The Hilbert Space Geometry of the Stochastic Rihaczek Distribution

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Abstract

Beginning with the Cramér-Loève spectral representation for a nonstationary discrete-time random process, one may derive the stochastic Rihaczek distribution as a natural time-frequency distribution. This distribution is within one Fourier transform of the time-varying correlation and the frequency-varying correlogram, and within two of the ambiguity function. But, more importantly, it is a complex Hilbert space inner product, or cross-correlation, between the time series and its one-term Fourier expansion. To this inner product we may attach an illuminating geometry. Moreover, the Rihaczek distribution determines a time-varying Wiener filter for estimating the time series from its local spectrum, the error covariance of the estimator, and the related time-varying coherence. The squared coherence is the magnitude-squared of the complex Rihaczek distribution, normalized by its time and frequency marginals. It is this squared coherence that determines the time-varying localization of the time series in frequency. Most of these insights extend to the characterization of time-varying and random channels, in which case the stochastic Rihaczek distribution is a fine-grained characterization of the channel that complements the coarse-grained characterization given by the ambiguity function.

1 Introduction

For every nonstationary, harmonizable random sequence there is a well-defined bivariate correlation function and bivariate generalized spectral density. Fourier transforms of these produce what might be called the stochastic Rihaczek distribution [1], or the Rihaczek spectrum [4]. In this paper we review the finding from [8] that this distribution is, in fact, a Hilbert space inner product, or cross correlation, between the random sequence and its one-term Fourier expansion.

We show that the stochastic Rihaczek distribution determines a complex time-varying Wiener filter for estimating a time series from a single value of its global Fourier transform. Moreover, the magnitude squared of the stochastic Rihaczek distribution determines the mean-squared error of the estimator, and therefore provides a basis for determining the local spectrum of the time series. Suitably normalized, the stochastic Rihaczek distribution is the coherence, or cosine-squared of the angle, between the time series random variable $x[n]$ and the Fourier transform random variable $\widehat{x}(\theta)$. Thus the correct interpretation of the stochastic Rihaczek distribution is that it distributes complex correlation over time and frequency, and not power or energy.

These insights extend to the analysis of time-varying and random channels. The stochastic Rihaczek distribution gives a fine-grained characterization of a channel to complement the coarse-grained characterization given by the ambiguity function.

2 The Stochastic Rihaczek Distribution and its Hilbert Space Geometry

Following the lead of [2], [3], [7], we shall begin with the Cramér-Loève spectral representation for a discrete-time zero-mean nonstationary time series $\{x[n], n \in \mathcal{Z}\}$:

$$x[n] = \int_{-\pi}^{\pi} e^{jn\theta} dX(\theta). \qquad (1)$$

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Here $X(\theta)$ is a complex valued stochastic set function with non-orthogonal increments, i.e.,

$$E\{dX(\theta_1)dX^*(\theta_2)\} = S_L(\theta_1,\theta_2)d\theta_1 d\theta_2/(2\pi)^2. \quad (2)$$

The function $S_L(\theta_1,\theta_2)$ is the bivariate generalized power spectrum for the time series and its bivariate Fourier transform is the bivariate correlation sequence for the time series:

$$r_L[n_1,n_2] = E\{x[n_1]x[n_2]^*\} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S_L(\theta_1,\theta_2)e^{j(n_1\theta_1-n_2\theta_2)}d\theta_1 d\theta_2$$

The subscripts L denote Loève spectrum and Loève correlation, after one of the discoverers of the Cramér-Loève spectral representation in Eq. (1). We shall assume that the nonstationary time series $\{x[n], n \in \mathcal{Z}\}$ is square summable, with probability one, so that it makes sense to Fourier transform realizations of the series.

In the stationary limit, $r_L[n_1,n_2] \to r[n_1-n_2]$, which implies $S_L(\theta_1,\theta_2) \to S(\theta_1)\delta(\theta_1-\theta_2)/(2\pi)^2$, where $S(\theta) = \sum_n r[n]\exp(-j\theta n); \quad |\theta| \le \pi$.

The Rihaczek distribution figures prominently in Flandrin's monograph "Temps-fréquence" [4] (see also the English translation [5]. Its stochastic version is obtained by first defining the frequency-varying spectrum $S(\delta,\theta) = S_L(\theta+\delta,\theta)$ and the time-varying correlation $r(n,k) = r_L(n,n-k)$, which are functions of global frequency θ, local frequency δ, global time n, and local time k. Then, these are Fourier transformed to produce the stochastic Rihaczek distribution [8]

$$V(n,\theta) = \sum_{k=-\infty}^{\infty} r[n,k]e^{-jk\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\delta,\theta)e^{jn\delta}d\delta.$$

The Fourier transform picture is completed by noting that $r[n,k]$ and $S(\delta,\theta)$ are Fourier transforms of $V(n,\theta)$, and of each other:

$$r[n,k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(n,\theta)e^{jk\theta}d\theta, \quad (3)$$

$$S(\delta,\theta) = \sum_{n=-\infty}^{\infty} V(n,\theta)e^{-jn\delta}. \quad (4)$$

It is easy to check that in the stationary limit, $V(n,\theta) \to S(\theta)$, so the conventional power spectral density for stationary processes is recovered.

Actually, the stochastic Rihaczek distribution is within one Fourier transform of the time- and frequency-domain correlations, and within two of the radar ambiguity function $A[\delta,k]$:

$$A(\delta,k) = \sum_n r[n,k]e^{-jn\delta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\delta,\theta)e^{jk\theta}d\theta.$$

$$\begin{tikzcd} r[n,k] \arrow[r, "k\to\theta"] \arrow[d, "n\to\delta"] & V[n,\theta] \arrow[d, "n\to\delta"] \\ A(\delta,k) \arrow[r, "k\to\theta"] & S(\delta,\theta) \end{tikzcd}$$

Figure 1: *Relation between time-frequency functions.*

In Fig. 1, we show the relation between the four key time-frequency functions, $r[n,k]$, $V[n,\theta]$, $A(\delta,k)$, and $S(\delta,\theta)$. Going from left to right implies a discrete Fourier transform (DTFT) from local time k to global frequency θ. Going from top to bottom implies a DTFT from global time n to local frequency δ. Note that while the stochastic Rihaczek distribution is a function of *global* time and frequency variables, the ambiguity function is a function of *local* time and frequency variables.

Let us recall that $r[n,k]$ is the expectation or inner product $E\{x[n]x^*[n-k]\}$ and $S(\delta,\theta)$ is the expectation or inner product $E\{\widehat{x}(\theta+\delta)\widehat{x}^*(\theta)\}$. When these identities are substituted into the definitions of the stochastic Rihaczek distribution, we find that

$$V(n,\theta) = E\{x[n](\widehat{x}(\theta)e^{j\theta n})^*\} = E\{(x[n]e^{-j\theta n})\widehat{x}^*(\theta)\},$$

where

$$\widehat{x}(\theta) = \sum_n x[n]e^{-jn\theta}.$$

This representation of $V(n,\theta)$ shows that the stochastic Rihaczek distribution is nothing more than the Hilbert space inner product between the time series $\{x[n]\}$ and its one-term Fourier expansion $\{\widehat{x}(\theta)e^{j\theta n}\}$, evaluated at time n. This inner product is in general complex valued, but its marginals are real and non-negative:

$$E\{|\ \widehat{x}(\theta)\ |^2\} = S(0,\theta) = \sum_n V(n,\theta) \ge 0, \quad (5)$$

$$E\{|\ x[n]\ |^2\} = r[n,0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(n,\theta)d\theta \ge 0. \quad (6)$$

The fact that marginals are non-negative and real does not in any way suggest that the stochastic Rihaczek distribution is a distribution of power or energy over time and frequency. Rather, it is a distribution of inner product or cross-correlation over time and frequency.

In order to develop this geometry further, let us construct the 2×1 complex vector

$$\mathbf{z}[n,\theta] = \left[\begin{array}{c} x[n] \\ \widehat{x}(\theta)e^{j\theta n} \end{array} \right], \qquad (7)$$

and define its second-order covariance matrix $\mathbf{K}[n,\theta]$:

$$\mathbf{K}[n,\theta] = E\{\mathbf{z}[n,\theta]\mathbf{z}^H[n,\theta]\} = \left[\begin{array}{cc} r[n,0] & V(n,\theta) \\ V^*(n,\theta) & S(0,\theta) \end{array} \right].$$

Under time delay n_0 and complex frequency modulation $e^{j\theta_0 n}$, this covariance matrix is transformed by the group action

$$\mathbf{K}[n,\theta] \longrightarrow \left[\begin{array}{cc} r[n-n_0,0] & V(n-n_0,\theta-\theta_0) \\ V^*(n,\theta-\theta_0) & S(0,\theta-\theta_0) \end{array} \right]. \qquad (8)$$

Thus this second-order description of the vector $\mathbf{z}[n,\theta]$ is co-variant under time delay and complex modulation, which is one of the key requirements of a time-frequency distribution [9], [10], [6].

From the covariance matrix $\mathbf{K}[n,\theta]$ we can now read out the Wiener filter $W(n,\theta)$ for estimating the complex time series value $x[n]$ from the one-term complex valued Fourier-expansion $\widehat{x}(\theta)\exp(jn\theta)$ as

$$W(n,\theta) = \frac{V(n,\theta)}{S(0,\theta)} = \frac{V(n,\theta)}{\sum\limits_n V(n,\theta)}. \qquad (9)$$

The mean-squared error $Q(n,\theta)$ of the estimator $W(n,\theta)\widehat{x}(\theta)e^{j\theta n}$ is readily found to be

$$Q(n,\theta) = r[n,\theta]\left[1 - \frac{|V(n,\theta)|^2}{r[n,0]S(0,\theta)}\right] =$$

$$= r[n,\theta]\left[1 - \frac{|V(n,\theta)|^2}{\left[\sum\limits_n V(n,\theta)\right]\frac{1}{2\pi}\int_{-\pi}^{\pi}V(n,\theta)d\theta}\right].$$

By defining a complex valued time-frequency coherence function as

$$\rho(n,\theta) = \frac{V(n,\theta)}{\left[\sum\limits_n V(n,\theta)\right]^{1/2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}V(n,\theta)d\theta\right]^{1/2}}, \qquad (10)$$

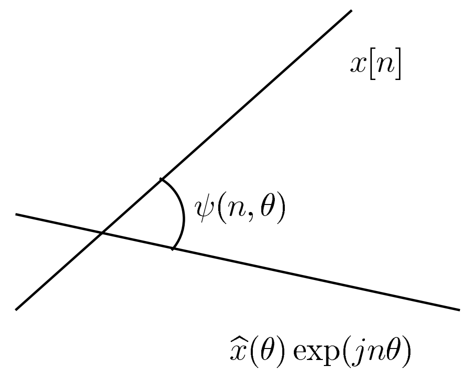

Figure 2: *Hilbert space geometry for stochastic Rihaczek distribution.*

we see that the mean-squared error can be written as

$$Q(n,\theta) = r[n,\theta]\left(1 - |\rho(n,\theta)|^2\right). \qquad (11)$$

Note that $0 \le |\rho(n,\theta)|^2 \le 1$ can also be interpreted as a correlation coefficient or a cosine squared.

This distribution of magnitude-squared coherence over time and frequency seems to be the fundamental descriptor of the Rihaczek time-frequency distribution. The reason is that it defines the cosine-squared of the angle between the complex variable $x[n]$ and its one-term Fourier transform variable $\widehat{x}(\theta)e^{j\theta n}$ in the Hilbert space of complex random variables. This cosine-squared is illustrated in Fig. 2, where $x[n]$ makes an angle $\psi(n,\theta)$ with $\widehat{x}(\theta)e^{j\theta n}$, and $\cos^2\psi(n,\theta) = |\rho(n,\theta)|^2$.

As long in time as the magnitude-squared coherence $|\rho(n,\theta)|^2$ is close to one, then the time series $x[n]$ is estimable from the the one-term Fourier expansion $\widehat{x}(\theta)e^{j\theta n}$. Equivalently, as long in θ as the the magnitude-squared coherence is close to one, the spectrum $\widehat{x}(\theta)$ is estimable from the one-term Fourier transform $x[n]e^{-j\theta n}$. This interpretation is illustrated in Fig. 3, where it is suggested that a time series $x[n]$ which hugs the phasor $\widehat{x}(\theta)e^{j\theta n}$ over a time interval $0 \le n \le N-1$ has a squared coherence near to one, and may be said to have local spectrum $\widehat{x}(\theta)$ over that interval.

3 Time-Varying Channels

Let us now extend this Hilbert space geometry to the analysis of a linear time-varying channel, whose impulse response at time n due to a unit pulse applied at time m is $h[n,m]$. Each of the variables n and m is a global time variable. The output of the channel in

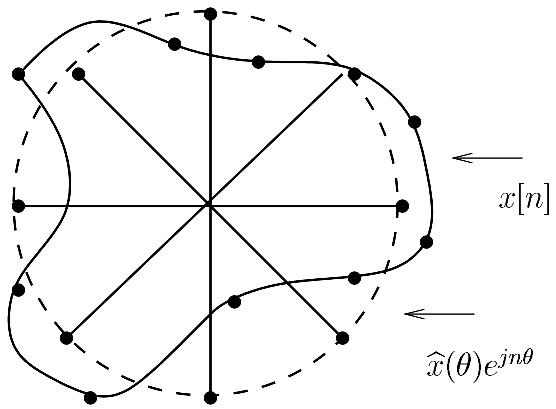

Figure 3: *Approximation of $x[n]$ (full line) by the rotating phasor $\widehat{x}(\theta)\exp(jn\theta)$ (dashed line), for $\theta = 2\pi/8$. Bullets indicate sampling points.*

response to an arbitrary input sequence $x[n]$ is

$$y[n] = \sum_m h[n,m]x[m]. \tag{12}$$

There are two transforms of this unit-pulse response $h[n,m]$ that will be of interest in this section and the next. The first, $H(\theta,m)$, characterizes the Fourier transform of the unit pulse response, and the second, $H(n,\theta)$, the channel response to a complex exponential input.

$$H(\theta,m) = \sum_n h[n,m]e^{-jn\theta}, \tag{13}$$

$$H(n,\theta) = \sum_m h[n,m]e^{jm\theta}. \tag{14}$$

$$H(\theta,\phi) = \sum_n H(n,\phi)e^{-jn\theta} = \sum_m H(\theta,m)e^{jm\phi}. \tag{15}$$

When this channel is excited by zero-mean, stationary white noise of unit variance, then the autocorrelation of the output is

$$\begin{aligned} r[n,k] &= E\{y[n]y^*[n-k]\} = \\ &= \sum_m h[n,m]h^*[n-k,m] \equiv \langle h[n,\cdot], h[n-k,\cdot]\rangle, \end{aligned}$$

which is an inner product between $h[n,m]$ and $h[n-k,m]$. In this formula, n is a global output time variable and k is a local output time variable. The variance of $y[n]$ is $r[n,0] = \sum_m |h[n,m]|^2 > 0$.

The Fourier transform of the autocorrelation function on the global variable n produces the ambiguity function

$$A(\delta,k) = \sum_n r[n,k]e^{-jn\delta}, \tag{16}$$

which cannot be written as an inner product. In this formula, δ and k are local frequency and time variables, respectively, as is usual for the ambiguity function.

The Fourier transform of the correlation sequence on the variable k produces the time-averaged Rihaczek distribution:

$$\begin{aligned} V(n,\theta) &= \sum_k r[n,k]e^{-jk\theta} = \\ &= \sum_m h[n,m]H^*(\theta,m)e^{-jn\theta} = \langle h[n,\cdot], H(\theta,\cdot)e^{jn\theta}\rangle, \end{aligned}$$

which is an inner product between the unit pulse response $h[n,m]$ and its one-term Fourier expansion $H(\theta,m)e^{jn\theta}$. In this formula, n is a global time variable and θ is a global frequency variable. In fact, we may say that $r[n,k]$ and $V(n,\theta)$ are a Fourier transform pair, from which it follows that the time marginal of $V(n,\theta)$ is non-negative:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} V(n,\theta)d\theta = r[n,0] = \sum_m |h[n,m]|^2 > 0. \tag{17}$$

The Fourier transform of $V(n,\theta)$ on the variable n, or $A(\delta,k)$ on the variable k, produces the spectral correlation

$$\begin{aligned} S(\delta,\theta) &= \sum_n V(n,\theta)e^{jn\delta} = \\ &= \sum_m H(\theta+\delta,m)H^*(\theta,m) = \langle H(\theta+\delta,\cdot)H(\theta,\cdot)\rangle \end{aligned}$$

which is an inner product between the complex frequency responses $H(\theta+\delta,m)$ and $H(\theta,m)$. In this function the variable θ is global frequency and the variable δ is local frequency. In fact $V(n,\theta)$ and $S(\delta,\theta)$ are a Fourier transform pair, from which it follows that the frequency marginal of $V(n,\theta)$ is non-negative:

$$\sum_n V(n,\theta) = S(0,\theta) = \sum_m |H(\theta,m)|^2 > 0. \tag{18}$$

In summary, there are four descriptions of a time-varying channel. The two variable correlation function $r[n,k]$ describes the autocorrelation of $h[n,m]$ at two nearby points n and $n-k$, and this is independent of m, because the input is white. The ambiguity function is actually not very illuminating. The time-averaged Rihaczek distribution describes the cross-correlation between the impulse response at time n and its Fourier transform at frequency θ. The spectral correlation describes the spectral autocorrelation between the complex Fourier transform $H(\theta, m)$ at two nearby frequencies θ and δ. The extent to which the impulse response $h[n, m]$ is sinusoidal is measured by the time-frequency coherence between it and the one-term Fourier expansion $H(\theta, m)e^{jn\theta}$.

4 Random Channels

In order to apply these results to random channels, we shall assume that the channel is a wide-sense stationary-uncorrelated sources (WSSUS) channel. The meaning and influence of this assumption will become clear in due course. We shall again be interested in a channel whose unit pulse response is $h[n, m]$.

The first object of interest is the correlation between the channel output at time n to a complex exponential input of frequency θ and a similar output at time $n-k$ to a complex exponential input of frequency $\theta + \delta$:

$$\begin{aligned} &E\{H(n,\theta)H^*(n-k,\theta+\delta)\} = \\ &\quad E\{\sum_m \sum_l h[n,m]e^{jm\theta}h^*[n-k,l]e^{-jl(\theta+\delta)}\}. \end{aligned}$$

Under the WSSUS assumption, the expectation of $h[n, m]h^*[n-k, l]$ is zero for mismatch between global input times m and l, and dependent only on the local difference in output times n and $n-k$:

$$E\{h[n,m]h^*[n-k,l]\} = r[m,k]\delta[m-l] \tag{19}$$

$$r[m,k] = E\{h[n,m]h^*[n-k,m]\} \tag{20}$$

In this equation, m is a global input time, k is a local output time, and $r[m, k]$ is the inner product between $h[n, m]$ and $h[n-k, m]$. This interpretation is very important, and it differs from the interpretation of $r[n, k]$ in the analysis of the time-varying channel with white noise input. For $m = l$ and $k = 0$, the function $r[m, 0] = E\{|h[n, m]|^2\}$ is the global time spread function for the input, which measures the temporal selectivity of the channel. This language is somewhat different than the language of [11], but the equations are the same. As long as the duration of $r[m, 0]$ is small compared with the symbol rate of the channel, then there is little temporal selectivity, or intersymbol interference (ISI), in the channel.

Under the WSSUS assumption, we see that the ambiguity function of the channel can be written as

$$\begin{aligned} &A(\delta,k) = E\{H(n,\theta)H^*(n-k,\theta+\delta)\} = \\ &\quad \sum_m r[m,k]e^{-jm\delta} = \langle H(n,\theta), H(n-k,\theta+\delta)\rangle. \end{aligned}$$

In this ambiguity function, which is the Fourier transform of the correlation function $r[m, k]$, the variable δ is a local frequency variable and the variable k is a local time variable. When evaluated at $k = 0$, this ambiguity function is the spectral coherence function $A(\delta, 0) = E\{H(n, \theta)H^*(n, \theta + \delta)\}$, which measures the frequency selectivity of the channel to sinusoids of nearby frequencies, separated by δ. Again, the language is somewhat different than the language of [11], but the results are the same. When the bandwidth of $A(\delta, 0)$ is larger than the bandwidth of a symbolling waveform, then there is little frequency selectivity in the channel. Of course, the global time spread function $r[m, 0]$ and the spectral coherence function $A(\delta, 0)$ constitute a Fourier transform pair.

The Fourier transform of the correlation function $r[m, k]$ is the stochastic Rihaczek distribution:

$$\begin{aligned} &V(m,\theta) = \sum_k r[m,k]e^{-jk\theta} = \\ &\quad = E\{h[n,m]H^*(\theta,m)e^{-jn\theta}\} = \langle h[n,m], H(\theta,m)e^{jn\theta}\rangle \end{aligned}$$

This inner product describes the cross-correlation between $h[n, m]$, the channel output at global time n to an input at global time m and a single term Fourier series expansion, $H(\theta, m)e^{jn\theta}$, using a single term in the Fourier transform of $h[n, m]$. The extent to which the impulse response $h[n, m]$ is sinusoidal is measured by the coherence between it and the one-term Fourier expansion $H(\theta, m)e^{jn\theta}$.

The Fourier transform of the Rihaczek distribution on the global time variable m, or the Fourier transform of the ambiguity function on the local time variable k gives the spectral correlation function

$$S(\delta,\theta) = \sum_m V(m,\theta)e^{-jm\delta} = \sum_k A(\delta,k)e^{-jk\theta}, \tag{21}$$

which cannot be written as an inner product.

In summary, there are four descriptions of a random channel. The two variable correlation function $r[m, k]$ describes the autocorrelation of $h[n, m]$ at two nearby outputs times n and $n-k$, and a common input time m. The ambiguity function describes the autocorrelation between the output at two nearby times n and $n-k$, when the inputs are complex exponentials with nearby frequencies θ and $\theta + \delta$. The stochastic Rihaczek distribution describes the cross-correlation between the impulse response at time n due to an excitation at time m and its Fourier transform at frequency θ. The extent to which the impulse response $h[n, m]$ is sinusoidal is measured by the coherence between it and the one-term Fourier expansion $H(\theta, m)e^{jn\theta}$.

5 Conclusions

The Cramér-Loève representation for a nonstationary time series suggests that the stochastic Rihaczek distribution $V[n, \theta]$ is the natural time-frequency distribution to associate with the time-series. In [8], drawing on the insights of [2], [3], [4], [6],[7], we showed that the stochastic Rihaczek distribution is, in fact, the Hilbert space inner product of the time series sample $x[n]$ and its global Fourier transform value $\widehat{x}(\theta)$. In this paper we have clarified this geometry by showing that the stochastic Rihaczek distribution determines the Hilbert space angle that the time series makes with its spectrum. We have reviewed this geometry from the point of view of a single time series value and a single spectrum value. But the arguments generalize easily to vectors of time series values within a time domain window and vectors of frequency domain values within a spectral window. Then the appropriate generalization is to canonical angles between a time domain subspace and a frequency domain subspace. This will be clarified in a forthcoming paper.

All of these insights have been extended to the description of time-varying channels and random channels that are WSSUS. For such channels the Rihaczek distribution gives a fine-grained description, which complements the coarse-grained description of the ambiguity function.

The theory of stochastic Rihaczek distributions has recently been extended to higher-order moment spectra [12]. Also in the higher-order case, we find important and useful Hilbert space interpretations.

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