GENERALIZED LAMPERTI TRANSFORMATION OF BROKEN SCALE INVARIANCE

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ABSTRACT

The Lamperti transformation is a powerful tool for studying nonstationary self-similar processes via associated stationary generators. After having discussed how the Lamperti transformation offers a new perspective on the representation of nonstationary self-similar processes, we propose here to enlarge its scope in two manners. First, we show that many kinds of broken self-similarity admit Lamperti counterparts which are weakened forms of stationarity: a prominent example is that of Discrete Scale Invariance (i.e., scale invariance for some preferred scaling factors only), which is the Lamperti image of cyclostationarity. Second, we introduce modified versions of the original Lamperti transformation in order to stationarize other forms of broken scale invariance such as local self-similarity or finite-size scale invariance based on warped dilations properties.

1. LAMPERTI TRANSFORMATION: A NEW READING

Dating back to 1962 and the first ideas on self-similarity, a classical result has been proved by J.W. Lamperti [11]: by means of a suitable invertible transformation, one can map stationary processes onto self-similar processes of index H (H-ss processes), i.e., processes that are invariance under renormalized dilations.

More precisely, the Lamperti transform is defined for a process $\{Y(t), t \in \mathbb{R}\}$ as $(\mathcal{L}_H Y)(t) := t^H Y(\log t)$, and the corresponding inverse Lamperti transform acts on a process $\{X(t), t > 0\}$ according to $({\mathcal{L}_H}^{-1} x)(t) := e^{-Ht} X(e^t)$. The central property is that, given the time-shift operator $(\mathcal{S}_\tau Y) := Y(t+\tau)$ and the (renormalized) dilation operator $(\mathcal{D}_{H,\lambda} X)(t) := \lambda^{-H} X(\lambda t)$, the transformation ensures that

$$ {\mathcal{L}_H}^{-1} \mathcal{D}_{H,\lambda} \mathcal{L}_H = \mathcal{S}_{\log \lambda}. \tag{1} $$

It is of common knowledge indeed that self-similarity and stationarity are mutually exclusive properties. Frequently, one adds a further assumption of stationary increments ("si"): this enables to use usual (stationary) methods of analysis to study H-ss processes from their increments or by means of a wavelet transformation, known to stationarize the wavelet coefficients of H-sssi processes [1]. The property (1) offers a different possibility to cope with nonstationary H-ss processes by studying their stationary generator. A first series of consequences of the aforementioned approach allows therefore to re-issue some problems on the representation of genuinely self-similar processes and their manipulation. Some recent advances in this direction have been proposed in [2, 15, 19], and we will here elaborate on those results.

2. REPRESENTATION OF NONSTATIONARY SELF-SIMILAR PROCESSES

2.1. Filters and generators

In classical linear system theory, it is well-known that linear *filters* are those linear operators $\mathcal{H}$ which are shift-covariant, i.e., such that $\mathcal{H}\mathcal{S}_\tau = \mathcal{S}_\tau \mathcal{H}$ for any $\tau \in \mathbb{R}$. By analogy, it is natural to introduce systems which preserve scaling properties [19, 18]. More precisely, a linear operator $\mathcal{G}$, acting on processes $\{X(t), t > 0\}$, will be said to be scale-covariant if it commutes with any renormalized dilation, i.e., if we have $\mathcal{G}\mathcal{D}_{H,\lambda} = \mathcal{D}_{H,\lambda}\mathcal{G}$ for any H and any $\lambda > 0$. As a consequence of this definition, it can be shown [8] that the Lamperti transform maps linear filters onto scale-covariant systems. Moreover, if an operator $\mathcal{G}$ is scale-covariant, then it necessarily acts on processes $\{X(t), t > 0\}$ as a multiplicative convolution, according to

$$ (\mathcal{G}X)(t) = \int_0^{+\infty} g(t/s)\, X(s)\, ds/s, \tag{2} $$

and scale-covariant operators preserve self-similarity.

Continuing along this analogy, H-ss processes can be represented as the output of scale-covariant systems, as stationary processes are outputs of linear filters. More precisely, stationarity being preserved by linear filtering, stationary processes admit a representation under a convolutive form:

$$ Y(t) = \int_{-\infty}^{+\infty} h(t-s)\, dB(s), \tag{3} $$

with $\mathbb{E} dB(t) dB(s) = \sigma^2 \, \delta(t-s) \, dt \, ds$.

Applying the Lamperti transformation to (3) ends up with the relation

$$(\mathcal{L}_H Y)(t) = \int_0^{+\infty} (\mathcal{L}_H h)(t/s) \, (\mathcal{L}_H dB)(s)/s \tag{4}$$

and, comparing with (2), this corresponds to the output of a linear scale-covariant system whose input is such that

$$\mathbb{E} (\mathcal{L}_H dB)(t) (\mathcal{L}_H dB)(s) = \sigma^2 \, t^{2H+1} \, \delta(t-s) \, dt \, ds. \tag{5}$$

It then follows that any H-ss process $\{X(t), t > 0\}$ can be represented as the output of a linear scale-covariant system of some impulse response $g(.)$:

$$X(t) = \int_0^{+\infty} g(t/s) \, dV(s)/s, \tag{6}$$

with $\mathbb{E} dV(t) dV(s) = \sigma^2 \, t^{2H+1} \, \delta(t-s) \, dt \, ds$.

From a dual perspective, stationary processes $\{Y(t), t \in \mathbb{R}\}$ are known to admit a Cramér representation [16]

$$Y(t) = \int_{-\infty}^{+\infty} e^{i2\pi f t} \, d\xi(f), \tag{7}$$

with spectral increments $d\xi(f)$ such that

$$\mathbb{E} d\xi(f) \overline{d\xi(\nu)} = \delta(f-\nu) \, d\mathbf{S}_Y(f) \, d\nu, \tag{8}$$

and $d\mathbf{S}_Y(f) = \Gamma_Y(f) \, df$ in case of absolute continuity with respect to the Lebesgue measure. Moreover, the Fourier transform of the spectral measure $d\mathbf{S}_Y(f)$ identifies to the covariance function which, in this case, takes necessarily on the form $\mathbf{R}_Y(t,s) := \mathbb{E} Y(t) Y(s) = \boldsymbol{\gamma}_Y(t-s)$, with $\boldsymbol{\gamma}_Y(.)$ a non-negative definite function, and we get from (3) that:

$$d\mathbf{S}_Y(f) = \sigma^2 \, |(\mathcal{F}h)(f)|^2 \, df, \tag{9}$$

where $\mathcal{F}$ stands for the Fourier transformation.

In a very similar way, the covariance function of H-ss processes $\{X(t), t > 0\}$ expresses necessarily as $\mathbf{R}_X(t,s) = (ts)^H \, c_H(t/s)$, with $c_H(\exp(.))$ some non-negative definite function. Given (6), we have explicitly

$$c_H(\lambda) = \sigma^2 \, \lambda^{-H} \int_0^{+\infty} g(\theta) \, g(\lambda\theta) \, d\theta / \theta^{2H+1}, \tag{10}$$

and the power spectrum density of the stationary counterpart $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ of $\{X(t), t > 0\}$ is simply given by

$$\Gamma_{\mathcal{L}_H^{-1} X}(f) = \sigma^2 \, |(\mathcal{M}g)(H + i2\pi f)|^2, \tag{11}$$

where

$$(\mathcal{M}g)(s) := \int_0^{+\infty} g(t) \, t^{-s-1} \, dt \tag{12}$$

stands for the Mellin transform.

The fact that the Mellin transform appears in this context just results from the Lamperti transformation of (7), thanks to which self-similar processes can be expanded on a basis of "chirps" $t^H \exp(i2\pi f \log t)$, obtained as the Lamperti image of "tones" $\exp(i2\pi f t)$.

2.2. Examples

2.2.1. Fractional Brownian motion

Fractional Brownian motion (fBm) $B_H(t)$ [12] is the most well-known example of a "H-sssi" process and it is in fact the only H-sssi process which is Gaussian. Its covariance function reads

$$\mathbf{R}_{B_H}(t,s) = \frac{\sigma^2}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right), \tag{13}$$

with $\sigma^2 := \mathbb{E} B_H^2(1)$ and $0 < H < 1$, thus offering an extension of ordinary Brownian motion $B(t) \equiv B_{1/2}(t)$ (known to have uncorrelated increments) to situations where increments may be correlated (negatively if $0 < H < 1/2$ and positively if $1/2 < H < 1$).

Since fBm is H-ss, its covariance function (13) can be factorized as $\mathbf{R}_{B_H}(t,s) = (ts)^H \, c_H(t/s)$, with

$$c_H(\lambda) = \frac{\sigma^2}{2} \left[\lambda^H + \lambda^{-H} \left(1 - |1-\lambda|^{2H} \right) \right], \tag{14}$$

and it follows that the covariance function of the inverse Lamperti transform $\{Y_H(t) := (\mathcal{L}_H^{-1} B_H)(t), t \in \mathbb{R}\}$ expresses as [8, 19]:

$$\boldsymbol{\gamma}_{Y_H}(\tau) = \sigma^2 \left(\cosh(H|\tau|) - [2\sinh(|\tau|/2)]^{2H}/2 \right). \tag{15}$$

Manipulating (nonstationary) fBm is usually made easier by resorting to its (stationary) increments, although this may be at the expense of facing *long-range dependence* (LRD) when $1/2 < H < 1$. In contrast, it follows from (15) that

$$\boldsymbol{\gamma}_{Y_H}(\tau) \propto e^{-\min(H, 1-H)\tau} \tag{16}$$

when $\tau \to \infty$, which means that the stationary counterpart of fBm is indeed *short-range dependent* for any $H \in (0,1)$, since its correlation function decreases exponentially fast at infinity.

As shown in [14, 15], using the Lamperti transformation in the context of linear estimation of self-similar processes makes possible a number of manipulations (such as whitening or prediction) which otherwise prove much more difficult to handle. Factoring the Fourier transform of (15) and using representations of H-ss processes as in (6), it then becomes possible to, e.g., re-derive representation formulæ for fBm on a finite interval using a finite interval of ordinary Bm (and vice-versa), as well as to get explicit prediction formulæ for fBm.

2.2.2. *Ornstein-Uhlenbeck processes*

If we let $H = 1/2$ in (15), we readily get for the (stationary) covariance function the simple exponential form $\gamma_{Y_{1/2}}(\tau) = \sigma^2 \exp\{-|\tau|/2\}$ which is characteristic of the Ornstein-Uhlenbeck process $\{Y_{1/2}(t), t \in \mathbb{R}\}$, known to be the Lamperti image of ordinary Brownian motion and solution of the Langevin equation:

$$dY(t) + \alpha\, Y(t)\, dt = dB(t), \tag{17}$$

with $\alpha = 1/2$. Lamperti transforming the general Langevin equation (17), and defining $X(t) := (\mathcal{L}_H Y)(t)$, it follows that the process $\{X(t), t > 0\}$ is H-ss and solution of

$$t\, dX(t) + (\alpha - H)\, X(t)\, dt = dV(t), \tag{18}$$

where $dV(t)$ is covariance-equivalent to the "modulated white noise" $d\tilde{V}(t) := t^{H+1/2}\, dB(t)$.

Alternatively, for a given $\alpha > 0$, Ornstein-Uhlenbeck processes admit the integral representation

$$Y_\alpha(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)}\, dB(s), \tag{19}$$

whose Lamperti transform reads

$$X_{\alpha,H}(t) = \int_0^{+\infty} [(t/s)^{H-\alpha}\, u(t/s - 1)]\, [s^{H+1/2}\, dB(s)]/s, \tag{20}$$

with $u(.)$ the unit step function.

We recognize in (20) an example of the general representation (6), with the identification $g(\theta) := \theta^{H-\alpha}\, u(\theta - 1)$ and $dV(t) := t^{H+1/2}\, dB(t)$. This H-ss process has for (nonstationary) covariance function:

$$\mathbf{R}_{X_{\alpha,H}}(t,s) = \sigma^2\, (\min(t,s))^{H+\alpha}\, (\max(t,s))^{H-\alpha}. \tag{21}$$

In the general case of arbitrary α and H, this makes of $\{X_{\alpha,H}(t), t > 0\}$ a versatile 2-parameter model [13], in which H controls self-similarity whereas α may be related to long-range dependence.

2.2.3. *Euler-Cauchy processes*

As a third example, we can briefly mention that (17) can be rewritten as a stochastic (first-order) differential equation, from which $Y(t)$ is usually interpreted as the output of a first-order linear system whose input is white noise $W(t)$. Such a system constitutes a building block for more complicated ones (with elementary sub-systems in cascade and/or in parallel), and it can also be generalized to higher orders, as in ARMA(p,q) processes. The (self-similar) Lamperti counterparts of such (stationary) processes are referred to as Euler-Cauchy processes, and we refer the interested reader to [19] (or [8]) for more details.

3. BROKEN SCALE INVARIANCE

In many cases, exact self-similarity (or exact stationarity) is however a too strong requirement for a proper modeling of real-life situations. In this respect, we propose to enlarge the Lamperti transformation in order to study processes which are not exactly self-similar.

Before pursuing this program, we can first remark that a process with any form of broken scale invariance can no longer be the Lamperti image of a stationary process. It is however always possible to apply the Lamperti transformation to nonstationary processes, resulting in specific classes of processes with broken scale invariance. As a generalization of (7), it is known that nonstationary processes $\{Y(t), t \in \mathbb{R}\}$ still admit a Crámer-like decomposition, but with correlated spectral increments:

$$\mathbb{E} d\xi(f)\overline{d\xi(\nu)} = d^2\Phi_Y(f,\nu), \tag{22}$$

i.e., with spectral masses which are not located along the only diagonal of the frequency-frequency plane. Provided that the Loève's condition [16]

$$\iint_{-\infty}^{+\infty} |d^2\Phi_Y(f,\nu)| < \infty \tag{23}$$

is satisfied, the corresponding nonstationary processes are referred to as *harmonizable*, and such that

$$\mathbf{R}_Y(t,s) = \iint_{-\infty}^{+\infty} e^{i2\pi(ft - s\nu)}\, d^2\Phi_Y(f,\nu). \tag{24}$$

As a follow-up, a companion concept of *multiplicative harmonizability* can be introduced [2] in the case of processes $\{X(t), t > 0\}$ deviating from exact self-similarity. This readily follows from the "lampertization" of (7) which, together with (23), leads to

$$(\mathcal{L}_H Y)(t) = \int_{-\infty}^{+\infty} t^{H+i2\pi\sigma}\, d\xi(\sigma), \tag{25}$$

whereas the restriction of this general expression to the special case of independent spectral increments leads to the representation considered, e.g., in [7, 9, 19].

3.1. Discrete scale invariance

An important example of broken self-similarity which enters the above framework of multiplicative harmonizability is that of *Discrete Scale Invariance* (DSI), i.e., scale invariance for some preferred scaling factors only. While routinely present in toy systems (think of the triadic Cantor set, for which exact replication can only be achieved for scale factors $\{\lambda = 3^k, k \in \mathbb{Z}\}$, or of the Mellin "chirps" of the form $t^H \exp(i2\pi f_0 \log t)$, for which scale invariance applies for $\{\lambda = (\exp(1/f_0))^k, k \in \mathbb{Z}\}$ only), DSI has in fact been recently put forward as a central concept in the study of many critical systems [17], and it has received much attention in a deterministic context. An extension of the DSI concept to stochastic processes has been proposed [2], according to which a process $\{X(t), t > 0\}$ is said to possess a discrete scale invariance of index H and of scaling factor $\lambda_0 > 0$ (or to be "(H, λ_0)-DSI") if $\{(\mathcal{D}_{H,\lambda_0} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}$ for some preferred scale factor λ_0 (and, hence, for any $\lambda = \lambda_0^k, k \in \mathbb{Z}$).

Given this definition, it is straightforward to establish [2] that the Lamperti transformation guarantees a one-to-one correspondence between DSI processes and *cyclostationary* (or *periodically correlated*) processes, i.e., those processes for which $\{(\mathcal{S}_{T_0} Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}$ for some preferred period T_0 only (and all integer multiples $T = kT_0, k \in \mathbb{Z}$) [10]. More precisely, if a process $\{Y(t), t \in \mathbb{R}\}$ is T_0-cyclostationary, its Lamperti transform $\{(\mathcal{L}_H Y)(t), t > 0\}$ is (H, e^{T_0})-DSI. Conversely, if a process $\{X(t), t > 0\}$ is (H, λ_0)-DSI, its inverse Lamperti transform $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ is $(\log \lambda_0)$-cyclostationary.

It is well-known that, in contrast with the stationary case for which the spectral distribution function is entirely concentrated along the main diagonal $\nu = f$ of the frequency-frequency plane, that of cyclostationary processes is also non-zero along equally spaced parallel lines [10]:

$$d^2\Phi_Y(f, \nu) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n(\nu)\, \delta(\nu - (f - n/T_0))\, df\, d\nu. \tag{26}$$

This is a direct consequence of the existence of a Fourier series expansion for the covariance function of cyclostationary processes. Therefore, since DSI processes result from a "lampertization" of cyclostationary processes, we readily get that (H, λ_0)-DSI processes $\{X(t), t > 0\}$ have a covariance function of the form:

$$\mathbf{R}_X(t, kt) = (kt)^H \sum_{n=-\infty}^{\infty} \mathbf{C}_n(\log k)\, t^{H + i2\pi n/T_0}, \tag{27}$$

with $T_0 = \log \lambda_0$.

Stochastic DSI processes can be considered from two different perspectives. On the one hand, "delampertizing" DSI processes make them enter the better known world of cyclostationary processes to which numerous tools have been dedicated (e.g., cyclic spectrum analysis). On the other hand, DSI processes can be viewed as resulting from the "lampertization" of cyclostationary processes, so that it can be imagined to apply to them specific tools which would themselves result from the lampertization of classical ones. We will here not comment further on both aspects, and we refer the interested reader to [3] for a more detailed account including analysis, synthesis, modeling and examples.

3.2. Local self-similarity

Another possibility that we will just mention is to connect *locally self-similar* processes, among them the multifractional Brownian motion (mfBm) (and its generalizations) [5], to *locally stationary* processes. As shown in [3, 8], this can be achieved via the introduction of a local Lamperti transformation around some time t_0, thanks to which a locally stationary generator of the mfBm can be obtained.

3.3. Finite size scale invariance

A third variation is concerned with the experimental evidence that scale invariance is usually only observed over some finite range for scales and/or fields. In order to cope with this issue, it is worth noting that, whereas the renormalized dilation operator $\mathcal{D}_{H,\lambda}$ usually acts on the "natural" representation of a process $X(t)$, other representations may prove more useful for a sake of generalization. Introducing indeed the modified representation $U_X := \log(X/X_0)$ (where $X(t) > 0$ and X_0 is some reference for the field [1]) and the associated logarithmic time warping $s := \log(t/t_0)$ (where $t > 0$ and t_0 is some reference for time), it happens that self-similarity is turned into the equality:

$$U_X(s) \stackrel{d}{=} U_X(s + \mu) - H\mu \tag{28}$$

or, in other words, into the stationarity of $U_X(s) - Hs$.

Following Nottale and Dubrulle [6], the "additive" representation obtained this way can then be modified so as to accommodate for finite size effects pertaining to scale and/or fields. To this end, the idea is to generalize addition to other composition laws so that restriction to a finite interval still guarantees a group structure. In this respect, if we first restrict scale s to belong to some finite interval $]s_-, s_+[$, the solution is given by the composition law [6]:

$$s_1 \odot s_2 := \frac{s_1 + s_2 - s_1 s_2 (1/s_- + 1/s_+)}{1 - s_1 s_2 / s_- s_+}, \tag{29}$$

which of course reduces to addition when the cutoffs s_- and s_+ go to infinity. A companion composition law $\otimes$ can be introduced the same way for fields, thus leading to define *finite size* scale invariance by generalizing (28) according to:

$$U_X(s) \stackrel{d}{=} U_X(s \odot \mu) \otimes g(\mu). \tag{30}$$

In this expression, the function $g(.)$ accounts for a renormalization analogous to the λ^{-H} pre-factor of the ordinary (renormalized) dilation, and it can be further expressed as $g(\mu) = S_\otimes^{-1}(-H S_\odot(\mu))$, where $S_\otimes$ and $S_\odot$ stand for the

[1] Due to space limitations, the discussion is limited here to positive processes, but this positivity restriction can be alleviated at the expense of considering more complicated groups based on modulus and sign [3].

morphisms from the groups defined by the considered interval (in scale or field, respectively) equipped with the corresponding composition law to the real line equipped with the addition [3].

The finite size dilation operator $\mathcal{D}^{fs}_{H,\lambda}$ involved in the right-hand side of (30) can therefore be given the additive representation:

$$(\mathcal{D}^{fs}_{H,\lambda}U_X)(s) := U_X(s \odot \mu) \otimes S_\otimes^{-1}(-HS_\odot(\mu)), \quad (31)$$

from which, mimicking (1), a generalized *finite size* Lamperti transform $\mathcal{L}^{fs}_H$ can be defined via the equivalence:

$$(\mathcal{L}^{fs}_H)^{-1}\mathcal{D}^{fs}_{H,\lambda}\mathcal{L}^{fs}_H = \mathcal{S}_{S_\odot(\log\lambda)}. \quad (32)$$

This transform can finally be written in terms of the original process indexed by the ordinary time variable:

$$(\mathcal{L}^{fs}_H Y)(t) = \exp S_\otimes^{-1}\left\{\log Y(S_\odot(\log t)) + HS_\odot(\log t))\right\}. \quad (33)$$

Specific examples of processes exhibiting such a finite size scale invariance as the result of this generalized lampertization of a given stationary process (e.g., an Ornstein-Uhlenbeck process) will be detailed in [4].

4. CONCLUSION

Because of the ubiquity of scale invariance in a large variety of fields ranging from physics or biology to human activity (network traffic [1], finance [17], . . .), self-similar processes deserve a special attention. Dealing with self-similar processes is however faced with two different difficulties. The first one is that self-similarity goes along with nonstationarity, and the second one is that truely self-similar processes are unlikely to be observed in real-world situations. We have here discussed general frameworks for overcoming such limitations by introducing suitable transformations aimed at putting scale invariance (complete or broken) in correspondence with stationarity (in a strict or weakened form). Among the many issues related to this approach and that have not been addressed here, it is worth to mention that switching from continuous-time to discrete-time is a challenging problem that has already received some partial attention [2, 3] but which will need further studies so as to make of theoretical considerations effective tools for the modeling, analysis or manipulation of processes with broken scale invariance.

5. REFERENCES

[1] P. Abry, P. Flandrin, M. Taqqu, and D. Veitch, "Wavelets for the analysis, estimation, and synthesis of scaling data," in *Self-Similar Network Traffic and Performance Evaluation* (K. Park and W. Willinger, eds.), pp. 39–88, Wiley, 2000.

[2] P. Borgnat, P. Flandrin, and P.-O. Amblard, "Stochastic discrete scale invariance," *IEEE Signal Proc. Lett.*, vol. 9, no. 6, pp. 181–184, 2002.

[3] P. Borgnat, *Modèles et outils pour l'invariance d'échelle brisée: Variations sur la transformation de Lamperti et contributions aux modèles statistiques de vortex en turbulence modeling*, PhD Thesis (in French), École normale supérieure de Lyon, France, 2002.

[4] P. Borgnat, P.-O. Amblard, and P. Flandrin, "Lamperti transformation for finite scale invariance," *Proc. Int. Conf. on Physics in Signal and Image Proc. PSIP-03*, Grenoble, 2003, to appear.

[5] S. Cohen, "Champs localement auto-similaires," in *Lois d'Échelle, Fractales et Ondelettes* (P. Abry, P. Gonçalves and J. Lévy-Vehel, eds.), vol. 1, pp. 107–139, Hermes, 2002.

[6] B. Dubrulle, F. Graner, and D. Sornette (eds.), *Scale Invariance and Beyond*, EDP Sciences - Springer, 1997.

[7] P. Flandrin, "Scale-Invariant Wigner spectra and self-similarity," in *Signal Processing V : Theories and Applications* (L. Torres *et al.*, eds.), pp. 149–152, Elsevier, 1990.

[8] P. Flandrin, P. Borgnat, and P.-O. Amblard, "From stationarity to self-similarity, and back : Variations on the Lamperti transformation," in *Long-Range Dependent Stochastic Processes: Theory and Applications* (G. Raganjaran, ed.), Springer, 2002, to appear.

[9] H.L. Gray and N.F. Zhang, "On a class of nonstationary processes," *J. Time Series Anal.*, vol. 9, no. 2, pp. 133–154, 1988.

[10] H.L. Hurd, "Representation of strongly harmonizable periodically correlated processes and their covariances," *J. Multiv. Anal.*, vol. 29, no. 1, pp. 53–67, 1989.

[11] J. Lamperti, "Semi-stable stochastic processes," *Trans. Amer. Math. Soc.*, vol. 104, pp. 62–78, 1962.

[12] B.B. Mandelbrot and J.W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Rev.*, vol. 10, pp. 422–437, 1968.

[13] E. Noret and M. Guglielmi, "Modélisation et synthèse d'une classe de signaux auto-similaires et à mémoire longue," in *Proc. Conf. Delft (NL) : Fractals in Engineering*, pp. 301–315, INRIA, 1999.

[14] C. Nuzman and H.V. Poor, "Transformed spectral analysis of self-similar processes," in *Proc. CISS'99*, 1999.

[15] C. Nuzman and H.V. Poor, "Linear estimation of self-similar processes via Lamperti's transformation," *J. of Appl. Prob.*, vol. 37, no. 2, pp. 429–452, 2000.

[16] M.B. Priestley, *Spectral Analysis and Times Series*, Academic Press, 1981.

[17] D. Sornette, "Discrete scale invariance and complex dimensions," *Physics Reports*, vol. 297, pp. 239–270, 1998.

[18] G. Wornell, *Signal Processing with Fractals: A Wavelet-Based Approach*, Prentice-Hall, 1996.

[19] B. Yazici and R. L. Kashyap, "A class of second-order stationary self-similar processes for $1/f$ phenomena," *IEEE Trans. on Signal Proc.*, vol. 45, no. 2, pp. 396–410, 1997.