following definitions gather the key-notions attached to multifractal analysis of functions. For thorough and complete overviews, the reader is referred to e.g.,.[4, 5, 14] Then, the wavelet leader based multifractal formalism is introduced and studied.

Hölder Exponent. Let $\{X(t)\}_{t\in\mathbb{R}}$ denote the sample path of the function or stochastic process of interest. Its local regularity is commonly studied via the notion of pointwise Hölder exponent.

Definition 1. *Let $t_0 \in \mathbb{R}$ and let $\alpha \geq 0$. A locally bounded function $X : \mathbb{R} \to \mathbb{R}$ belongs to $C^\alpha(t_0)$ if there exists a constant $C > 0$ and a polynomial P satisfying deg$(P) < \alpha$ and such that, in a neighbourhood of t_0,*

$$|X(t) - P(t-t_0)| \leq C|t-t_0|^\alpha. \tag{11}$$

The Hölder exponent of X at t_0 is

$$h(t_0) = \sup\{\alpha : X \in C^\alpha(t_0)\}.$$

If $h(t_0) < 1$, then the polynomial $P(t-t_0)$ simplifies to $X(t_0)$. A famous though simple example is given by the function $X(t) = A + B|t-t_0|^h$, whose Hölder exponent (sometimes also called the *singularity strength*) at t_0 is simply h (when h is not a even integer) ; $A + B|t-t_0|^h$ is commonly referred to as a *cusp-type* singularity.

Wavelet Leaders and Hölder Exponent. The wavelet characterization of the Hölder exponent requires a regularity hypothesis which is slightly stronger than continuity: X is said to be uniform Hölder if $\exists \epsilon > 0$ such that $X \in C^{\epsilon}(\mathbb{R})$, i.e.

$$\exists C > 0 \quad \text{such that} \quad \forall t, s \in \mathbb{R}, \ \ |X(t) - X(s)| \leq C|t-s|^{\epsilon}.$$

The following theorem is a restatement of a result in[15] and allows to characterise the pointwise regularity by a decay condition of the $\mathrm{L}_j(t_0)$ when $j \to +\infty$.

THEOREM 1. *Let $h > 0$. Let ψ_0 denote mother wavelet with compact time support and such that $N > h$. If X is $C^h(t_0)$, then there exists $C > 0$ such that*

$$\forall j \geq 0, \qquad L_j(t_0) \leq C 2^{jh}. \tag{12}$$

Conversely, if (12) holds and if X is uniform Hölder, then $\exists C > 0$ and a polynomial P satisfying deg$(P) < h$ such that, in a neighbourhood of t_0,

$$|X(t) - P(t-t_0)| \leq C|t-t_0|^h \log(1/|t-t_0|).$$

Multifractal (or Singularity) Spectrum. The fluctuations, or the irregularity, of the Hölder exponent $h(t)$ along along the path $\{X(t)\}_{t \in \mathbb{R}}$ are usually described through the so-called *multifractal (or singularity) spectrum*, hereafter labelled $D(h)$.

DEFINITION 2. *We denote by E_h the set of points where the Hölder exponent takes the value h. The spectrum of singularities $D(h)$ of X consists of the Hausdorff dimension of E_h. (By convention, dim$(\emptyset) = -\infty$.)*

For the definition of the Hausdorff dimension the reader is referred to e.g.,.[4, 5] By definition of the Hausdorff dimension, the multifractal spectrum takes values in $\{-\infty\} \cup [0, 1]$. Furthermore, we will assume without loss of generality that $D(h)$ differs from $-\infty$ in a finite range of Hölder exponents:

$$D(h) \neq -\infty, \ \ h \in [h_*^-, h_*^+]. \tag{13}$$

In applications, $D(h)$ has often been used as a tool to analyse/classify empirical time series (to detect pathologies in medecine, to model information fluxed in Internet traffic,[7] ...). Roughly speaking, it pictures the roughness or irregularity along time of the analysed time series. For instance, the range $[h_*^-, h_*^+]$ of existing singularity strengths is often used to classify data.

Multifractal Formalism. The determination of the singularity spectrum from empirical data is crucial for applications. A numerical computation straight from the definition is obviously not feasible: For an interesting multifractal process, the Hölder exponent will vary widely from point to point making its numerical measurement extremely unstable and actual empirical data come with practical limitations such as discrete time sampling and finite resolution. The way out consists in obtaining the desired multifractal spectrum via auxiliary functions (called the *structure functions*) that can more easily be computed. This procedure is referred to as the *multifractal formalism*, after the analogy with thermodynamic formalism first introduced in a seminal work in hydrodynamic turbulence.[16] The original proposition was based on continuous sums of the L^q-norm of increments $X(t + a\tau_0) - X(t)$ of X. The remainder of this section aims at showing the theoretical, practical and pedagogical benefits gained in replacing increments with wavelet leaders.

Wavelet Leader Multifractal Formalism. Let $S_L(q, j)$ denote the wavelet leader based structure functions:

$$S_L(q, j) = \frac{1}{n_j} \sum_{k=1}^{n_j} |L_X(j, k)|^q, \tag{14}$$

where n_j is the number of available $L_X(j, k)$ at octave j. Roughly (i.e., up to border effects), $n_j \simeq n_0 2^{-j}$. In the setting of Section 1, it corresponds to Eq. (1) with the non standard choice $F(d_X(j, k)) = L_X(j, k)$.

Theorem 1 can loosely been interpreted as stating that, if the Hölder exponent of X at t_0 is h, then the wavelet leader $L_X(j,t_0)$ of X corresponding to $\lambda = \lambda_j(t_0)$ will have size $|L_j(t_0)| \sim 2^{jh}$. Hence, the points with Hölder exponent h brings a contribution $\sim 2^j 2^{jqh} 2^{-jD(h)}$ to $S_L(q,j)$. Therefore, $S_L(q,j)$ will behave as $\sim c_q(2^j)^{\zeta_L(q)}$ and a standard steepest descent argument yields a *Legendre transform* relationship between the multifractal spectrum $D(h)$ and the scaling exponents $\zeta_L(q) = \inf_{h\in[h_*^-,h_*^+]}(1+qh-D(h))$:

$$S_L(q,j) \sim_{2^j\to 0} c_q(2^j)^{\zeta_L(q)}.$$

This leads to a Wavelet Leader based Multifractal Formalism:

$$\zeta_L(q) = \liminf_{j\to 0}\left(\frac{\log_2 S_L(q,j)}{j}\right), \qquad (15)$$

$$D(h) = \inf_{q\neq 0}\left(1+qh-\zeta_L(q)\right). \qquad (16)$$

It is actually mathematically proven that $\inf_{q\neq 0}(1+qh-\zeta_L(q))$ acts as a sharp upper bound for $D(h)$ for all functions or processes (on condition that they satisfy the mild uniform Hölder regularity condition)[14]:

$$D(h) \leq \inf_{q\neq 0}\left(1+qh-\zeta_L(q)\right). \qquad (17)$$

2.3. Wavelet coefficients vs wavelet leaders

In most introductions dedicated to the wavelet based multifractal analysis, this latest is based on wavelet coefficients $d_X(j,k)$ instead of wavelet leaders $L_X(j,k)$. In this section, we put the emphasis on the improvement brought by the wavelet leader approach.

Wavelet Coefficient Multifractal Formalism. Wavelet coefficient based structure functions and scaling exponents are defined as:

$$S_d(q,j) = \frac{1}{n_j}\sum_{k=1}^{n_j}|d_X(j,k)|^q. \qquad (18)$$

In the setting of Section 1, it corresponds to Eq. (1) with the common choice $F(x) = x$. Arguments similar to those in previous section yield that $S_d(q,j)$ behave as power laws of the scale in the limit of small scales:

$$S_d(q,j) \sim_{2^j\to 0} c_q(2^j)^{\zeta_d(q)}. \qquad (19)$$

The Wavelet Coefficient based Multifractal Formalism is standardly stated as:

$$\zeta_d(q) = \liminf_{j\to 0}\left(\frac{\log_2 S_d(q,j)}{j}\right), \qquad (20)$$

$$D(h) = \inf_{q\geq q_o}\left(1+qh-\zeta_d(q)\right), \qquad (21)$$

where q_o is a positive value of q such that $\zeta_d(q) = 1$ (cf.[5, 17]). This wavelet coefficient based multifractal formalism suffers from two major drawbacks discussed below: it fails to operate correctly for negative qs and for processes that contain oscillating singularities.

Negative qs. For most functions or processes of interest, by definition or nature of the wavelet transforms, a significant number of wavelet coefficients $d_X(j,k)$ will have *close to* 0 values. This implies that the computation of $S_d(q,j)$ for negative qs will be numerically unstable. In a stochastic framework, it can be rephrased into the fact that the wavelet coefficients are random variables with a strictly positive probability density function at the origin and hence infinite moments of order $q < -1$. In both cases, it implies practically that wavelet coefficient based structure functions with $q < -1$ cannot, must not and should not be used to infer the multifractal properties of X. A striking example is the one provided by fractional Brownian motion (fBm)[18, 19] with self-similarity parameter

H. The Wavelet Coefficient based Multifractal Formalism would yield the following uncorrect determination of the multifractal spectrum:

$$\begin{aligned} D(h) &= 1 - h + H \quad && \text{if } h \in [H, H+1], \\ &= -\infty && \text{else;} \end{aligned}$$

whereas the Wavelet Leader based Multifractal Formalism yields the correct one

$$\begin{aligned} D(h) &= 1 \quad && \text{if } h = H, \\ &= -\infty && \text{else.} \end{aligned}$$

Generally speaking, the Wavelet Coefficient based Multifractal Formalism will miss the part of $D(h)$ that describes the singularities with the highest values of h. For bell-shaped $D(h)$, it *will not see* the singularities with $h \geq h_*^*$, where h_*^* is such that $D(h_*^*)$ is the maximum of D.

Oscillating or chirp-type singularities. For cusp-type singularities $A + B|t - t_0|^h$, it has been shown that the argument $|d_X(j, t_0)| \sim 2^{jh}$ holds for wavelet coefficients. This is the crucial step that led to the formulation of the Wavelet Coefficient based Multifractal Formalism. However, there are known counterexamples, the most famous ones being *chirp*-type functions (or *oscillating* singularities)[14] of the form:

$$X(t) = |t - t_0|^h \sin\left(1/|t - t_0|^\beta\right), \text{ with } h, \beta > 0. \tag{22}$$

For such functions, if $\lambda = \lambda_{j,t_0}$ then $d_\lambda = o(2^{j\gamma})$ $\forall \gamma > 0$ so that the standard statement of the multifractal formalism will fail to yield a correct analysis of the multifractal properties of X. There exists a bound:

$$D(h) \leq \inf_{q \geq q_o} \left(1 + qh - \zeta_d(q)\right),$$

similar to that obtained using $\zeta_L(q)$ (cf. Eq. (17)), but it is far from being as sharp, see.[14] Thus a safer way to derive the multifractal formalism is to base the structure function on the wavelet leaders instead of the wavelet coefficients. Furthermore, discrepancies potentially observed for $q > 0$ between empirically observed $\zeta_L(q)$ and $\zeta_d(q)$ might provide us with an interesting tool to detect the existence of oscillating singularities in empirical data. Whether such singularities exist in hydrodynamic turbulence is an open issue,[2, 16] they are also expected to exist in gravitational waves.[20] Up to our knowledge, so far, no oscillating singularity has ever been evidenced in actual empirical data, the design of relevant detection tools constitutes hence a major challenge.

Conclusions. The Wavelet Leader Multifractal Formalism overcomes the two drawbacks described above: It holds both for positive and negative qs and whether the function or process under study embodies chirp-type singularities or not. Therefore, it provide us with the multifractal spectrum of the studied process over its whole range and must certainly be preferred theoretically and conceptually to the Wavelet Leader Multifractal Formalism. However, note that, from a more practical point of view, numerically computing $\zeta_L(q)$s requires the knowledge of wavelet coefficients on a deeper range of scales than that necessary to get $\zeta_d(q)$s: indeed, in order to be meaningful, the computation of L_λ at a given scale requires that of the wavelet coefficients d_λ over several scales below.

2.4. Further comments

A number of complementary comments and comparisons are in order.

Higher dimensions. For sake of simplicity, the presentation was proposed here for $1D$ processes or functions. However, the wavelet leader multifractal formalism can be straightforwardly extended to arbitrary higher dimensions $d \geq 1$, $\{X(t)\}_{t \in \mathbb{R}^d}$, simply by adapting the definitions of the wavelet coefficients and leaders as well as that of the Legendre transform, $\inf_{q \neq 0}(d + hq - \zeta_L(q))$.

Computational costs. On the computational side, the leader approach, based on a decomposition on an orthogonal wavelet basis can be implemented using the fast pyramidal algorithm underlying the Discrete Wavelet Transform and has thus a very low computational cost.

Modulus Maxima of the Wavelet Transform. The negative qs issue had already been addressed and solved by a multifractal formalism based on the Modulus Maxima of the Wavelet Transform initially introduced by S. Mallat[6] and developed in the context of multifractal analysis by Arneodo et al..[8,9] Let $\{T_X(a,t)\ a > 0,\ t \in \mathbb{R}\}$ denote the continuous wavelet transform:

$$T_X(a,t) = \frac{1}{a}\int X(u)\psi\left(\frac{t-u}{a}\right) du.$$

The Modulus Maxima Wavelet Transform consists first in extracting for each given scale a, the local maxima along time t of the functions $t \to |T_X(a,t)|$, second in chaining those maxima along scales at given time positions, third in selecting only the largest coefficients on a maxima line towards finer scales. Structure functions are then based on this skeleton. In the setting of Eq. (1), it corresponds to another non linear choice for F.
The wavelet leader approach is highly reminiscent of the Modulus Maxima Wavelet Transform technique because the leaders also consists of the largest coefficients along scales at each time position while avoiding the (painfull) maxima tracking and chaining operations. The analogy between *leaders* and *modulus maxima* also indicates that in this latter technique the largest coefficient selection along scale is probably far more important than the time maxima tracking phase.
The Modulus Maxima Wavelet Transform has also been shown to work on examples containing oscillating singularities however no general result is so far available.[9] Indeed, in this technique, the spacing between local maxima need not be of the order of magnitude of the scale a or even be regularly spaced. Therefore, the *modulus maxima* scaling exponents may differ from the *leader* ones (see[5,14] where counterexamples are constructed). It follows that no mathematical result such as the one in Eq. (16) has so far been proven.
From a practical point of view, the modulus maxima technique involves a Continuous Wavelet Transform plus maxima tracking and chaining operations. This results in a high computational cost whereas the leader approach benefits from a significantly much lower one. Furthermore, as already mentioned, the wavelet leader approach can be easily theoretically and practically generalised to higher dimensions. This is far less the case for the MMWT method, indeed, it requires a d-dimensional Continuous Wavelet Transform plus a dD extension of the notion of modulus maxima.[21]

3. BACK TO SCALING, MULTIFRACTAL AND CASCADES

The aim of this section is to revisit the relationships between multifractal, scaling and multiplicative cascades with the wavelet Leader approach as an enlightening guide. We will see how it clarifies potential confusions between the various functions of q encountered so far: $\eta(q), \zeta_L(q), \zeta_d(q), \xi(q), \phi(q)$.

3.1. Multifractal vs Scaling

Let us start by examining the connections between the multifractal spectrum and the scaling exponents.

Legendre Transform of the Multifractal Spectrum: $D(h)$ vs $\eta(q)$. The Multifractal spectrum is defined as a Hausdorff dimension, it is hence a positive function, except where $D = -\infty$, that commonly lives on a finite support $h \in [h_*^-, h_*^+]$. Let us define

$$D_*^- = D(h_*^-),\ \ D_*^+ = D(h_*^+). \tag{23}$$

Because, η is defined through a Legendre transform,

$$\eta(q) = \inf_{h\in[h_*^-,h_*^+]} (1 + qh - D(h)), \tag{24}$$

it implies that:

$$\begin{aligned} q > q_*^+, \quad \eta(q) &= d - D_*^- + qh_*^-, &(25)\\ q < q_*^-, \quad \eta(q) &= d - D_*^+ + qh_*^+, &(26)\end{aligned}$$

where q_*^-, q_*^+ are defined respectively as:

$$q_*^+ = \left(\frac{dD}{dh}\right)_{h=h_*^-}, \tag{27}$$

$$q_*^- = \left(\frac{dD}{dh}\right)_{h=h_*^+}. \tag{28}$$

Note that q_*^-, q_*^+ are possibly infinite. In words, the above set of equations means that, by definition, the function $\eta(q)$ necessarily consists of straight lines for large and small qs.
Finally, let us remark that the Legendre transform of $\eta(q)$ will yield the convex hull of $D(h)$ and hence the inequality:

$$D(h) \leq \inf_{q \neq 0}(d + qh - \eta(q)). \tag{29}$$

Legendre Transform of the Multifractal Spectrum vs Scaling Exponents: $\eta(q)$ vs $\zeta_L(q)$. The wavelet Leader multifractal formalism developed above suggests that relevant structure functions, as defined in Eq. (14), are to be based on wavelet leaders. In the notations of Section 1, it means that the function F is chosen such that: $L_X(j,k) = F(d_X(j,k))$. With this proper choice, for all uniform Hölder functions or stochastic processes, we have the very general inequality:

$$D(h) \leq \inf_{q \neq 0}(d + qh - \zeta_L(q)). \tag{30}$$

Furthermore, on condition that the function or process enters the large class of *self-similar functions* defined and studied thoroughly in,[17] this inequality is turned into an identity:

$$\zeta_L(q) \equiv \eta(q), \; q \in \mathbb{R}. \tag{31}$$

Let us put the emphasis on the fact that the above equality is valid for most stochastic multifractal processes studied in the literature and used in applications (multiplicative martingales, random wavelet series, fractional Brownian motion, Lévy processes,[22] ...) because their sample paths fall into this broad class of functions. This means that their functions η and ζ_L need not be distinguished and those two different notations were introduced here mainly for pedagogical purposes and to highlight both the multifractal (cf. Eq. (4)) and the scaling (cf. Eq. (1)) starting points.

Scaling Exponents: $\zeta_L(q)$. Rephrased with different words, this implies that the wavelet leader based structure functions behave in general as power laws of the scale according to:

$$S_L(q,j) \sim_{2^j \to 0} c_q |2^j|^{\zeta_L(q)} = c_q |2^j|^{\eta(q)} = c_q |2^j|^{\inf_{h \in [h_*^-, h_*^+]}(d+qh-D(h))}. \tag{32}$$

This put the emphasis on a major practical implication largely overlooked in the literature and in applications. The scaling exponents, whose measurement is often the goal of many estimation/analysis practical procedures, necessarily behave as linear functions of q when q is outside the interval $q \in [q_*^-, q_*^+]$, as stated in Eq. (25) and Eq. (26) above. Note moreover that the definition of this interval (cf. Eq. (27) and Eq. (28) above) depends only on the theoretical multifractal properties of the process under analysis and not on practical estimation limitations (finite size, finite observation duration, finite resolution,...). Those issues were further studied in.[23, 24] From an application point of view, it means that it is pointless and even dangerous to estimate scaling exponents for $q \notin [q_*^-, q_*^+]$. Particularly, the discrimination between various multifractal models in competition for the description of data must not be based on $q \notin [q_*^-, q_*^+]$.

The cusp-type specific case: $\eta(q)$ vs $\zeta_d(q)$. For this paragraph, we restrict the discussion to the specific class of processes that contain only cusp-type singularities (they do not present any chirp-type singularity). For such a subclass, let us consider the wavelet coefficient multifractal formalism and the corresponding $\zeta_d(q)$ scaling exponents. It is known that the wavelet coefficient structure functions also behave as power laws of the scale, as in Eq. (19), with scaling exponents $\zeta_d(q)$. It has been shown, for this specific case (cf. e.g.,[4,5]), that we have the following identity:

$$\zeta_d(q) = \zeta_L(q) \equiv \eta(q), \;\; q \in [0, +\infty). \tag{33}$$

In other words, for cusp-type processes, $\zeta_d(q)$ and $\eta(q)$ coincide essentially for $q > 0$ but significantly differ otherwise, $\zeta_d(q)$ with $q \leq 0$ must not be used to analyse the multifractal properties of X.

3.2. Multiplicative Cascades: the pedagogical example of compound Poisson cascades

Now, we want to further investigate the relationships between multiplicative cascades, scaling and multifractal. The most famous multiplicative cascades are the ones originally introduced in the field of hydrodynamic turbulence (see for instance, the seminal works in[2,3]). Most of the multiplicative cascades described in the literature fall into the general class of multiplicative martingales[25] and therefore share a large set of their deepest properties. Rather than the celebrated Mandelbrot canonical cascades, we will use here, for the purpose of our argumentation, the pedagogical example of compound Poisson cascades very recently introduced and studied in[26] ; a major quality of these cascades being that their scaling and multifractal properties are theoretically known.

Compound Poisson Cascades. The definition of compound Poisson cascades requires the combination of the following ingredients. Let I denote the upper half-strip $\mathbb{R}\times]0,1]$. Let $(t_i, r_i)_{i\in I}$ stand for a 2D Poisson point process with *control measure* $dm(r,t)$. Let $\mathcal{C}_r(t)$ denote a subpart of I and referred to as the *cone of influence*. Let $\{W_i\}_{i\in I}$ be independent identically distributed positive random variables, independent of the point process $(t_i, r_i)_{i\in I}$. They are usually called the *multipliers* of the cascade and the set $(t_i, r_i, W_i)_{i\in I}$ defines a marked (or compound) Poisson process. The Compound Poisson Cascade $Q_r(t)$ is then defined at time t as the product of all the multipliers W_i associated to points (t_i, r_i) that fall in the cone of influence $\mathcal{C}_r(t)$ located at time t:

$$Q_r(t) = \exp\left[(1 - \mathbb{E}W) m(\mathcal{C}_r(t))\right] \prod_{i,/(t_i,r_i)\in\mathcal{C}_r(t)} W_i, \tag{34}$$

with

$$m(\mathcal{C}_r(t)) = \int_{\mathcal{C}_r(t)} dm. \tag{35}$$

By construction, $\{Q_r(t), t \in \mathbb{R}\}$ is a positive process and $\mathbb{E}Q_r(t) \equiv 1, \forall r \,/\, 1 > r > 0, \forall t \in \mathbb{R}$. Furthermore, the choice of a time-shift invariant control measure:

$$dm(r,t) = g(r) dr dt, \tag{36}$$

combined to the fact that the $\{W_i, i \in I\}$ are independent identically distributed, ensures that $\{Q_r(t), t \in \mathbb{R}\}$ is a stationary process. Moreover, the combination of the specific choices,

$$dm(r,t) = \frac{c}{r^2} dr dt, \;\; c > 0, \tag{37}$$

together with

$$\mathcal{C}_r(t) = \left\{ (t', r') \,:\, r' > r \text{ and } t' - \frac{r'}{2} < t < t' + \frac{r'}{2} \right\}, \tag{38}$$

ensures that $\{Q_r(t), t \in \mathbb{R}\}$ has scaling and multifractal properties, all controlled via the key relation*:

$$\mathbb{E}Q_r(t)^q = r^{\varphi(q)}, q \in \mathbb{R}, \tag{39}$$

where we assumed, for sake of simplicity that, $\forall q \in \mathbb{R}, \mathbb{E}W^q < \infty$ and where

$$\varphi(q) = (\mathbb{E}W^q - 1) - q(\mathbb{E}W - 1). \tag{40}$$

*Note that in itself, this relation is not the signature of a scaling property since it does not involve a dependence with an analysing scale but with the construction resolution r.

A further simple, yet key, property of Q_r is that if forms a multiplicative martingale.[25] For detailed analysis of the properties of the compound Poisson cascades, the reader is referred to.[10, 26]

A variety of interesting multifractal processes can be defined from Q_r, for a review, see e.g.,.[24] For sake of simplicity and pedagogy, we will concentrate here on the simplest one, the compound Poisson motion, labelled here X (for the coherence of notation with the remainder of the paper) and defined as:

$$X(t) = \lim_{r \to 0} \int_0^t Q_r(u) du. \tag{41}$$

The properties of $\{Q_r(t), t \in \mathbb{R}\}$ imprint those of $\{X(t), t \in \mathbb{R}\}$. Therefore, this is a process with stationary increments (i.e., the statistical properties of $\{X(t + a\tau_0) - X(t), t \in \mathbb{R}\}$, $a > 0, \tau_0 > 0$ do not depend on t). It exhibits scaling and multifractal properties. For the remainder of this section, it is assumed that X is a compound Poisson motion as defined above and for ease of notation, let us define

$$\begin{aligned} \phi(q) &= q + \varphi(q), & (42) \\ f(h) &= \mathrm{Inf}_{q \neq 0}(1 + qh - \phi(q)). & (43) \end{aligned}$$

Compound Poisson cascades vs Multifractal spectrum. It has been proven that the process $\{X(t), t \in \mathbb{R}\}$ defined above has multifractal sample paths, with multifractal spectrum[26]:

$$\left.\begin{aligned} D(h) &= f(h), \quad \text{if } f(h) \geq 0, \\ &= -\infty, \quad \text{else.} \end{aligned}\right\} \tag{44}$$

Since by definition $\phi(q)$ is a convex function, it yields immediately that $\phi(q)$ and $\eta(q)$ coincide when $q \in [q_*^-, q_*^+]$ but differ outside this interval:

$$\left.\begin{aligned} \phi(q) &= \eta(q) \quad q \in [q_*^-, q_*^+], \\ \phi(q) &\neq \eta(q) \quad q \notin [q_*^-, q_*^+]. \end{aligned}\right\} \tag{45}$$

Compound Poisson Cascades vs Scaling (1/3). The argument developed in Section 3.1 and Eq. (31) immediately imply that:

$$\left.\begin{aligned} \phi(q) &\equiv \eta(q) \equiv \zeta_L(q), \quad q \in [q_*^-, q_*^+], \\ \phi(q) &\neq \eta(q) \equiv \zeta_L(q), \quad q \notin [q_*^-, q_*^+]. \end{aligned}\right\} \tag{46}$$

Compound Poisson Cascades vs Scaling (2/3). Compound Poisson cascades, in particular, and multiplicative martingales in general, belong to a specific subclass of stochastic processes characterised by the fact that they contain no chirp-type singularities. As detailled in Section 3.1 above, this implies that:

$$\phi(q) = \zeta_d(q) = \eta(q) \equiv \zeta_L(q), \; q \in [0, q_*^+]. \tag{47}$$

Compound Poisson Cascades vs Scaling (3/3). The example of the compound Poisson cascades enables us to address a third issue related to scaling and usually totally overlooked. Because of the multiplicative martingale nature of Q_r, it can be shown that the (positive) moments of order q of X are infinite for $q > q_c^+$:

$$\begin{aligned} \mathbb{E}|X(t + a\tau_0) - X(t)|^q &= +\infty, \; q > q_c^+, & (48) \\ q_c^+ &= \sup\{q \geq 1 : \phi(q) \geq 1\}. & (49) \end{aligned}$$

From the definitions of q_c^+ and q_*^+, it is straightforward to show that:

$$1 \leq q_*^+ < q_c^+. \tag{50}$$

Detailed studies[10] of the compound Poisson cascades also showed with the specific choices of time shift-invariant power law control measure $dm(r,t) = c/r^2 drdt$ together with the triangle-shaped cone of influence, the following result holds[†]:

$$\mathbb{E}|X(t+a\tau_0) - X(t)|^q \simeq c_q |a|^{\phi(q)}, \ 0 \leq a\tau_0 \leq 1, \ q \in [0, q_c^+]. \tag{51}$$

This is a scaling relation that can be immediately transferred, together with stationarity, to the wavelet coefficients as,

$$\mathbb{E}|d_X(j,k)|^q \simeq c_q |2^j|^{\phi(q)}, \ 0 \leq 2^j \leq 1, \ q \in [0, q_c^+], \tag{52}$$

and can be compared to Eq. (2) and the heuristic approach developed in Section 1. The corresponding scaling exponents read:

$$\xi(q) = \phi(q), \ q \in [0, q_c^+]. \tag{53}$$

Now, because the process X has stationary increments and wavelet coefficients, the wavelet coefficient structure functions $S_d(q,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} |d_X(j,k)|^q$, that consist of time average, can naturally be read as estimates of the ensemble averages $\mathbb{E}|d_X(j,k)|^q$. However, because compound Poisson cascades are multiplicative martingales and hence contain only cusp-type singularities, we also have (cf. Eq. (33) above):

$$S_d(q,j) \sim_{2^j \to 0} c_q 2^{j\zeta_L(q)}, \ q \geq 0.$$

Comparing the two above relationships yields a surprising, and so far not fully understood, result: despite stationarity and finiteness, ensemble averages and time averages exhibit the same scaling only within a finite range of values of q: $q \in [0, q_*^+]$ but have scaling that significantly differ otherwise:

$$\xi(q) = \phi(q) \quad = \quad \zeta_d(q) = \zeta_L(q) \equiv \eta(q), \ q \in [0, q_*^+], \tag{54}$$

$$\xi(q) = \phi(q) \quad \neq \quad \zeta_d(q) = \zeta_L(q) \equiv \eta(q), \ q \in [q_*^+, q_c^+]. \tag{55}$$

An equivalent distinction between $\xi(q)$ and $\zeta_L(q)$ can heuristically be supposed to exist for $q \leq 0$ with q_*^- acting as a frontier (in our setting, $q_c^- = -\infty$). However, no theoretical result for the moments of negative order for compound Poisson cascades is so far available.

Additional comment. It is also crucial to note that the *disagreement* between $\phi(q)$ (or $\xi(q)$) and $\zeta_L(q)$ for $q \notin [q_*^-, q_*^+]$ does not result from an infiniteness of moments argument, since we necessarily have[24]:

$$q_c^- \leq q_*^- \leq 0 \leq 1 \leq q_*^+ \leq q_c^+,$$

but is deeply related to the $\zeta_L(q)$ being non linear functions of q. Indeed, for a process like fractional Brownian motion, where $\zeta_L(q) = qH, q \in \mathbb{R}$, one has $q_*^+ = q_c^+ = +\infty$ and hence this discrepancy disappears.

3.3. Summary

Let us summarise the relationships between $D(h)$ and the different functions of q.

[†]In this specific case, $f(x) \simeq g(x)$ is used to indicate that there exists two constants c_1 and c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$, cf.[24] for further details and proofs.

In all generality (i.e., for all uniform Hölder functions or processes), the relationships between multifractal spectrum and scaling can be written:

DEFINITIONS

$$
\begin{array}{rcl}
D(h) \text{ defined} & \text{on} & h \in [h_*^-, h_*^+], \\
D_*^+ = D(h_*^+) & \text{and} & D_*^- = D(h_*^-), \\
q_*^+ = \left(\dfrac{dD}{dh}\right)_{h=h_*^-} & \text{and} & q_*^- = \left(\dfrac{dD}{dh}\right)_{h=h_*^+},
\end{array}
$$

PROPERTIES

$$
\begin{array}{rcl}
\eta(q) & = & \inf_{h\in[h_*^+, h_*^-]} (d + qh - D(h)), \\
q \in [q_*^+, +\infty),\ \eta(q) & = & d - D_*^+ + qh_*^+, \\
q \in (-\infty, q_*^-],\ \eta(q) & = & d - D_*^- + qh_*^-,
\end{array}
$$

ANALYSIS

$$
\begin{array}{rcl}
S_L(q,j) & = & \dfrac{1}{n_j}\displaystyle\sum_{k=1}^{n_j} |L_X(j,k)|^q, \\
\zeta_L(q) & = & \liminf_{j\to 0} \left(\dfrac{\log_2 S_L(q,j)}{j}\right), \\
q \in \mathbb{R},\ \eta(q) & \equiv & \zeta_L(q), \\
D(h) & \leq & \inf_{q\neq 0} (d + qh - \zeta_L(q)).
\end{array}
$$

For processes that contain cusp-like singularities (and no chirp-type — or oscillating — singularities), one has:

ANALYSIS

$$
\begin{array}{rcl}
S_d(q,j) & = & \dfrac{1}{n_j}\displaystyle\sum_{k=1}^{n_j} |d_X(j,k)|^q, \\
\zeta_d(q) & = & \liminf_{j\to 0} \left(\dfrac{\log_2 S_d(q,j)}{j}\right), \\
q \geq 0, \quad \zeta_d(q) = & & \zeta_L(q) \equiv \eta(q).
\end{array}
$$

For compound Poisson motion (and all multiplicative martingales, on condition that the definition of $\phi(q)$ is tuned to that of the martingales), one has:

DEFINITIONS

$$
\begin{array}{rcl}
q \in \mathbb{R},\ \phi(q) & = & q + (\mathbb{E}W^q - 1) - q(\mathbb{E}W - 1), \\
q_c^+ & = & \sup\{q \geq 1 : \phi(q) \geq 1\}, \\
q \in [0, q_c^+],\ \mathbb{E}|d_X(j,k)|^q & = & (2^j)^{\xi(q)},
\end{array}
$$

ANALYSIS

$$
\begin{array}{rcl}
q \in [0, q_*^+],\ \ \eta(q) \equiv \zeta_L(q) = \zeta_d(q) & = & \phi(q) = \xi(q), \\
q \in [q_*^+, q_c^+],\ \eta(q) \equiv \zeta_L(q) = \zeta_d(q) & \neq & \phi(q) = \xi(q), \\
q \in [q_*^-, q_*^+],\ \eta(q) \equiv \zeta_L(q) & = & \phi(q).
\end{array}
$$

3.4. Estimation Issues

From a practical point of view, analysing scaling properties in actual empirical data essentially means estimating scaling exponents via the power law behaviours of structure functions. With this respect, it is crucial to note that the measured scaling exponents necessarily behave as linear functions of q outside an interval $q \notin [q_*^-, q_*^+]$ and that only the $\zeta_L(q)$ with $q \in [q_*^-, q_*^+]$ are related to the multifractal properties of the data under analysis. For instance, it is pointless and meaningless to use values of q outside that interval to discriminate between various multifractal processes that are potential candidates to model data. Furthermore, when one intends to describe data with a multiplicative cascade, it is a key point to note that the estimated scaling exponents and the moments of the multipliers underlying the cascade shall be related only in this same finite interval of values of q. If one is interested in estimating the moments of the multipliers for larger values of q one has to abandon the multifractal formalism procedure and the use of structure functions based on multiresolution quantities.
This is why one should recast the usual goal of multifractal analysis: *estimate scaling exponents* into a more relevant one: *estimate the critical points* $q_*^\pm, h_*^\pm, D_*^\pm$*, and estimate the scaling exponents for* $q \in [q_*^-, q_*^+]$. We have developed a methodology for the estimation of the q_*^+, h_*^-, D_*^- critical points.[23, 24] Its extension to q_*^-, h_*^+, D_*^+ is under current investigation.

4. CONCLUSIONS

In this article, we gave a detailed presentation of the wavelet leader based multifractal analysis and of the corresponding multifractal formalism. We explained why the replacement of the usual wavelet coefficients with wavelet leaders brings substantial theoretical, conceptual and practical improvements. It enables the formulation of a new multifractal formalism valid for all uniform Hölder processes or functions, including those that contain not only cusp-type singularities but also chirp-type ones. This multifractal formalism yields the determintation of the multifractal spectrum of the analysed process over its entire range and hence allows to correctly study its full multifractal and scaling properties.
This wavelet leader multifractal formalism sheds a new insightful light on the relationships between the multifractal, scaling and multiplicative cascades (or martingales) ; the main result being that the observation there is only a finite range of orders $q \in [q_*^-, q_*^+]$ within which the various functions of q independently defined from the scaling, the multifractal and the multiplicative cascades frameworks coincide. The mixing up of those functions can at best cause unclear and unprecise analysis and at worst yield the drawing of erroneous and misleading conclusions. Multifractal formalisms are tied to the multifractal and scaling properties of the process and cannot, by nature, capture the statistical properties of the multipliers underlying the cascade construction beyond the interval $q \in [q_*^-, q_*^+]$ of statistical orders.
A set of MATLAB routines available upon request has been develop to implement the wavelet leader multifractal formalism as well as estimation procedures for the critical points, mainly for q_*^+.

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