Intrapartum Fetal Heart Rate Variability
Early Acidosis Detection
Multiscale Analysis

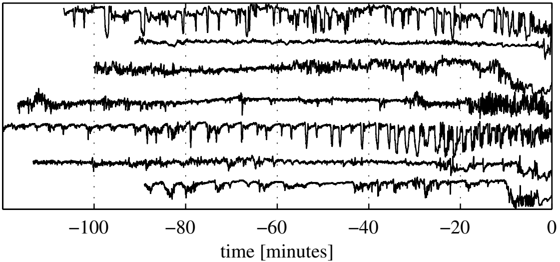
P. Abry,
FETUSES Research Program, ANR Grant

(1) CNRS, ENSLyon, Physics Dept., France
(2) Hôpital Femme-Mère-Enfant, HCL, Univ. Lyon I, France

Outline

IntraPartum Fetal Heart Rate Monitoring

- Routine Monitoring, Academic Hospital, Lyon, France,
- Scalp Electrode Measurement (STAN),
- RR pic arrivals, $\{t_n, n = 1, \ldots, N\}$,
- $\Rightarrow$ Beat-per-Minute Time Series, BpM
- $\simeq$ 3000 patients, data collected from 2001-2014.
- pH – measured after delivery
- FHR signals aligned by delivery time



IntraPartum Fetal Heart Rate: Acidosis Detection

- IntraPartum Acidosis Detection:

  Early detection of ongoing hypoxia/asphyxia ⇒ acidosis

  ⇒ Prevent adverse labor outcome (FIGO Criteria)
  (brain injury, neonatal death)

  High level of False Positives

  ⇒ Unnecessary operative delivery

  ⇒ Severe consequences for mother and newborn

- Statistical Analysis to Decrease False Positive Rate?
- Test Database:
  3-class (of 15 subjects each) highly documented database

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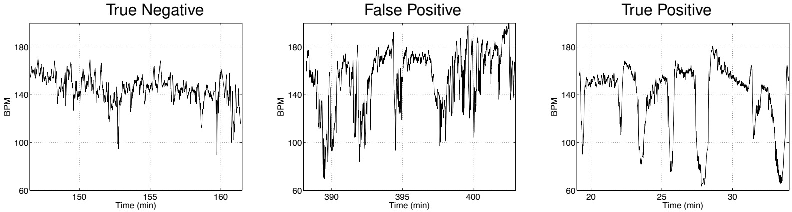

Fetal-Heart Rate Variability: Statistical Analysis ?

- Time Domain:
  - Long Term Variability
  - Short Term Variability
  - *good* Variability ⇒ large enough (?)
- Frequency Domain:
  - Power Spectral Density (Spectrum),
  - LF and HF bands, LF/HF ratio,
  - ⇒ Controversial (?)
- Misc.:
  - Entropy, entropy rates
  - Dynamical systems
  - ...
- Multiscale Analysis:
  - From Wavelets to Fractal and Multifractal analysis
  - From Wavelets to Scattering Transform

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Outline

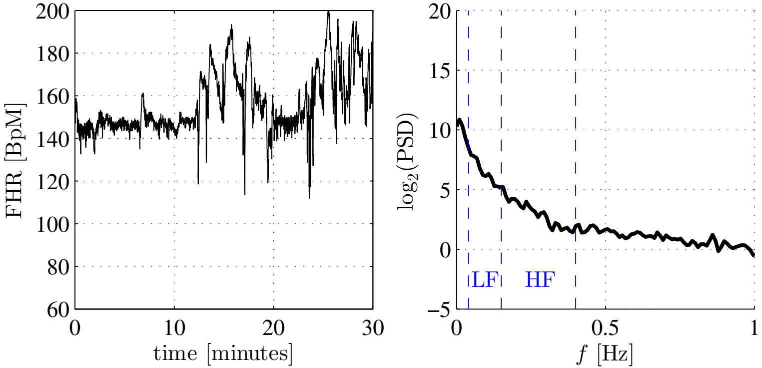
(Fetal-)Heart Rate Variability: Temporal Analysis

- Long Term Variability:
  - Largest oscillation within a window of size a:
    $\mathcal{O}_a(t) = \sup\{|X(u) - X(v)|, (u, v) \in [t - a/2, t + a/2]^2\}$
  - $LTV(t) = \mathcal{O}_a(t)$, with $a = 60$s,
  - Average LTV (BpM): $LTV = (1/n_a) \sum_k \mathcal{O}_a(k)$.
  - *good* Long Term Variability $\Rightarrow$ $LTV > 5$ BpM.

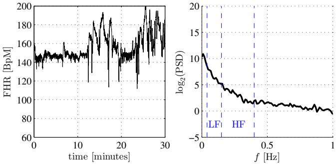
- Short Term Variability:
  - $N_a(t)$: Number of beats per unit interval of size a,
  - $STV(t) = a/N_a(t)$, with $a = 3.75$s,
  - Average STV (ms): $STV = (1/n_a) \sum_k a/N_a(k)$.
  - *good* Short Term Variability $\Rightarrow$ large enough (?)

(Fetal-)Heart Rate Variability: Spectral Analysis

- Power Spectral Density (Spectrum):
  - $X(t)$ is a (2nd-order) stationary process,
  - $\Gamma_X(f)$ Power Spectral Density (PSD or Spectrum),
  - f: frequency
  - $\Rightarrow$ Spectrum Estimation



(Fetal)-Heart Rate Variability: HF/LF decomposition



- Adult HRV:
  - $f < 1$ or 1.5 Hz,
  - Respiratory rhythms,
  - Central Nervous System:

    Sympathic/Parasympathic competing instances

$\Rightarrow$ HF: $f \in (0.15, 0.4)$, LF: $f \in (0.04, 0.15)$,

$\Rightarrow$ LF/HF Ratio $=$ Energy in LF Band / Energy in HF Band.

- Intrapartum Fetal HRV :
  - $f < 2.5$ or 3 Hz,
  - Nervous system: Not Documented/Controversial
  - Respiratory mechanisms: Not Documented/Controversial
  - What frequency bands: HF: ??, LF: ??
  - Is the LF/HF Ratio meaningful ?

Outline

A stochastic process to model scale invariance

- Self-Similar :
  $\{X(t)\}_{t\in\mathcal{R}} \stackrel{fdd}{=} \{a^H X(t/a)\}_{t\in\mathcal{R}}, \forall a > 0, 1 > H > 0,$

- Long Range Dependency, when $H > 1/2$:

  $\Rightarrow$ "Spectrum" : $\Gamma_X(f) \sim C|f|^{-(2H-1)},\ |f| \to 0$

Continuous Wavelet Transform

$$X(t) \to T_x(a, t) = \langle \frac{1}{a}\psi\left(\frac{u-t}{a}\right) | X \rangle$$

Interpretation: Joint time and frequency energy content

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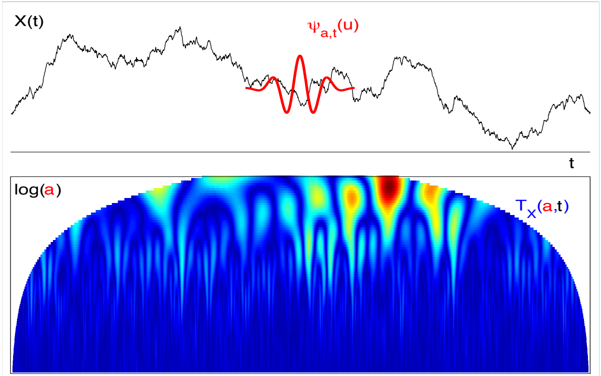

Interpretation: Joint time and frequency energy content

Self-Similarity and Wavelet Coefficients

- Theory:

  H : Self-Similarity or Hurst parameter
  Power Laws: $\mathbf{E}|T_X(a,t)|^q = C_q a^{qH}$
  For all scales: $\forall a > 0$,
  For all orders: $q > -1$,
  A single parameter qH

- Practice:

  Time averages: $S(a,q) = 1/n_a \sum_k |T_X(a,k)|^q$
  Time $\rightarrow$ Ensemble averages: $S(a,q) \rightarrow \mathbf{E}|T_X(a,k)|^q$.
  Empirical Power Laws: $S(a,q) \simeq a^{qH}$
  Estimation of H:
  $\hat{H}$ = Linear Regression $\log_2 S(a,q)$ vs. $\log_2 a/q$

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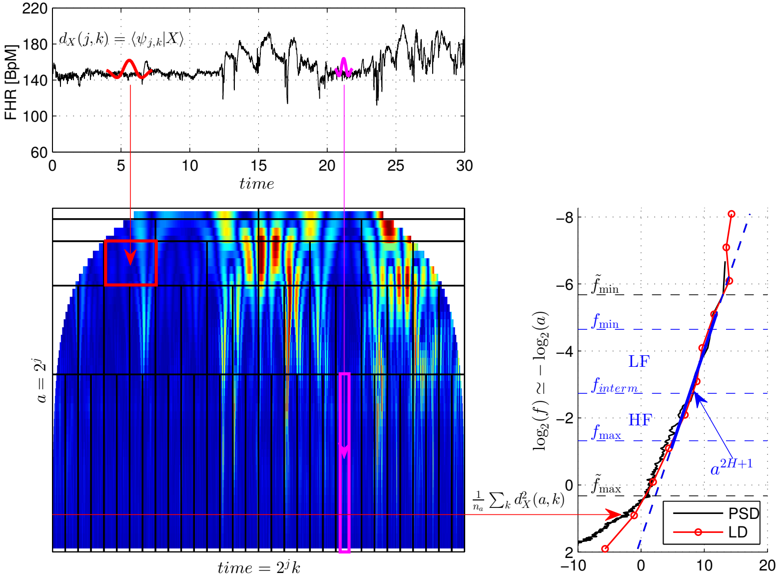
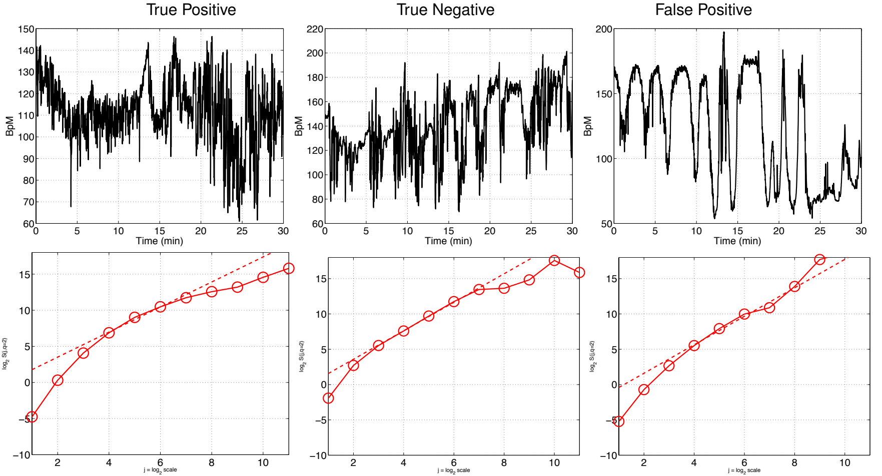


Outline

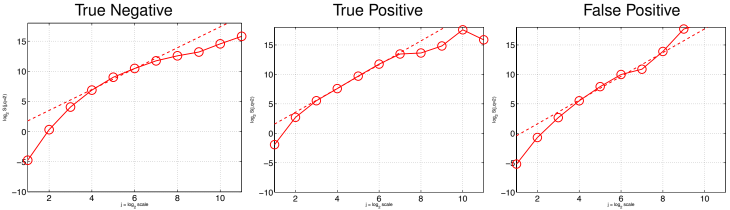
Scale Invariance in Intrapartum HRV



Power-law behavior: $a_m = 2^3 \simeq 1s \leq a \leq a_M = 2^8 \simeq 60s$

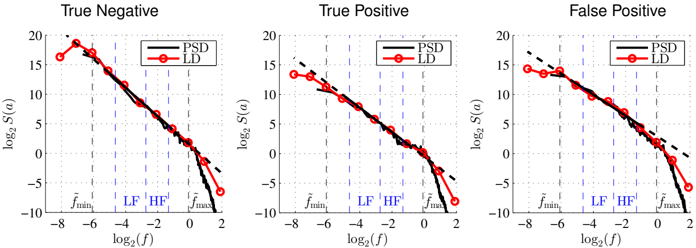
Temporal → Fractal Variability

- 1 - Multiresolution Quantities:
  $\mathcal{O}_a(t)$, $N_a(t)$ (oscillation, count) $\rightarrow T_X(a,t)$ Wav. Coeffs
- 2 - Scales:
  - Specific: $a_{LTV} = 60$s, $a_{STV} = 3.75$s $\rightarrow$ All scales $a > 0$
  - $a_m \simeq 1.5s \simeq a_{STV} \leq a \leq a_M \simeq 60s \simeq a_{LTV}$
- 3 - *Good* Variability $\equiv$ large variability ?
  - Amplitude $V_a > 5$ BpM
  - $\Rightarrow$ Scale invariance: $|T_X(a,t)| \simeq c(t)a^H$
  - $\Rightarrow$ $S_X(a,2) = S_0 a^{2H}, \forall a > 0$



Spectral → Fractal Variability

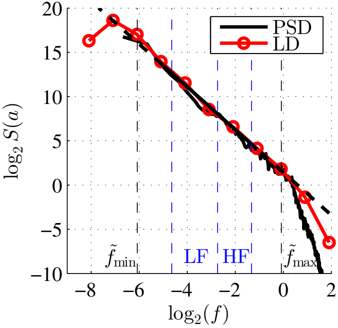
- Wavelet Spectrum
  - $\mathbf{E}|T_X(a,k)|^2 = \int \Gamma_X(f)|\tilde{\Psi}(af)|^2 df$
  - Wavelet Spectrum: $S_X(a, q=2) = (1/n_a)\sum_k |T_X(a,k)|^2$
  - Wavelet Spectrum: $S_X(a, q=2))$ estimates $\Gamma_X(f = f_0/a)$

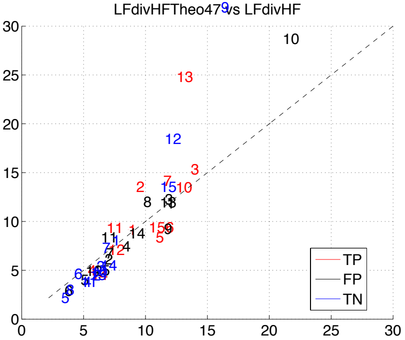


- Scale Invariance:
  - Self-Similar Model: Power Law Spectrum:
    $\Gamma_X(f) = C|f|^{-(2H-1)} \rightarrow S_X(a,2) = S_0 a^{2H}$
  - $a_m = 2^3 \leq a \leq a_M = 2^8 \rightarrow f_m \simeq 0.02Hz \leq f \leq f_M \simeq 1.25Hz$

Scaling Exponent H vs. HF/LF ratio: Adult Bands

- LF/HF Ratio : $\rho = \int_{f_m}^{f_l} \hat{\Gamma}(f)df / \int_{f_l}^{f_M} \hat{\Gamma}(f)df$
- $\hat{\rho} = (f_l^{2-2\hat{H}} - f_m^{2-2\hat{H}})/(f_M^{2-2\hat{H}} - f_l^{2-2\hat{H}})$,





- Why this choice for LF and HF bands ?

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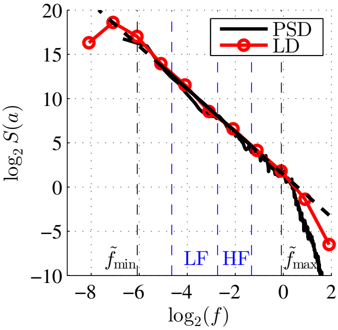
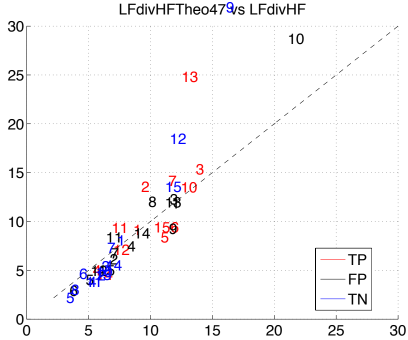
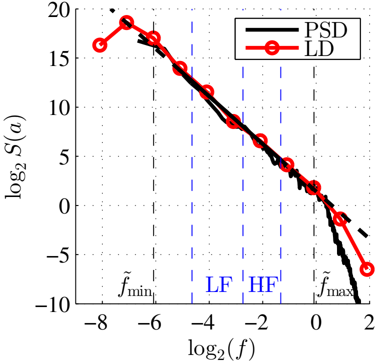


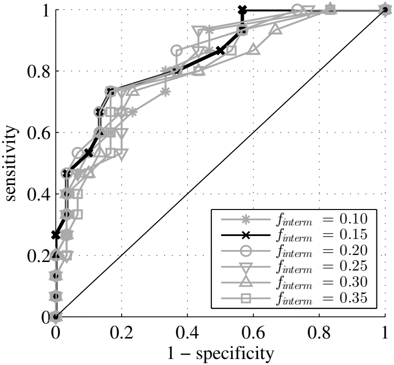


- Why this choice for LF and HF bands ?

Intermediate frequency ?

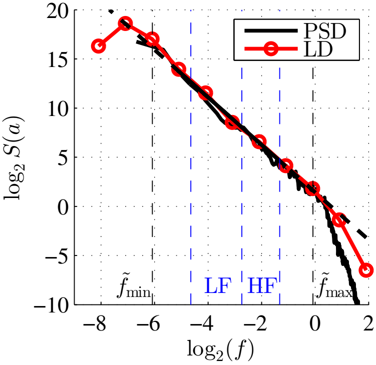
- Adult bands : HF: $f \in (0.15, 0.4)$, LF: $f \in (0.04, 0.15)$
- Why $f_{interm} = 0.04Hz$?
- Vary f_{interm} ?

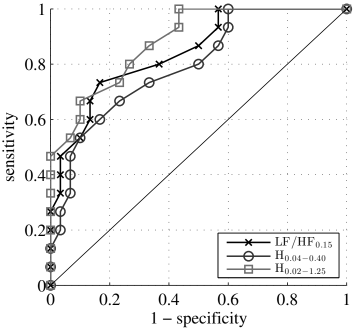




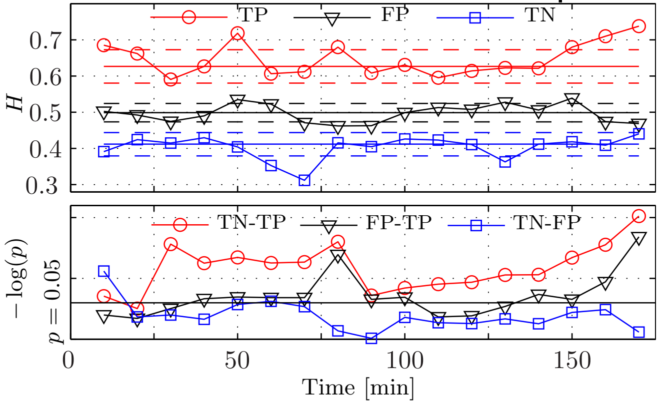
Larger frequency Bands?

- Adult band: HF: $f \in (0.15, 0.4) \Rightarrow j \in [4, 5]$
- Adult band: LF: $f \in (0.04, 0.15) \Rightarrow j \in [6, 7]$
- Scale invariance range: $j \in [3, 8] \Rightarrow f \in (0.02, 1.25)$
- Estimate $\hat{H}_{4-7} \equiv \hat{H}_{0.04-0.40}$ and $\hat{H}_{3-8} \equiv \hat{H}_{0.02-0.1.25}$?





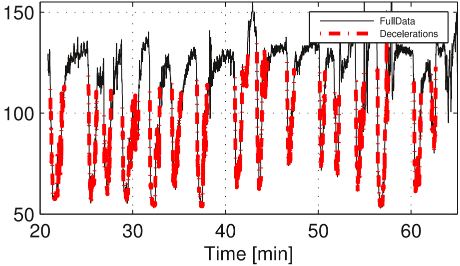
Acidosis Detection: Test on Hurst exponent H

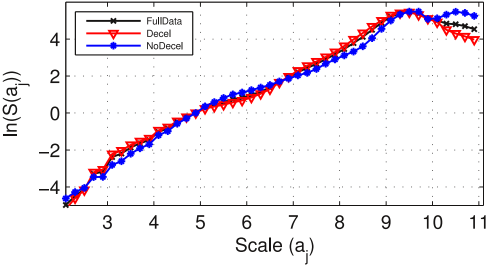


- H: a LF/HF ratio matching Scale Invariance
- No intermediate frequency - Adaptive Scaling range
- Discrimination Healthy - Non healthy
- Non Healthy - Larger H - Decreased Variability
- Discrimination an hour before delivery ?

Scaling and Decelerations

- Estimation of *H* conditioned to decelerations

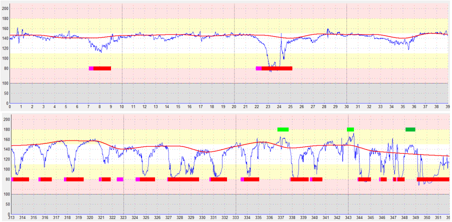


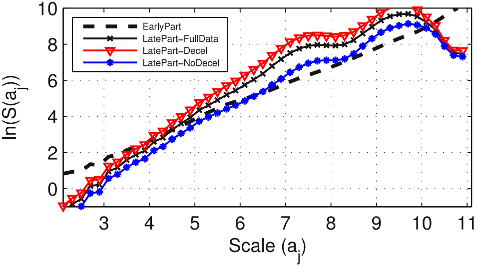


- Change of *H* and thus changes of temporal dynamics exist both during and in-between decelerations

  One same mechanism induces a change of temporal dynamics and decelerations

Scaling and Decelerations





- TP Subject - Non Healthy

  30 min 5 hours before delivery - Small $\hat{H}$ - Healthy

  30 min just before delivery - Larger $\hat{H}$ - Non Healthy

  Larger $\hat{H}$ both during and in-between decelerrations

Outline

Temporal → MultiFractal Variability

- 1 - Multiresolution Quantities:
  $\mathcal{O}_a(t)$, $N_a(t)$ (oscillation, count) → $T_X(a,t)$ Wav. Coeffs
- 2 - Scales:
  - Specific: $a_{LTV} = 60\text{s}$, $a_{STV} = 3.75\text{s} \rightarrow a_m \leq a \leq a_M$
- 3 - *Good* Variability ≡ large variability ?
  - Amplitude $V_a > 5$ BpM

  ⇒ Scale invariance: $|T_X(a,t)| \simeq c(t) a^{h(t)}$

  ⇒ h: Hölder exponent

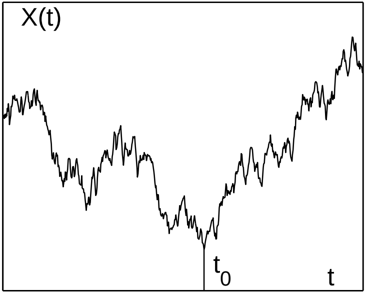
  ⇒ $S_X(a,q) = (1/n_a) \sum_k |T_X(a,t)|^q \simeq S_0 a^{\zeta(q)}$,

  ⇒ $\zeta(q) \neq qH$ concave in q
- 4 -Multiresolution Quantities:
  - Wav. Coeff. $|T_X(a,t)| \Rightarrow L_X(a,k)$ ▸ Wavelet Leaders
  - $S_X(a,q) = (1/n_a) \sum_k L_X(a,t)^q \simeq S_0 a^{\zeta(q)}$,
  - $\zeta(q) \neq qH$ concave in q
  - ▸ Multifractal Spectrum: $\text{Inf}_q(1 + qh - \zeta(q)) \geq D(h)$

Multifractal Analysis

- **Local regularity** of $X(t)$ at $t_0 : 0 < \alpha < 1$

  Compare: $|X(t) - X(t_0)| < C|t - t_0|^\alpha$



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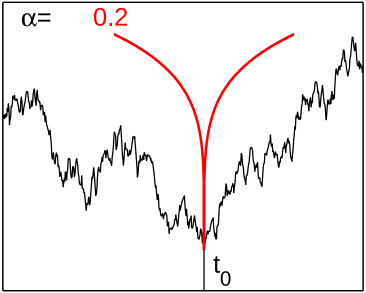

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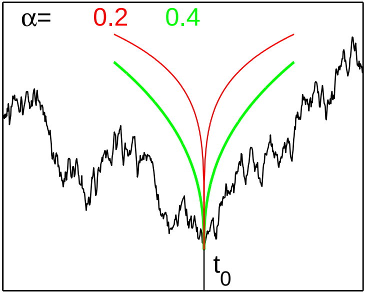

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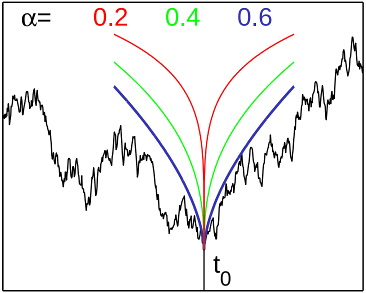

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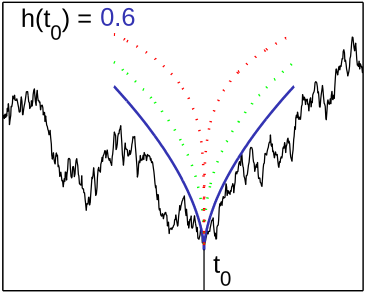
- **Hölder Exponent** : $h(t_0) = \sup_{\alpha}\{\alpha : X \in C^{\alpha}(t_0)\}$

  Extend differentiability to non integer : $0 < h(t_0) < 1$

  $\lim_{|\mathbf{t}-\mathbf{t_0}|\to 0} \frac{|X(\mathbf{t})-X(\mathbf{t_0})|}{|\mathbf{t}-\mathbf{t_0}|^{h(t_0)}} = C$



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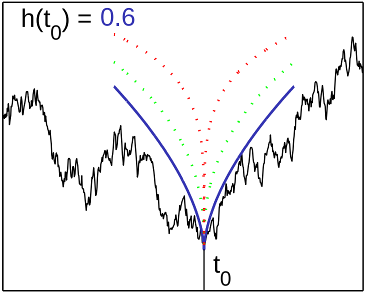
  $h(t_0) \to 1 \Rightarrow$, smooth, very regular,

  $h(t_0) \to 0 \Rightarrow$, rough, very irregular



Multifractal (or singularity) spectrum

- Data: a collection of singularities
  $|X(\mathbf{t}) - X(\mathbf{t_0})| \leq C|\mathbf{t} - \mathbf{t_0}|^{h(\mathbf{t_0})}$
- Fluctuations of local regularity: $h(\mathbf{t})$?
- not interested in h for each $(\mathbf{t})$!
- Instead, set $E(h)$ of points $\mathbf{t}$ with same h: $h(\mathbf{t}) = h$,
- Fractal dimension of $E(h)$,
- Actually Hausdorff dimension of $E(h)$, Hausdorff
- Multifractal spectrum: $D(h)$

$$D(h) = \dim_{\text{Haussdorf}}(E(h)).$$

$$0 \leq D(h) \leq d,$$
$$D(h) = -\infty \text{ if } E(h) = \{\emptyset\},$$

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- Instead, set $E(h)$ of points $\mathbf{t}$ with same h: $h(\mathbf{t}) = h$,
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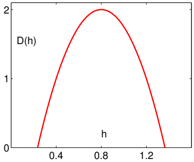

⇒ Global (geometrical) description of the fluctuations of the local regularity

- How to measure $D(h)$ from a single finite length observation? ⇒ Multifractal formalism.

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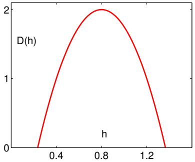


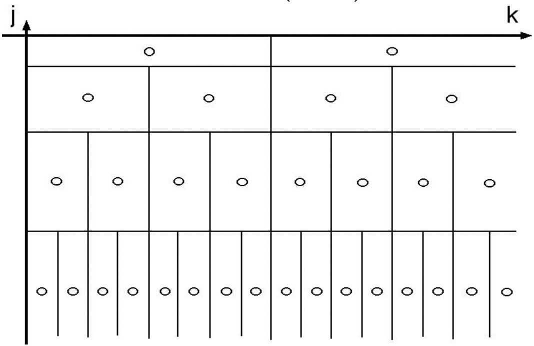
$\Rightarrow$ Global (geometrical) description of the fluctuations of the local regularity

- How to measure $D(h)$ from a single finite length observation? $\Rightarrow$ Multifractal formalism.

Wavelet Leaders

- Discrete Wavelet Transform: $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$$d_X(j,k) = \langle \frac{1}{2^j} \psi \left(\frac{t - 2^j k}{2^j} \right) | X(t) \rangle ,$$



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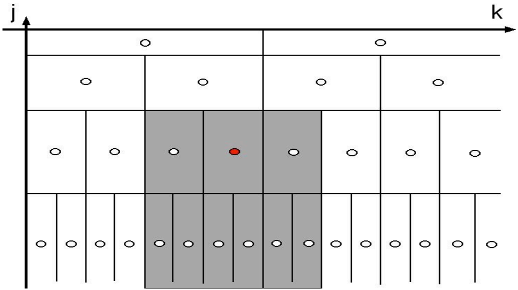

Multifractal formalism

- Multiresolution Quantities : $L_X(a,t)$,
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$$\begin{aligned} S_n(a, q) &\simeq a^d \sum_h a^{-D(h)} a^{hq}, \\ &\simeq \sum_h a^{d-D(h)+hq}, \\ &\sim_{a \to 0} c_q a^{\zeta(q)} \end{aligned}$$

Saddle-point argument: $\Rightarrow$ Legendre transform
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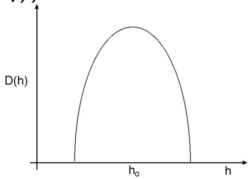

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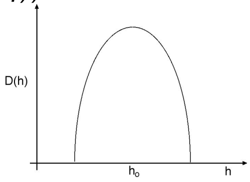

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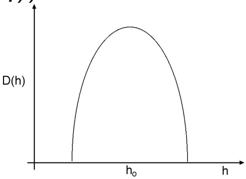

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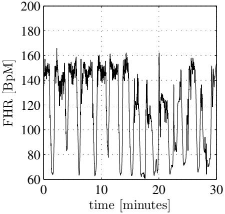
$X(t) \rightarrow d_X(a, t) \rightarrow L_X(a, t)$



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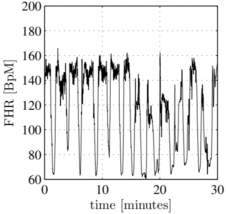

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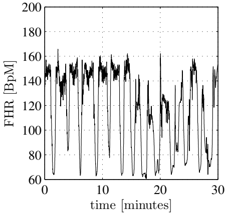

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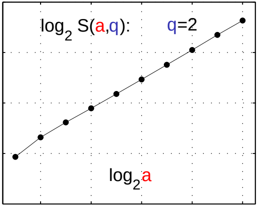

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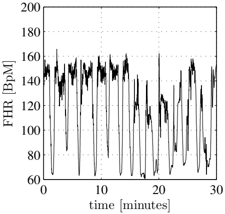

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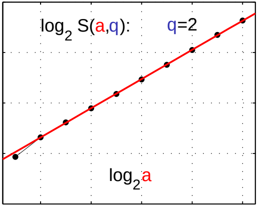

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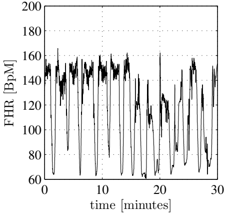

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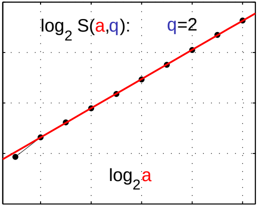
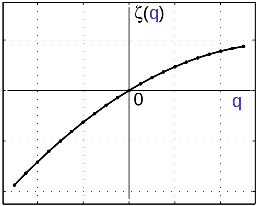


$\Longrightarrow$



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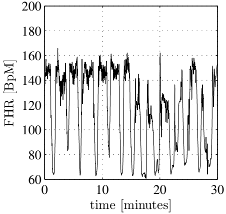

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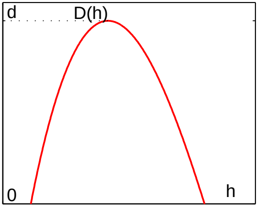

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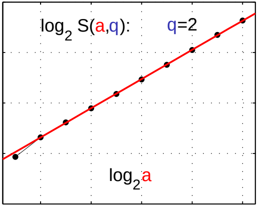

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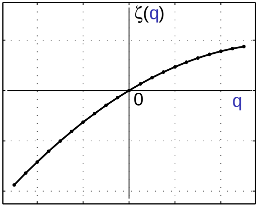
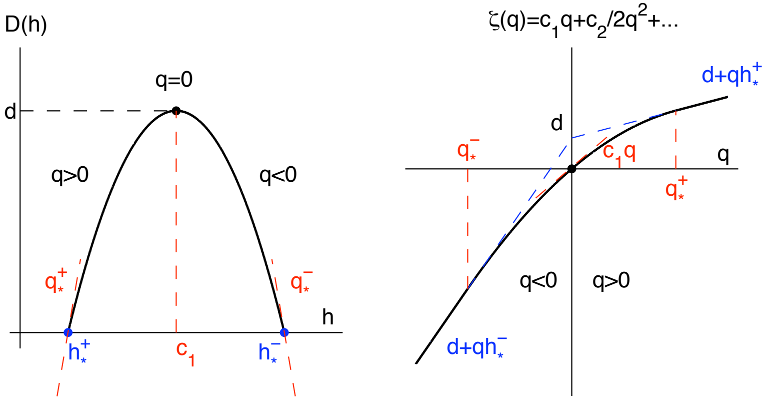


Legendre transform

Linearization effect



Log-cumulants

- Polynomial expansion:

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- c_1 : Location of max,
- $c_2 < 0$: width
- c_3 : asymmetry (hard to estimate)
- $h_{\min}$ Minimun regularity, $h_{\max}$ Maximum regularity
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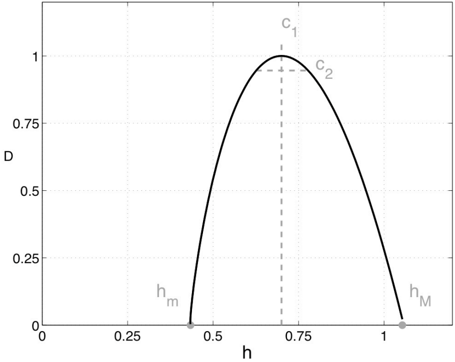

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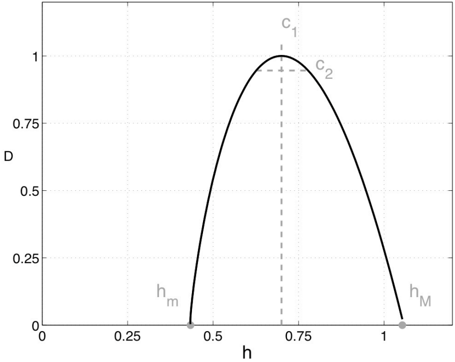

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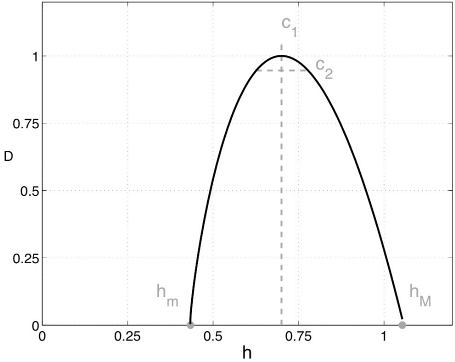
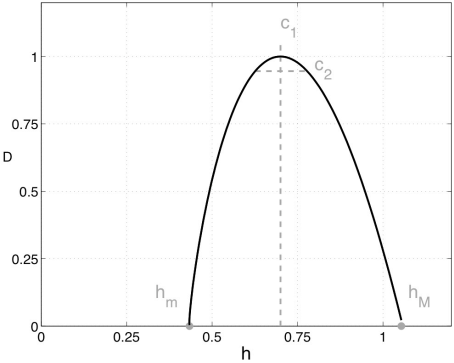
- $D(h) \simeq 1 + \frac{c_2}{2} \left(\frac{h - c_1}{c_2} \right)^2$



Multifractal Spectrum

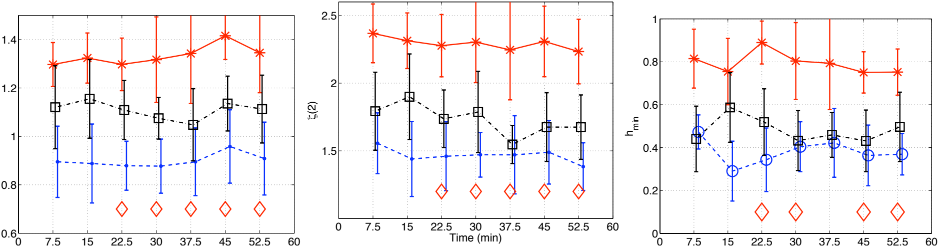
- c_1 : Location of max,
- $c_2 < 0$: width
- c_3 : asymmetry (hard to estimate)
- h_{min} Minimun regularity, h_{max} Maximum regularity
- $D(h) \simeq 1 + \frac{c_2}{2} \left(\frac{h - c_1}{c_2} \right)^2$



Outline

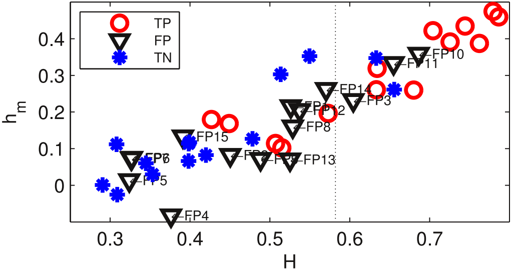
Acidosis Detection: Test on MF Attributes

- TP ; TN ; FP ; 10min long sliding window
- Wilcoxon RankSum Test between classes



- Discrimination Healthy - Non healthy
- Non Healthy - Larger *h* - Decreased Variability
- Discrimination an hour before delivery ?

Acidosis Detection: Classification

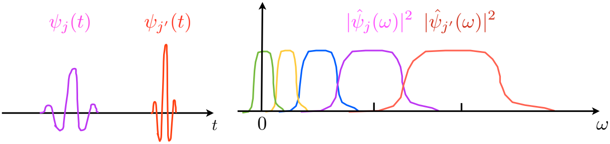


- False Positive with Low variability, low reactivity
  - are correctly classified as *Healthy*
  - Low variability, low reactivity do not actually mean change in temporal dynamics
- False Positive with *variable* and *complicated-shape* decelerations remain ill-classified
  - Do decelerations biais scaling analysis ?

Outline

Scattering Transform: First-order and Wavelets

- Wavelet Transform:



  - Complex mother wavelet $\psi(t)$
  - Dilated and translated templates $\psi_{j,k}(t) = 2^{-j}\psi(2^{-j}(t-k))$
  - Wavelet Coefficients: $X \star \psi_{j,k}$

- Scattering Transform:
  - First-order scattering coefficients:
    local time averages of absolute values of wavelet coefficients
    $SX(j,k) = N^{-1}\sum_{l=k}^{k+N} |X \star \psi_{j,l}|$

Scattering Transform: First-order and Wavelets

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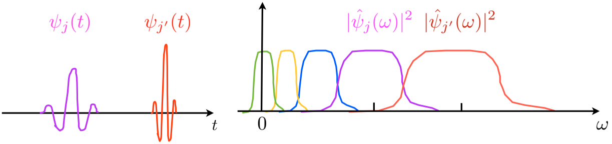

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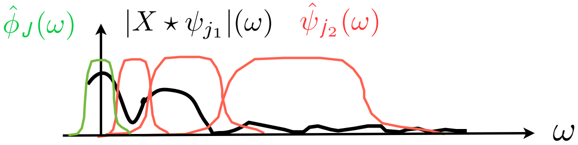
Scattering Transform: Second-order, beyond Wavelets

- 2nd order:



  - Wavelet transform of absolute values of wavelet coefficients
    $SX(j_1, j_2) = N^{-1} \sum_{t=1}^{N} ||X \star \psi_{j_1}| \star \psi_{j_2}(t)|, \ j_2 > j_1$

  - Renormalize 2nd order by 1st order:
    $\widetilde{S}X(j_1, j_2) = \frac{SX(j_1, j_2)}{SX(j_1)} \approx \frac{\sum_{t=1}^{2^J} ||X \star \psi_{j_1}| \star \psi_{j_2}(t)|}{\sum_{t=1}^{2^J} |X \star \psi_{j_1}(t)|}$

  - Non linear analysis

  ⇒ Beyond wavelet transform

  ⇒ Explore dependence beyond correlation (or spectrum)

- 3rd, 4th orders: further beyond
  not explored here

Scattering Transform: Second-order, beyond Wavelets

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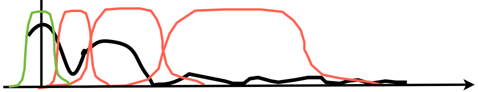

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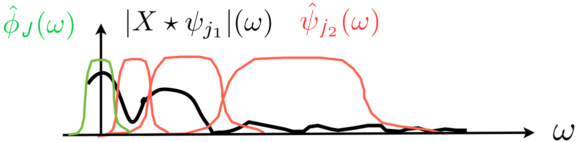

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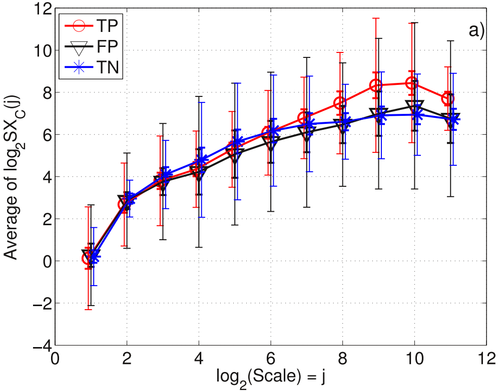
Outline

Fractal Temporal Dynamics in IntraPartum Fetal HRV

- Fetuses Data: 1st Order



- Fractal behavior:

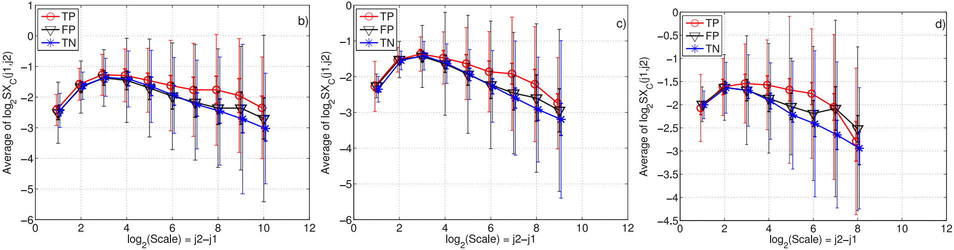
$\Rightarrow$ Time scales ranging from 1s $\leq a = 2^j \leq$ 60s

$\Rightarrow$ Estimate $\hat{H}$ for each subject, last 20 min. before delivery

Fractal Temporal Dynamics in IntraPartum Fetal HRV

- Fetuses Data: 2nd Order, for different j_1

$j_1 = 1$ $\qquad$ $j_1 = 2$ $\qquad$ $j_1 = 3$



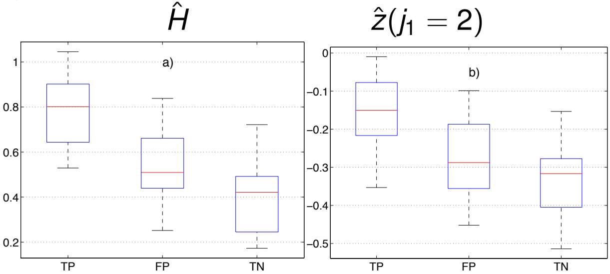
- Fractal behavior:

$\Rightarrow$ Time scales ranging from 1s $\leq a = 2^j \leq$ 60s

$\Rightarrow$ Estimate $\hat{z}(j_1)$ for each subject, last 20 min. before delivery

Discriminating Healthy from Non Healthy ?

- Box plots:



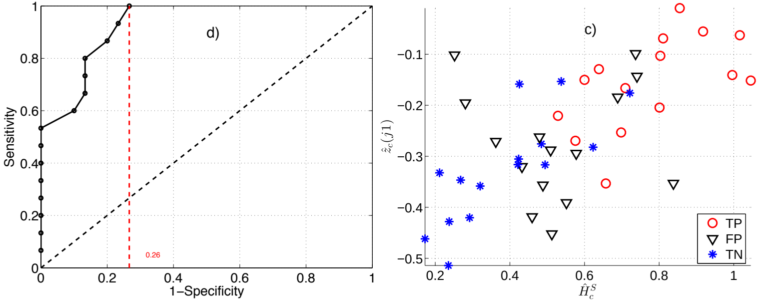
- p-Values of Ranksum tests (Null: Healthy $\equiv$ Non Healthy)

c	$\hat{H}_c$	$\hat{z}(j_1 = 1)$	$\hat{z}(j_1 = 2)$	$\hat{z}(j_1 = 3)$
TP/TN	0.00	0.00	0.00	0.02
TP/FP	0.00	0.01	0.01	0.09
FP/TN	0.02	0.17	0.25	0.15

$\Rightarrow$ $\hat{H}$ and $\hat{z}(j_1 = 2)$ discriminate Healthy from Non Healthy

Classification and typology

- Performance: ROC curve and Scatter plot



- Typology
  - FIGO-FPs with Low variability, low reactivity are correctly classified by $(\hat{H}, \hat{z}(j_1 = 2))$
  - FIGO-FPs with severe deceleration are not.

Sample path in $\hat{H}$ versus $\hat{z}(2)$ plans: Movies

FIGO-TN: Healthy

Sample path in $\hat{H}$ versus $\hat{z}(2)$ plans: Movies

FIGO-TP: Non Healthy

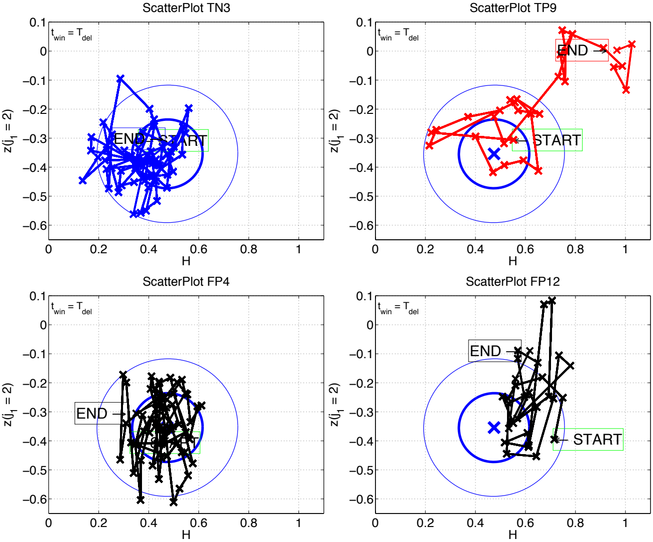
Sample path in $\hat{H}$ versus $\hat{z}(2)$ plans: Movies

FIGO-FP: Healthy

Sample path in $\hat{H}$ versus $\hat{z}(2)$ plans: Movies

FIGO-FP: Healthy

Sample path in $\hat{H}$ versus $\hat{z}(2)$ plans: Movies



Outline

Low Dimensional Embedding ?

- For each time window k:
  - Compute cattering Coefficients: $\{\mathcal{S}X(k)\}, k = 1, 2, \ldots, K$
  - $\mathcal{S}X(k)$ has dimension $N = J + J \times (J-1)/2 - 1 = 55$.
  - $\left(\{\log SX(j,k)\}_{1 \le j \le J}\,,\ \{\log \widetilde{S}X(j_1, j_2, k)\}_{1 \le j_1 < j_2 \le J}\right)$

- Dimension of the embedding space:
  - Do the K windows really live:
    in a space of dimension N ?
    or on a Low Dimensional Manifold of size $D \ll N$?

Embedding Construction

- Sliding Covariance amongst Scatt Coeff.:

$$\widehat{\mathbf{C}}(k) = \sum_{l=k-L}^{k+L} (\mathcal{S}X(l) - \widehat{\boldsymbol{\mu}}(k))^T (\mathcal{S}X(l) - \widehat{\boldsymbol{\mu}}(k))$$

- Riemannian metric between pairs $\mathcal{S}X(k), \mathcal{S}X(l)$:
$d(k,l) = (\mathcal{S}X(k) - \mathcal{S}X(l))^T (\mathbf{C}(k) + \mathbf{C}(l))^{-1} (\mathcal{S}X(k) - \mathcal{S}X(l))$

- Create a similarity matrix to create a graph of relations between the time-windows:

$$W_{kl} = \exp\left\{-\frac{d(l,k)}{\varepsilon}\right\}, \quad k,l = 1,\ldots,K.$$

  ε: Arbitray Reference distance

- Apply Spectral Clustering to the Graph:
  Normalize: $\mathbf{W}^{\mathrm{norm}} = \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{-1/2}$, $D_{kk} = \sum_l W_{kl}$
  EigenValue Dec.: $\mathbf{W}^{\mathrm{norm}} \Rightarrow \lambda_i$ and $\boldsymbol{\nu}_i$
  D-dimensional embedding ($D \ll N$):
  $\mathcal{S}X(k) \mapsto (\nu_1(k), \nu_2(k), \ldots, \nu_D(k))$

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Outline

Embedding Results

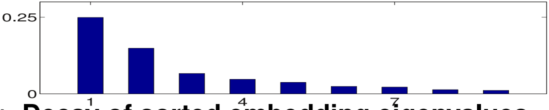

Figure : **Decay of sorted embedding eigenvalues** $\Rightarrow D = 3$

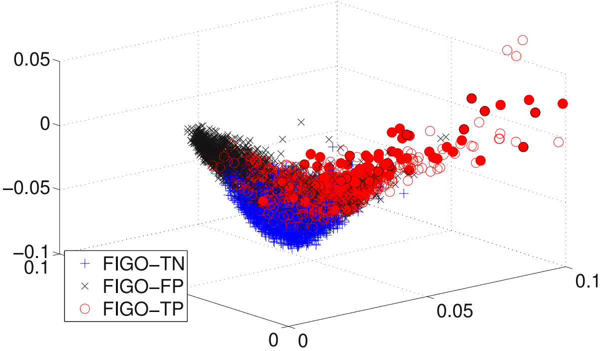

Figure : **Low-Dimensional Manifold Time-Window Embedding.**

Classification Performance and analysis

% Mean (std)	Sensitivity	Specificity	MCC	Error-rate
FIGO	100 (–)	50 (–)	50 (–)	33 (–)
Emb+NN	66 (29)	89 (15)	62 (29)	18 (13)
SVM	60 (27)	93 (10)	59 (26)	18 (10)

- Nearest-Neighbors (very simple classifier) on Low Manifold $D = 3 \ll N = 55$
- As good as SVM (very sophisticated classifier) in Space of Dimension N

$\Rightarrow$ Low Dimensional Embedding is relevant !

- FIGO-FP with *Low-Variability* or *Low-reactivity* : Well-Classified by Embedding
- FIGO-FP with *complicated-shape* and *severe* decelerations : Still mis-classified

Trajectory Embedding

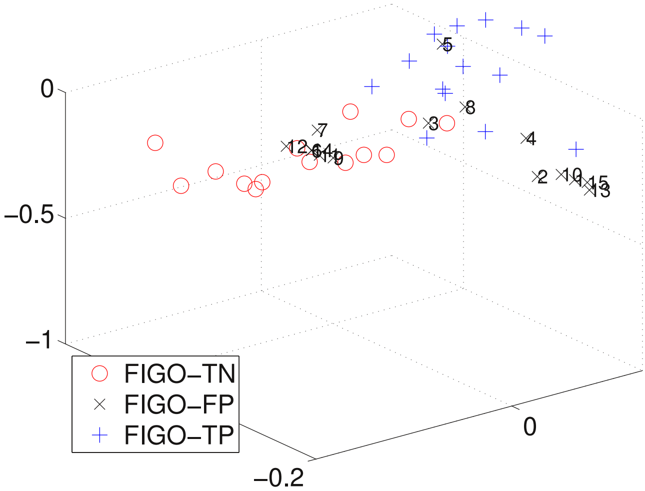

Figure : **Low-Dimensional Manifold Trajectory Embedding.**

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SVM	60 (27)	93 (10)	59 (26)	18 (10)
EmbTraj	56 (28)	96 (07)	63 (22)	17 (10)
SVMTraj	58 (27)	95(08)	60 (25)	17 (09)

- Trajectory classification better than independent time window classification
- Nearest-Neighbors on Low Manifold $D = 3 \ll N = 55$ Better than SVM in Space of Dimension N

$\Rightarrow$ Low Dimensional Embedding is relevant !

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Outline

Conclusions

- Scale Invariance:
  - Models Temporal Dynamics as Fractal Variability
  - Gathers time and spectral variabilities (linear analysis) within a single multiscale (wavelet) framework,
  - A range of scales rather than specific scales,
  - Extend to MultiFractal Variability (non Linear)
  - Extend to Scattering Variability (non Linear)
  - Can enter Acidosis detection/classification,
  - Re-Identify correctly some FIGO-FP, but not all.
- Continuations:
  - Large data base ?
  - Adults HRV ?
  - wavelet p-Leaders ⇒ R. Leonarduzzi's talk
- References:
  - patrice.abry@ens-lyon.fr
  - perso.ens-lyon.fr/patrice.abry/ ⇒ MF Toolbox
  - perso.ens-lyon.fr/patrice.abry/FETUSES/

Outline

Thanks to (co-authors)

- P. Borgnat, S. Roux, N. Pustelnik, ENS Lyon, France
- P. Gonçalvès, ENS Lyon, France,
- M. Doret, HFME French academic hospital, France
- J. Spilka, V. Chudacek, Praha, Cezch republic,
- S. Mallat, J. Anden., ENS Paris, France,
- T. Talmon, Technion, Israel,
- M.-E. Torres, R. Leonarduzzi, Univ. de Entre-Rios, Argentina.

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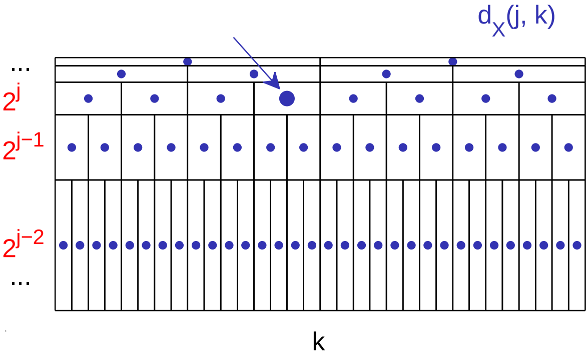
Questions?

Wavelet Leaders

- $d_X(j,k) \longrightarrow L_X(j,k)$:



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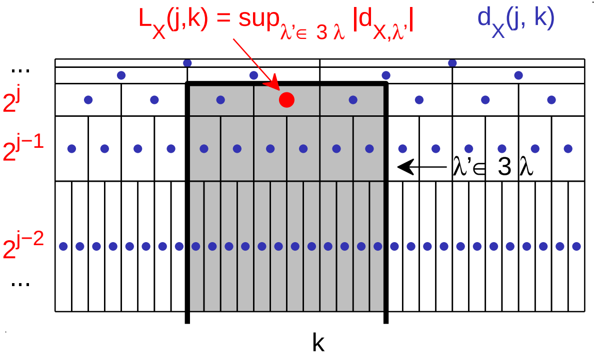
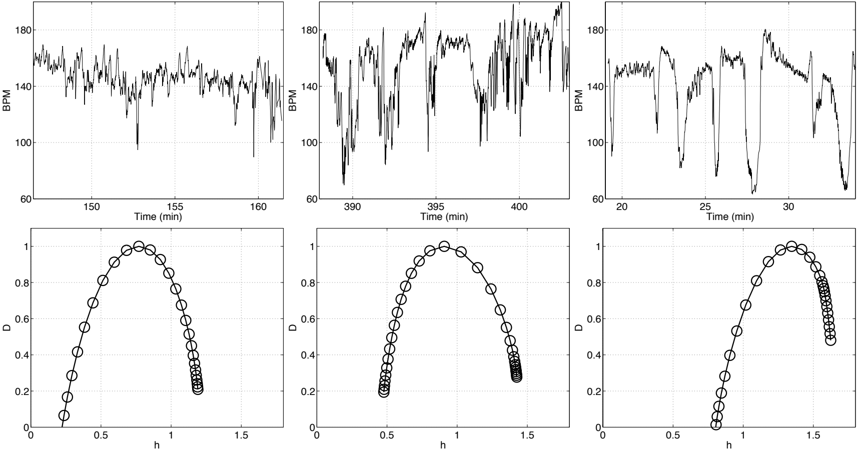


Multifractal: Typical Spectra

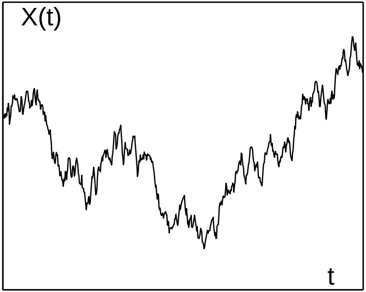
From Left to Right, TN, FP and TP typical examples

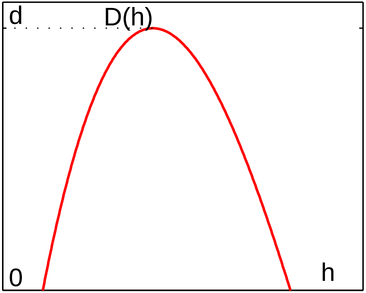


Multifractal Spectrum

- **Multifractal Spectre** $D(h)$:
  - Irregularity: Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set:

$$D(h) = \dim_H \{t : \; h(t) = h\}$$





Multifractal Spectrum

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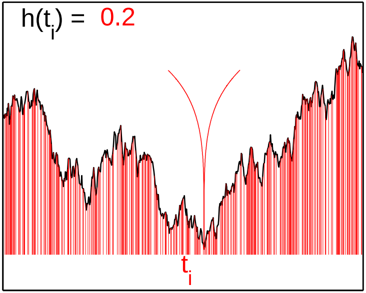
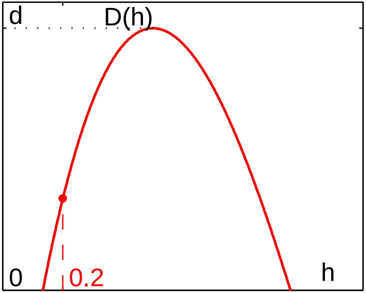
  - Irregularity: Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set:

$$D(h) = \dim_H\{t: \ h(t) = h\}$$





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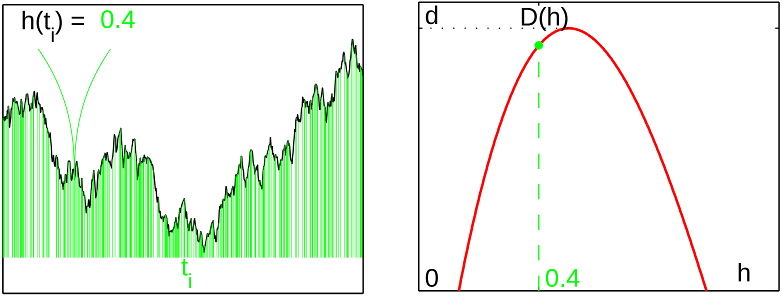

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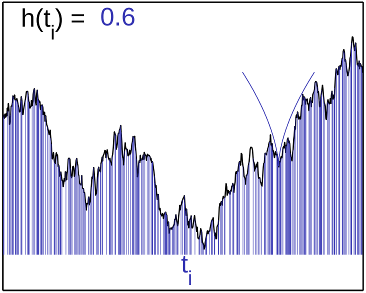

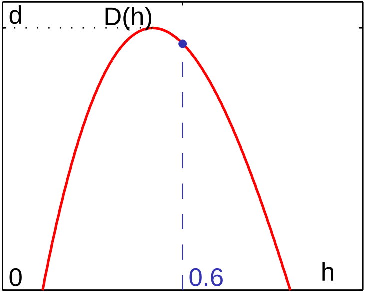

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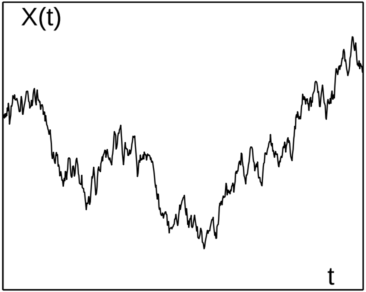
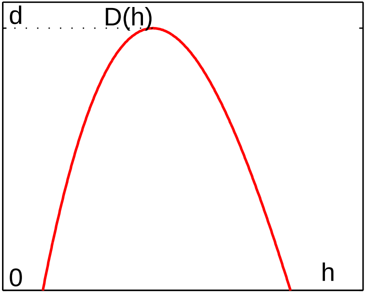
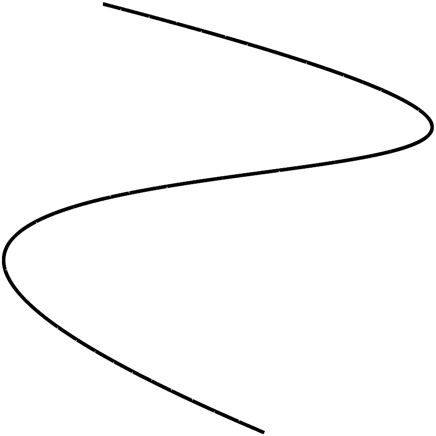
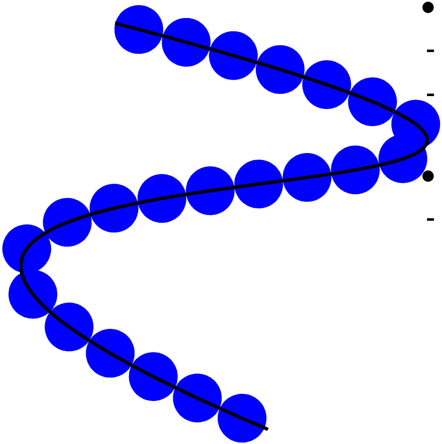




Dimension of a geometrical set

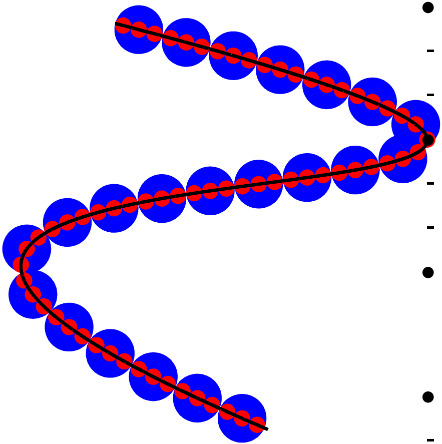


Euclidean dimension



- Let
  - $a (= 1)$ be the analysis scale,
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$

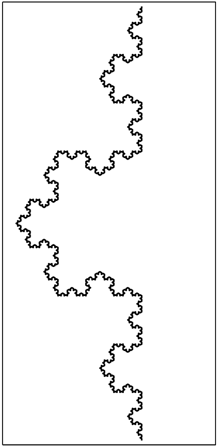
Euclidean dimension



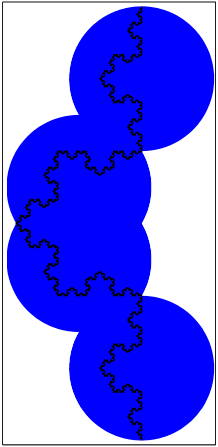
- Let
  - $a (= 1)$,
  - $a (= 1/3)$,
- hence,
  - $N = \frac{a}{a} \cdot N (= 3 \cdot N)$,
  - $L = N \cdot a = L = N \cdot a = L_0$,
- donc

  $L(a)$ does not depend on a nor on a !
- and
  - $L(a) = N(a) \cdot a = L_0$,
  - $\boxed{N(a) = L_0 / a = L_0 \cdot a^{-1}}$.

Dimension of a geometrical set

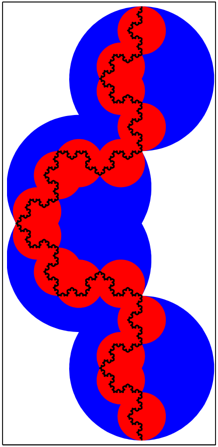


Fractal dimension



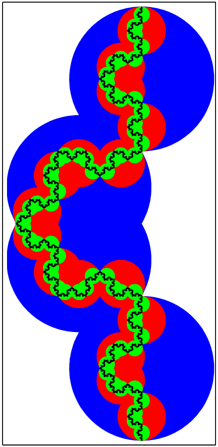
- Let
- a, be the analysis scale
- N denote the number of covering boxes with size a,
- Then
- Length is : $L = N \cdot a$
- Here,
- $a = 1/3$,
- $N = 4$,

Fractal dimension



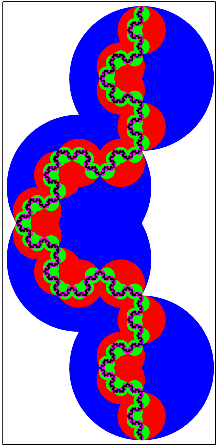
- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
- Then,
  - $N = 4$,
  - $N = 16$,
- Hence
  - $L = N \cdot a \neq L = N \cdot a$! ,

Fractal dimension



- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
  - $a = 1/27$,
- Then,
  - $N = 4$,
  - $N = 16$,
  - $N = 64$,
- donc
  - $L = N \cdot a \neq L = N \cdot a \neq L = N \cdot a$!

Fractal dimension



- One shows:
- $a(n) = (1/3)^n$,
- $N(n) = 4^n$,
- hence
- $L(a) = N(a) \cdot a$,
- $L(a)$ does depend on a !
- with,
- $\boxed{N(a) = a^{-D}}$,
- $L(a) = L_0 \cdot a^{1-D}$,
- D : fractal dimension,
- $1 < D < 2$,
- non integer = Frac-.

Haussdorf Dimension

- Intuition:

  Fractal dimension,
  Non integer extension of the natural *Euclidean* dimension, $0 \leq D \leq d$.
  Cover a set A with balls of size ϵ, Count how many you need $N(\epsilon)$.
  Assume a power law behaviour $N(\epsilon) \sim \epsilon^{-D}$.
  Define $D = \lim_{\epsilon \to 0} - \log N(\epsilon) / \log \epsilon$.

- Definition:

  $A \in \mathcal{R}^d$,
  $\epsilon > 0$, R ϵ-covering of A with a countable collection of bounded sets A_i, $|A_i| \leq \epsilon$,
  $\delta \in [0, d], M_\epsilon^\delta(A) = \inf_R \left(\sum_i |A_i|^\delta \right), M^\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta(A)$,
  D is such that $\delta > D, M^\delta(A) = 0, \delta < D, M^\delta(A) = \infty$.

Thermodynamic analogy (Parisi-Frisch, 85)

Thermodynamic	Multifractal
- $Z_\beta(U) = \sum_k e^{-\beta E_k}$,	- $S(a,q) = \sum_k \|T_X(a,k)\|^q$ $S(a,q) = \sum_k e^{q \log \|T_X(a,k)\|}$
$U = \langle E_k \rangle = \partial \log Z_\beta / \partial \beta$	- $\|T_X(a,k)\| = a^{h_k}$, - $S(a,q) = \sum_k e^{q h_k \log a}$
- β	- q
- $E_k = \epsilon_k \delta V$,	- $h_k \log a$, - $S(a,q) = a^{\zeta(q)}$,
- $F = -\ln Z_\beta$	- $\zeta(q) \log a = \log S(a,q)$,
- Entropy: $F = U - S/\beta$ (Legendre transform)	- Spectrum: $D(h) = qh - \zeta(q)$ (Legendre transform)

◂ to MF Form.

Rényi entropy
Strange attractors and chaotic systems (Kadanoff, 75)

- Rényi entropy: $Z_\alpha(a) = \sum_k P_k(a)^\alpha$,
- Rényi information: $I_\alpha(a) = \log Z_\alpha(a)/(1-\alpha)$,
- Generalized dimensions: $D_\alpha = \lim_{a\to 0} I_\alpha(a)/(-\log a)$,

$\Rightarrow$ $(1-\alpha)D_\alpha = \lim_{a\to 0} \log Z_\alpha(a)/\log a \equiv \zeta(\alpha)$!

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Log-Cumulants

- For certain classes of processes :
  - $\mathbf{E} L_X(j,\cdot)^q = F_q |2^j|^{\zeta(q)}$
- 2^{nd} characteristic function $\ln L_X(j,\cdot)$:
  - $\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \sum_p C_p^j \frac{q^p}{p!} = \ln F_q + \zeta(q) \ln 2^j$

    C_p^j: cumulant of order $p \geq 1$ de $\ln L_X(j,\cdot)$
- $\Rightarrow \forall p \geq 1 : \quad C_p^j = c_p^0 + c_p \ln 2^j$
  - $\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \underbrace{\sum_{p=1}^{\infty} c_p^0 \frac{q^p}{p!}}_{\ln F_q} + \underbrace{\sum_{p=1}^{\infty} c_p \frac{q^p}{p!}}_{\zeta(q)} \ln 2^j,$
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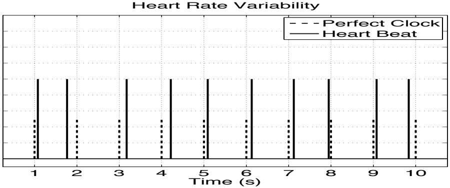
Heart Beat Variability?

- R-R Intervals: $t_k - t_{k-1}$ in ms



- Beat-per-Minute time series:
  $X(t) =$ Interp $(60000/(t_k - t_{k-1})), f_s = 8$Hz.