A Time-Frequency Formulation of Optimum Detection

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Abstract—**A time-frequency formulation is proposed for the optimum detection of Gaussian signals in white Gaussian noise. By choosing the Wigner–Ville distribution as the basic time-frequency tool, it is shown that the corresponding receivers generally take the form of a correlation between time-frequency structures, matching mathematical optimality with a physically meaningful interpretation. The case of low SNR is examined in some detail and various examples are considered: deterministic signal, Rayleigh fading signal, random jitter, and random time-varying channel. A general class of time-frequency receivers is also proposed, which admits as limiting cases different known structures, and whose suboptimum performance is evaluated. Possible extensions to more elaborate situations (including parameter estimation) are briefly mentioned.**

I. Introduction

THE optimum detection of signals in noise has already been so often considered in the literature that it could appear somewhat vain to address again classical problems for which solutions are well known. Nevertheless, if *solutions* are given, it is clear that a remaining question is that of their *realization*: the purpose of this paper is therefore to stress the importance of alternative detector configurations which have received little attention in the past, namely, *optimum time-frequency receivers*.

In fact, the realization problem for optimum receivers possesses at least two different aspects: one related to *interpretation* and the other one to *implementation*. Concerning interpretation, new insights can be gained by making use of realizations which match mathematical optimality with the simplest or the most relevant physical description of the considered situation. This is especially clear when we consider the detection of nonstationary processes, a problem for which intuition suggests comparison of time-frequency descriptions, but for which there is no explicit standard procedure for doing so. Moreover, it is expected that a new approach, while leading to known results under a more suggestive formulation w.r.t. interpretation, will also help in handling open problems. Concerning implementation, the choice of a realization is generally imposed by practical considerations related both to feasibility and simplicity: in this respect too, time-frequency receivers can be of some interest, for instance, in the cases where classical architectures must be discarded (such as, e.g., in some animal sonar systems where time-frequency implementations seem more appropriate, as models, than correlators or classical matched filters [1]).

One further interest in formulating optimum detection in terms of time-frequency receivers is at last to mix together interpretation and implementation by using the same tools (time-frequency representations) for both a *description* of signals of interest and the *decisions* which can be inferred from their observation. In fact, this point of view enters a general philosophy of time-frequency representations which deal with signal descriptions equivalent to the classical ones (in the time or frequency domain), but which are more convenient for further processing, since they are displayed in a representation space (the time-frequency plane) fit to the considered situation. Hence, when considering time-frequency representations, the two degrees of freedom of the time-frequency plane should permit one (for instance) to aim the detection procedure at only some subregions of interest in the time-frequency plane, performing at once sophisticated preprocessing (such as time-varying filtering) and optimum detection.

Although not extensively discussed, this problem of performing optimum detection in the time-frequency domain has already been considered in the literature. However, one difficulty encountered in the early approaches was related to the choice of the time-frequency distributions used as descriptions. By using the Rihaczek distribution [2], it was shown [3] that optimum detection can be achieved in terms of time-frequency correlations but, unfortunately, at the expense of a time-frequency description which shares very few of the properties that intuition attaches to a time-frequency representation. On the other hand, the use of the intuitive and classical spectrogram has been advocated [4], [5] for constructing time-frequency receivers (''spectrogram correlators''), but, in this case, the price to be paid was the loss of optimality, requiring additional deconvolution procedures for an exact equivalence with classical solutions. A major achievement has resulted from the comprehensive studies performed recently on time-frequency distributions, and pertaining to both deterministic and random signals: the key point of these studies is that a suitable candidate for matching mathematical optimality with physical interpretation is the Wigner–Ville distribution [6], [7].

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This paper, which extends some previous results, is intended to provide a coherent framework for Wigner–Ville based time-frequency receivers and is organized as follows. Section II is first devoted to addressing the general problem of detecting a nonstationary Gaussian signal in stationary white Gaussian noise and to reviewing the classical solution. Considering then Cohen's class of time-frequency distributions, it is shown that an equivalent time-frequency formulation can be given with any distribution which satisfies a certain inner product invariance law. It is justified that this formulation is restricted to the Wigner–Ville distribution and two special cases are derived: the detection of a deterministic signal and of a Rayleigh fading signal. Section III is concerned with a more detailed discussion of the receiver structure when the SNR is small. The general detection procedure, which is shown to reduce to a time-frequency correlation between two Wigner–Ville representations, is illustrated in two examples: the detection of a randomly jittered signal and that of the output of a random time-varying channel. The solution obtained suggests the definition of a general class of receivers which is studied in Section IV. This class depends on an arbitrary smoothing function and, according to the choice of this function, it is shown that the corresponding time-frequency receiver can vary continuously in between very different structures. The degradation in performance w.r.t. quadrature matched filtering is illustrated on an important example. At last, Section V proposes some possible extensions of the present approach: the reconfiguration of time-frequency detectors for nonwhite noise; the detection of signals with imperfectly known structure and, hence, the possibility of parameter estimation.

II. The Detection Problem

A. Formulation

The detection problem that we address is the following:

$$\left.\begin{aligned} H_0 &: r(t) = w(t) \\ H_1 &: r(t) = w(t) + s(t) \end{aligned}\right\} t \in (T), \tag{1}$$

where the observed (complex) signal $r(t)$ is known on the time interval (T), $w(t)$ is zero-mean complex white Gaussian noise (whose real and imaginary parts are independent and of equal power spectral densities) such that

$$E[w(t)] = 0; \quad E[w(t)\, w^*(u)] = N_0 \delta(t-u), \tag{2}$$

and $s(t)$ is the (complex) nonstationary Gaussian signal to be detected, and characterized by

$$E[s(t)] = m(t); \quad E[(s(t) - m(t))(s(u) - m(u))^*] = K_s(t, u). \tag{3}$$

In (2) and (3), E stands for the expectation operator and the star for complex conjugation.

B. Classical Solution

The addressed problem possesses a well-known solution (see, e.g., [8]) by means of the Karhunen–Loève expansion associated with the covariance function of $s(t)$. More specifically, $s(t)$ admits a doubly orthogonal decomposition

$$s(t) = \sum_n s_n \varphi_n(t) \tag{4}$$

such that

$$E[(s_n - m_n)(s_k - m_k)^*] = \lambda_n \delta_{nk}, \tag{5a}$$

and

$$\int_{(T)} \varphi_n(t)\, \varphi_k^*(t)\, dt = \delta_{nk} \tag{5b}$$

[where δ_{nk} is the Kronecker symbol and

$$m_k = \int_{(T)} m(t)\, \varphi_k^*(t)\, dt \tag{5c}$$

is the projection of $m(t)$ onto $\varphi_n(t)$], if the λ_n and $\varphi_n(t)$ are chosen as the eigenvalues and eigenfunctions, respectively, of the covariance function $K_s(t, u)$, i.e., if

$$\int_{(T)} K_s(t, u)\, \varphi_n(u)\, du = \lambda_n \varphi_n(t); \quad t \in (T). \tag{6}$$

It can then be shown that the optimum detector of $s(t)$ can be written as

$$l_R + l_D \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{7}$$

where γ is a threshold and

$$l_R = \frac{1}{N_0} \sum_n \frac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t)\, \varphi_n^*(t)\, dt \right|^2, \tag{8a}$$

$$l_D = 2 \sum_n \frac{1}{\lambda_n + N_0} \operatorname{Re}\left\{ \left[\int_{(T)} r(t)\, \varphi_n^*(t)\, dt \right] \cdot \left[\int_{(T)} m(t)\, \varphi_n^*(t)\, dt \right]^* \right\}. \tag{8b}$$

Both terms in (8) involve inner products between the observed realization and known characteristics of the process to be detected: the first one, which is a *bilinear* function of the observation, is related to the second-order properties of $s(t)$ around its mean-value (random fluctuations); whereas the second one, which is a *linear* function of the observation, is more specifically aimed at detecting this mean value [which can be thought of as a deterministic component of $s(t)$].

C. Equivalent Time-Frequency Formulation

Since the detection structure (7) involves inner products in the time domain, it is clear that equivalent formulations will be possible in the time-frequency domain with any transformation which preserves inner products. Among all the possible time-frequency representations, we will restrict ourselves here to shift-invariant bilinear distributions, i.e., to members of Cohen's class [9], [11]. This means that if $x(t)$ and $y(t)$ are any complex-valued signals, their cross-time-frequency distribution is of the form

$$C_{xy}(t, \omega; \Pi) = \iint_{-\infty}^{\infty} \Pi(t - t', \omega - \omega') \cdot W_{xy}(t', \omega')\, dt' \frac{d\omega'}{2\pi}, \quad (9)$$

where

$$W_{xy}(t, \omega) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau \quad (10)$$

is their cross-Wigner–Ville distribution [10] and $\Pi(t, \omega)$ is an arbitrary (but normalized) function.

It is worthwhile to point out that numerous proposed time-frequency distributions can be obtained from (9) by specifying the arbitrary function $\Pi(t, \omega)$. Of course, setting $y(t) = x(t)$ and

$$\Pi(t, \omega) = \Pi_W(t, \omega) = 2\pi\delta(t)\,\delta(\omega) \quad (11)$$

leads to the classical (auto) Wigner–Ville distribution [10]

$$C_{xx}(t, \omega; \Pi_W) = W_{xx}(t, \omega). \quad (12)$$

Setting

$$\Pi(t, \omega) = \Pi_R(t, \omega) = 2e^{-j2\omega t} \quad (13)$$

leads to the Rihaczek distribution [2]

$$R_{xx}(t, \omega) = x(t)\, X^*(\omega) e^{-j\omega t}, \quad (14)$$

whereas the choice

$$\Pi(t, \omega) = \Pi_S(t, \omega) = W_{hh}(t, \omega) \quad (15)$$

(where $h(t)$ is some time window) corresponds to the classical spectrogram

$$S_{xx}(t, \omega) = \left| \int_{-\infty}^{\infty} x(u)\, h^*(t - u) e^{-j\omega u}\, du \right|^2. \quad (16)$$

Therefore, in order to obtain formulations by means of formulas similar to (8), but expressed as inner products in the time-frequency plane, a natural requirement is to deal only with distributions (9) which satisfy an inner product conservation law of the type

$$\iint_{-\infty}^{\infty} C_{x_1x_2}(t, \omega; \Pi)\, C^*_{x_3x_4}(t, \omega; \Pi)\, dt \frac{d\omega}{2\pi} = \left[\int_{-\infty}^{\infty} x_1(t)\, x_3^*(t)\, dt \right] \times \left[\int_{-\infty}^{\infty} x_2(t)\, x_4^*(t)\, dt \right]^* \quad (17)$$

for any signals $x_k(t)$; $k = 1, \cdots, 4$. This should provide a natural counterpart, in the time-frequency plane, to usual correlation operations in the time domain.

It turns out that such a relationship holds only for time-frequency distributions characterized by a function $\Pi(t, \omega)$ whose two-dimensional Fourier transform is unimodular, i.e., such that

$$|\pi(\eta, \tau)| = 1, \quad (18)$$

where

$$\pi(\eta, \tau) = \iint_{-\infty}^{\infty} \Pi(t, \omega) e^{-j(\eta t + \omega\tau)}\, dt \frac{d\omega}{2\pi}. \quad (19)$$

A consequence of this result is that classical spectrograms must be discarded for exact time-frequency formulations of (8) since, according to (15), they are characterized by

$$\pi_S(\eta, \tau) = A_{hh}(\eta, \tau), \quad (20)$$

where

$$A_{xy}(\eta, \tau) = \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) y^*\left(u - \frac{\tau}{2}\right) e^{j\eta u}\, du \quad (21)$$

is the (cross) ambiguity function between $x(t)$ and $y(t)$. From (21), it is clear that the (auto) ambiguity function (20) of the window $h(t)$ is a quantity whose modulus cannot be forced to be one everywhere.

According to (18), both the Rihaczek [(14)] and the Wigner–Ville distribution [(12)] seem equally suitable candidates. However, the Wigner–Ville distribution possesses at least two advantages over Rihaczek's:

1) it is a real-valued function; and

2) it possesses the least amount of spread in the time-frequency plane [12], [14], which will permit us later to restrict time-frequency inner products to concentrated subregions of the plane. [An example of this time-frequency concentration property is given below by (26) and (27).]

For these reasons, the Wigner–Ville distribution will be retained as the basic time-frequency tool in the rest of this paper.

Starting again from (8), and using (17), it is straightforward to obtain the equivalent time-frequency formulation (equation (22a) can be found in [13]):

$$l_R = \frac{1}{N_0} \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega) \cdot \left[\sum_n \frac{\lambda_n}{\lambda_n + N_0} W_{\varphi_n\varphi_n}(t, \omega) \right] dt \frac{d\omega}{2\pi}, \quad (22a)$$

$$l_D = 2 \int_{-\infty}^{\infty} \int_{(T)} \operatorname{Re}\left\{ W_{rm}(t, \omega) \right\} \cdot \left[\sum_n \frac{1}{\lambda_n + N_0} W_{\varphi_n\varphi_n}(t, \omega) \right] dt \frac{d\omega}{2\pi}. \quad (22b)$$

Both terms involve an inner product, in the time-frequency plane, between time-frequency structures of the observation and of a reference (depending on known properties of the signal to detect). One can remark that the bilinear (respectively, linear) character of l_R (respectively, l_D) is translated to an auto (respectively, cross) Wigner–Ville distribution of the observation.

D. Two Special Cases

We will now illustrate how the general structure (22) can be simplified in two classical circumstances: that of detecting either a known deterministic signal or a zero-mean (random) Rayleigh fading signal.

1) Deterministic Signal: In this first case, we suppose that

$$s(t) = f(t), \tag{23}$$

where $f(t)$ is deterministic and known, i.e., such that

$$m(t) = f(t); \quad K_s(t, u) = 0. \tag{24}$$

Hence, denoting by E_f the energy of $f(t)$ on (T), we obtain from (22) that

$$l_R = 0 \tag{25a}$$

and

$$l_D = \frac{2}{E_f(N_0 + E_f)} \int_{-\infty}^{\infty} \int_{(T)} \mathrm{Re}\left\{W_{rf}(t, \omega)\right\} \cdot W_{ff}(t, \omega)\, dt\, \frac{d\omega}{2\pi}. \tag{25b}$$

The signal to detect being nonrandom, the optimum detector is linear: it consists of an inner product between the Wigner–Ville distribution of $f(t)$ and the cross-Wigner–Ville distribution of $r(t)$ and $f(t)$, a form proposed in [7].

An example which clearly exhibits the superiority of the Wigner–Ville distribution over Rihaczek's is provided when $f(t)$ is a linear FM sweep. In fact, if

$$f(t) = e^{j(\omega_0 t + (\alpha/2)t^2)}, \tag{26}$$

we have

$$W_{ff}(t, \omega) = 2\pi\delta(\omega - (\omega_0 + \alpha t)), \tag{27}$$

whereas $R_{ff}(t, \omega)$ is spread over the whole time-frequency plane [14]. In this case, (25b) reduces to

$$l_D = \frac{2}{T(N_0 + T)} \int_{(T)} \mathrm{Re}\left\{W_{rf}(t, \omega_0 + \alpha t)\right\} dt, \tag{28}$$

which is only a path integration in the time-frequency plane along the instantaneous frequency line. A localized evaluation of the inner product (25b) is then sufficient, whereas using Rihaczek's distribution would have led to an optimum detector too, but involving the computation of an inner product over the whole time-frequency plane.

2) Rayleigh Fading Signal: In this case, we suppose that

$$s(t) = bf(t), \tag{29}$$

where $f(t)$ is deterministic and known, and b is a zero-mean complex Gaussian random variable (whose real and imaginary parts are independent and have equal variances) such that

$$E[b] = 0; \quad E[|b|^2] = 2\sigma_b^2. \tag{30}$$

It follows from (29) and (30) that

$$m(t) = 0; \quad K_s(t, u) = 2\sigma_b^2 f(t) f^*(u) \tag{31}$$

and, denoting again by E_f the energy of $f(t)$ on (T), (22) reduces to

$$l_R = \frac{2\sigma_b^2}{N_0(N_0 + 2\sigma_b^2 E_f)} \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega) \cdot W_{ff}(t, \omega)\, dt\, \frac{d\omega}{2\pi} \tag{32a}$$

and

$$l_D = 0, \tag{32b}$$

which corresponds to the form given in [6].

The detector is now bilinear ($s(t)$ is in fact purely random) and, according to (17), it provides an alternative interpretation of quadrature matched filtering in terms of an inner product, or *correlation*, of Wigner–Ville distributions since

$$\int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, W_{ff}(t, \omega)\, dt\, \frac{d\omega}{2\pi} = \left| \int_{(T)} r(t) f^*(t)\, dt \right|^2. \tag{33}$$

(Equation (33), which is a special case of (17), is usually referred to as *Moyal's formula* [10]).

Again, in the case of a linear FM sweep, the detection procedure takes on an especially simple form of path integration [6].

III. Low Energy Coherence Case

In this section, we will focus on the detection of a zero-mean Gaussian signal when the SNR is small, a situation referred to as the *low energy coherence case* [15], or *locally optimum detection* [16].

Since $s(t)$ is supposed to be zero mean, the optimum detector reduces to l_R, and hence to (22a). According to the low energy coherence assumption, we have

$$\lambda_n \ll N_0 \tag{34}$$

for all n and, hence, (22a) admits as first-order approximation (and up to a constant factor) the quantity [21]

$$l_1 = \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, E[W_{ss}(t, \omega)]\, dt\, \frac{d\omega}{2\pi}. \tag{35}$$

This follows from the fact that, if the random signal $s(t)$ is decomposed as in (4) and (5), then the expected value of its Wigner–Ville distribution reduces to

$$E[W_{ss}(t, \omega)] = \sum_n \lambda_n W_{\varphi_n \varphi_n}(t, \omega). \tag{36}$$

Thus, for small SNR's, we obtain the meaningful (and intuitive) result that optimum detection is achieved by comparing the time-frequency inner product between the Wigner–Ville distribution of the observation and the expected Wigner–Ville distribution (or *Wigner–Ville spectrum* [17]) of the signal to detect with a threshold: the optimum detector is of the shape of a *time-frequency correlator*.

A. Two Examples

The general detection procedure (35) can be illustrated by two examples for which classical solutions will be recovered: the detection of a randomly jittered signal and that of the output of a random time-varying channel.

1) Random Jitter: In this first example, we suppose that

$$s(t) = bf(t - t')e^{j\omega' t}, \tag{37}$$

where $f(t)$ is deterministic and known, t' and ω' are random variables with a probability density function $G(t', \omega')$, and b is as in (30).

It follows then from the properties of the Wigner–Ville distribution that

$$W_{ss}(t, \omega) = |b|^2 W_{ff}(t - t', \omega - \omega') \tag{38}$$

and, hence,

$$E[W_{ss}(t, \omega)] = 2\sigma_b^2 C_{ff}(t, \omega; G). \tag{39}$$

The resulting optimum detector is, therefore, up to the constant factor $2\sigma_b^2$,

$$l_1 = \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, C_{ff}(t, \omega; G)\, dt\, \frac{d\omega}{2\pi}. \tag{40}$$

If we compare this result to (32a) and (9), we see that, in the jitter case, the time-frequency reference is simply the Wigner–Ville distribution of the known signal, but *smeared* by the probability density function of the jitter, which is exactly what intuition suggests.

A straightforward computation [20] shows that (40) is equivalent to

$$l_1 = \iint_{-\infty}^{\infty} G(\tau, \eta) \left|A_{rf}(\eta, \tau)\right|^2 d\tau\, \frac{d\eta}{2\pi}, \tag{41}$$

which is the classically derived solution [4].

2) Random Time-Varying Channel: In this second example, we suppose now that $s(t)$ is the output of a random time-varying channel [with time-varying impulse response $h(t, u)$] whose input is a deterministic and known signal $f(t)$.

We have then

$$s(t) = \int_{-\infty}^{\infty} h(t, u) f(u)\, du. \tag{42}$$

If we suppose furthermore that

$$E[h(t, u)] = 0; \quad E[h(t_1, u_1)\, h^*(t_2, u_2)] = k(t_1 - t_2, u_1)\, \delta(u_1 - u_2), \tag{43}$$

we readily obtain

$$E[W_{ss}(t, \omega)] = C_{ff}(t, \omega; S), \tag{44}$$

where

$$S(t, \omega) = \int_{-\infty}^{\infty} k(t, u) e^{-j\omega u}\, du \tag{45}$$

is the *scattering function* of the channel [18].

This means that the Wigner–Ville spectrum of the output of the considered channel is nothing other than the Wigner–Ville distribution of the input, *smeared* in both time and frequency by the scattering function, which is in direct accordance with physical interpretation. It should be noted that the result (44) appears also in [19] and, of course, a smearing relation of this type still holds for any distribution linearly related to the Wigner–Ville distribution, such as the spectrogram [4]. Nevertheless, the key role played in this context by the Wigner–Ville distribution is again related to its maximum energy concentration properties [24], which could be of interest for identifying scattering functions by means of (44).

Once (44) has been established, it is clear that the corresponding optimum detector is

$$l_1 = \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, C_{ff}(t, \omega; S)\, dt\, \frac{d\omega}{2\pi}, \tag{46}$$

which is of the same structure as (40).

IV. A General Class of Receivers

The last two examples exhibited a similar receiver structure that computed the time-frequency correlation between the Wigner–Ville distribution of the observation and a smoothed Wigner–Ville distribution of a reference signal, which is in fact a generalization of the time-frequency equivalent quadrature matched filtering (32a).

In this section, we will be primarily interested in considering receivers like (40) or (46) for themselves, interpreting them and evaluating their performances.

This will provide a coherent framework for dealing in a unique and physically meaningful formulation with receivers which, otherwise, could be thought of as totally different and unrelated.

A. Interpretations and Limiting Cases

Given an observation $r(t)$ on a time interval (T) and some reference signal $f(t)$, we consider the general class of receivers [20]

$$\Lambda = \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, C_{ff}(t, \omega; \Pi)\, dt\, \frac{d\omega}{2\pi}, \tag{47}$$

where $\Pi(t, \omega)$ is some arbitrary smoothing function.

The first interpretation, which has already been mentioned, is to compare the time-frequency structure of the observation to a smoothed time-frequency structure of a reference signal, in order to handle, for instance, an *a priori* distribution of parameters. If the smoothing function $\Pi(t, \omega)$ is real valued and even (which is not a re-

TABLE I
DETECTOR CONFIGURATIONS RESULTING FROM THE CHOICE OF DIFFERENT SMOOTHING FUNCTIONS IN (47)

$\Pi(t, \omega)$	Λ	Type
$2\pi\delta(t)\,\delta(\omega)$	$\left\| \int_{(T)} r(t) f^*(t)\, dt \right\|^2$	Quadrature matched filtering
$\delta(t)$	$\int_{(T)} \|r(t)\|^2 \|f(t)\|^2\, dt$	Time intensity correlator
$2\pi\delta(\omega)$	$\int_{-\infty}^{\infty} \|R(\omega)\|^2 \|F(\omega)\|^2 \frac{d\omega}{2\pi}$	Spectral density correlator
1	$\int_{(T)} \|r(t)\|^2\, dt \cdot \int_{(T)} \|f(t)\|^2\, dt$	Energy detector

strictive assumption), it is straightforward to verify that we have in an equivalent way

$$\Lambda = \int_{-\infty}^{\infty} \int_{(T)} C_{rr}(t, \omega; \Pi)\, W_{ff}(t, \omega)\, dt\, \frac{d\omega}{2\pi}. \quad (48)$$

This leads to a second interpretation in which the function $\Pi(t, \omega)$ plays the role of an *a posteriori* smoothing on the observation, as it is known to be necessary if we want at the same time to make an estimate of the time-frequency structure of the observation [17].

In both cases, it is interesting to look at what the unique formulation (47) or (48) is equivalent to, when the smoothing function is forced to simple limiting cases: this is summarized in Table I (examples of the second and third configurations can be found, respectively, in [22] and [23]).

We see from this that very different receivers can be viewed in fact as special cases of a unique formulation. In any other case different from those mentioned, the corresponding receiver lies somewhere in between these limiting structures. For instance, if we assume some smearing in frequency according to

$$\Pi(t, \omega) = \delta(t)\, H(\omega), \quad (49)$$

we obtain a detector structure of the type

$$\Lambda = \iint_{(T)} [r(t) f^*(t)][r(u) f^*(u)]^* h(u - t)\, dt\, du, \quad (50)$$

where $h(t)$ is the inverse Fourier transform of $H(\omega)$.

B. Performances

Performances in this general class of receivers can be evaluated by means of a deflection criterion [8]

$$d = \frac{\left| E[\Lambda | H_1] - E[\Lambda | H_0] \right|}{\left(\text{var}\,[\Lambda | H_0]\right)^{1/2}}. \quad (51)$$

According to the definition (47), we obtain as a result

$$d = \frac{1}{N_0} \frac{\int_{-\infty}^{\infty} \int_{(T)} W_{ff}(t, \omega)\, C_{ff}(t, \omega; \Pi)\, dt\, \frac{d\omega}{2\pi}}{\left(\int_{-\infty}^{\infty} \int_{(T)} C_{ff}^2(t, \omega; \Pi)\, dt\, \frac{d\omega}{2\pi} \right)^{1/2}}. \quad (52)$$

If we apply Schwarz inequality to the numerator of (51), we conclude that

$$d \leq \frac{E_f}{N_0}, \quad (53)$$

where, as previously, E_f is the energy of the reference signal $f(t)$ on the considered time interval (T).

As expected, it follows from (53) that the general receiver (47) is suboptimum w.r.t. quadrature matched filtering (33) for which (53) is an equality.

The loss of performance which occurs when passing from (33) to (47) can be quantitatively evaluated in some cases. For instance, if we consider the FM signal with a Gaussian envelope

$$f(t) = e^{-\pi(1 - j\Omega_f T_f/2)(t/T_f)^2} \quad (54)$$

and the doubly Gaussian smoothing function

$$\Pi(t, \omega) = e^{-\pi((t/T_\pi)^2 + (\omega/\Omega_\pi)^2)}, \quad (55)$$

we obtain, when $\Omega_f T_f >> 1$, the following results:

$$T_\pi = (2\pi T_f/\Omega_f)^{1/2}, \quad \Omega_\pi = (2\pi \Omega_f/T_f)^{1/2} \Rightarrow d = \frac{E_f}{N_0} \cdot (2\pi)^{1/4} O\left((\Omega_f T_f)^{-1/4}\right) \quad (56)$$

and

$$T_\pi = T_f, \quad \Omega_\pi = \Omega_f \Rightarrow d = \frac{E_f}{N_0} 2(2\pi^2/9)^{1/4} \cdot O\left((\Omega_f T_f)^{-1/2}\right). \quad (57)$$

This gives an indication of the asymptotic behavior, for large bandwidth-duration products of $f(t)$, of the smoothed time-frequency receiver (48) when the smoothing is either matched to the FM rate [(56)] or equivalent to its time-frequency spread [(57)]. This latter situation corresponds essentially to an energy detector.

V. POSSIBLE EXTENSIONS

The theory developed up to this point has focused on the detection of a Gaussian signal of known structure in white Gaussian noise. In fact, it can be also adapted to take into account more elaborate situations and to give new insights into some decision problems.

In this section, we will mention, without entering into details, two of the natural extensions suggested by the preceeding discussion: the reconfiguration of time-frequency detectors for nonwhite noise; and the detection of signals with imperfectly known structure and, hence, the possibility of parameter estimation.

A. Nonwhite Noise

A first possible modification is to replace in (1) $w(t)$ by some colored noise

$$n(t) = c(t) + w(t), \quad (58)$$

with

$$K_n(t, u) = K_c(t, u) + N_0 \delta(t - u). \quad (59)$$

Classically [8], the detection of the Rayleigh fading signal (29) is then achieved by preserving the structure (33), but with $f(t)$ replaced by its whitened version

$$g(t) = \int_{(T)} K_n^{(-1)}(t, u) f(u)\, du, \quad (60)$$

where $K_n^{(-1)}(t, u)$ is the inverse kernel of the covariance $K_n(t, u)$ of the colored noise, i.e., the function which satisfies

$$\int_{(T)} K_n^{(-1)}(t, u)\, K_n^*(u, v)\, du = \delta(t - v). \quad (61)$$

Given $K_n(t, u)$, and hence $K_n^{(-1)}(t, u)$, a detector equivalent to (33) is then given by

$$l = \left| \int_{-\infty}^{\infty} \int_{(T)} W_{rf}(t, \omega)\, E[W_{nn}^{(-1)}(t, \omega)]\, dt \frac{d\omega}{2\pi} \right|^2, \quad (62)$$

with

$$E[W_{nn}^{(-1)}(t, \omega)] = \int_{-\infty}^{\infty} K_n^{(-1)}\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau. \quad (63)$$

In fact, denoting the eigenvalues and eigenfunctions of the covariance function $K_c(t, u)$ by μ_n and $\psi_n(t)$, respectively, we have

$$E[W_{nn}^{(-1)}(t, \omega)] = \sum_n \frac{1}{\mu_n + N_0} W_{\psi_n\psi_n}(t, \omega) \quad (64)$$

and, hence, labeling by L_D the complex quantity upon which the real part operator is acting in (22b), we get

$$l = \tfrac{1}{4} |L_D|^2, \quad (65)$$

where $f(t)$ plays the role of the deterministic component of $s(t)$, and $K_c(t, u)$ that of its centered covariance function.

Thus, the result (62) is similar to (22b), but with the real part operator replaced by a square law device for taking into account the multiplicative random variable b.

B. Parameter Estimation

Considering again the Rayleigh fading signal (29), it has been shown [6] that the corresponding optimum time-frequency detector (33) could be extended to the case

$$s(t) = bf(t; \boldsymbol{\theta}), \quad (66)$$

where $\boldsymbol{\theta}$ is a vector of unknown parameters. The detector structure is then based on a *generalized likelihood ratio test* [25] and takes on the form

$$l(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} \int_{(T)} W_{rr}(t, \omega)\, W_{ff}(t, \omega; \boldsymbol{\theta})\, dt \frac{d\omega}{2\pi}; \quad (67a)$$

$$\max_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \quad (67b)$$

where $W_{ff}(t, \omega; \boldsymbol{\theta})$ stands for the Wigner–Ville distribution of $f(t; \boldsymbol{\theta})$.

This permits one to perform simultaneously a maximum likelihood estimation $\boldsymbol{\theta}_{ML}$ of $\boldsymbol{\theta}$ according to

$$\boldsymbol{\theta}_{ML} = \left\{ \arg \max_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) | H_1 \right\}. \quad (68)$$

Wigner–Ville based receivers can then be used for both the optimum detection of signals with some unknown parameters and the maximum likelihood estimation of these parameters.

VI. Conclusion

We have shown that classical receiver structures designed for the optimum detection of Gaussian signals in Gaussian noise admit equivalent formulations in the time-frequency plane. These alternative realizations take on the general form of a time-frequency correlator which demonstrates the great advantage of matching optimality with physical interpretation if the Wigner–Ville distribution is used as the basic time-frequency tool. A versatile general class of time-frequency receivers has also been proposed, which admits as special cases different receivers which, otherwise, would be thought of as completely unrelated.

Apart from unifying interpretations, and providing possible implementations when classical structures are not feasible, the proposed approach permits one to handle simultaneously the *analysis* of signals and the *decisions* inferred from their observation. This can be of special interest when some learning process is necessary prior to the detection itself, such as, e.g., in fault detection problems for which the method has already proved useful [26]. More generally, it is believed that the time-frequency formulation of optimum detection can provide new hints for handling open problems in a comprehensive way.

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