On the Spectrum of Fractional Brownian Motions

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Abstract —Fractional Brownian motions (FBM's) provide useful models for a number of physical phenomena whose empirical spectra obey power laws of fractional order. However, due to the nonstationarity of these processes, the precise meaning of such spectra remains generally unclear. Two complementary approaches are proposed which are intended to clarify this point. The first one, based on a time–frequency analysis, takes into account the nonstationary nature of FBM and puts emphasis on time-averaged measurements; the second one, based on a time–scale analysis, is matched to self-similarity properties of FBM and reveals an underlying stationary structure relative to each time-scaling.

I. Introduction

In a number of physical phenomena which involve long-term dependencies, measured spectra exhibit laws of the type $|\omega|^{-2H-1}$, $0 < H < 1$, over a wide range of frequencies (see, e.g., [9], [13] and references therein). An apparent contradiction exists between the stationarity assumption upon which usual spectral concepts are based and the fact that such spectral behaviors cannot be associated with stationary processes: thus, the precise meaning of the notion of spectrum needs to be clarified.

Powerful models with self-similarity properties, namely fractional Brownian motions (FBM's), have been proposed [11] for getting better insight into such issues and have been applied successfully in different areas [5], [8], [10]. The question of associating a spectrum has not been explicitly addressed, however. A common way of approaching it has been the use of the following fundamental property of FBM: although FBM is not itself stationary, its increments are (and hence its derivative); this allows one to associate a well-defined spectral representation [11], [14] with the increments. Starting with this observation, a valid approach is given by Mandelbrot and Van Ness [11, p. 436]:

> This suggests that [FBM] has a "spectral density" proportional to $|\omega|^{-2H-1}$. Spectral densities of nonstationary random functions are, however, difficult to interpret. It is tempting to differentiate [FBM] and claim that [its derivative] has a spectral density proportional to $|\omega|^{1-2H}$.

Clearly, this is not a fully satisfactory interpretation; moreover, spectral densities of nonstationary random functions are not obviously defined, and their definition is a natural prerequisite to addressing the problem. A further remark is that, even if a well-defined (time-dependent) spectrum is given, its measurement is another problem which also must be taken into account to establish meaningful links with data stemming from experiments.

The purpose of this correspondence is to make precise the concept of spectrum for FBM's by making use of measurement-based analyses. This is achieved by first performing a time–frequency analysis, which is well-suited to the nonstationary character of FBM's, and then a time–scale analysis aimed at examining their self-similarity properties.

II. Fractional Brownian Motions

Let H be a parameter such that $0 < H < 1$ and $B(t)$ is the ordinary Brownian motion. Following [1], we adopt as a definition of fractional Brownian motion the following (slightly specialized) version of the definition proposed by Mandelbrot and Van Ness in [11]:

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left[\int_{-\infty}^{0}\left(|t-s|^{H-1/2} - |s|^{H-1/2}\right) dB(s) + \int_0^t |t-s|^{H-1/2}\, dB(s)\right]. \quad (1)$$

This clearly generalizes ordinary Brownian motion, the latter appearing as a special case of FBM with parameter $H = 1/2$. FBM possesses numerous interesting properties; among them, we can mention the fact that its increments are stationary and self-similar. This means that, for any scale parameter $a > 0$ and any time t,

$$\left(B_H(t+a\tau) - B_H(t)\right) \overset{d}{=} \left(a^H B_H(\tau)\right) \quad (2)$$

in the sense of equality in distribution.

However, FBM's themselves are not stationary processes: this can be seen by inspection of their covariance function, which reads [1]

$$r_{B_H}(t,s) = E[B_H(t) B_H(s)] = (V_H/2)\cdot\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right], \quad (3)$$

with

$$V_H = \Gamma(1-2H)\frac{\cos \pi H}{\pi H}. \quad (4)$$

As a consequence, the question of associating a spectrum to FBM cannot be solved directly by the use of a standard power spectral density. One possible way out of the difficulty is based on introducing a time-dependent spectrum by performing a time–frequency analysis.

III. Time–Frequency Analysis of FBM

Although in nonstationary cases no unique tool exists for performing a time-dependent spectral analysis (see, e.g., the discussion in [7]), a series of recent works has emphasized the usefulness of the so-called Wigner–Ville spectrum [12].

By definition, the Wigner-Ville spectrum (WVS) of a nonstationary process $x(t)$ with covariance function $r_x(t,s)$ is given by

$$W_x(t,\omega) = \int_{-\infty}^{+\infty} r_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-i\omega\tau}\, d\tau. \quad (5)$$

This (not necessarily positive) quantity possesses numerous interesting properties [12], generalizes in a natural way the concept of power spectral density for stationary cases, and reduces to it if the process happens to be stationary.

If we apply this definition to FBM, we get, by substituting (3) into (5), the following expression:

$$W_{B_H}(t,\omega) = \left(1 - 2^{1-2H}\cos 2\omega t\right)\cdot\frac{1}{|\omega|^{2H+1}}. \quad (6)$$

A number of interesting properties can now be deduced from this time–frequency description.

A. Self-Similarity

If we introduce scaled FBM's

$$B_{H;a}(t) = B_H(at), \qquad a > 0, \quad (7)$$

we know from the properties of the WVS that

$$W_{B_{H;a}}(t,\omega) = \frac{1}{a} W_{B_H}\left(at, \frac{\omega}{a}\right). \quad (8)$$

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It follows from (6) that

$$W_{B_{H;a}}(t,\omega) = W_{a^H B_H}(t,\omega), \tag{9}$$

which is a second-order manifestation of FBM's self-similarity.

B. Fractional Gaussian Noise

In a way similar to the definition of white Gaussian noise as the derivative of ordinary Brownian motion it is possible to define fractional Gaussian noise (FGN) as the derivative of FBM [11]. Denoting by x' the derivative of a process x, we can deduce from the structure of the WVS that

$$W_{x'}(t,\omega) = \omega^2 W_x(t,\omega) + \frac{1}{4}\frac{\partial^2}{\partial t^2} W_x(t,\omega). \tag{10}$$

In the case of FBM this yields

$$W_{B'_H}(t,\omega) = \frac{1}{|\omega|^{2H-1}}. \tag{11}$$

The quantity obtained represents the desired spectral behavior of FGN [11]. It does not depend upon time, and this fact expresses the stationarity of the process considered, resulting from that of the increments of FBM.

Stationarity of the increments is also attested to by the WVS: if we let

$$X_{H;T}(t) = B_H(t+T) - B_H(t), \tag{12}$$

we obtain, from the definition (5),

$$W_{X_{H;T}}(t,\omega) = 4\left(\sin\frac{\omega T}{2}\right)^2 \cdot \frac{1}{|\omega|^{2H+1}}. \tag{13}$$

This is a time-independent nonnegative quantity which corresponds to the power spectral density of a stationary process. Noting that the derivative of FBM can be viewed as

$$B'_H(t) = \lim_{\delta\to 0} \frac{1}{\delta} X_{H;\delta}(t), \tag{14}$$

we can check from (13) that

$$\lim_{\delta \to 0} W_{(1/\delta)X_{H;\delta}}(t,\omega) = \frac{1}{|\omega|^{2H-1}}, \tag{15}$$

which is exactly the expression (11) obtained before, and corresponding to the spectrum of FGN.

C. Average Spectra

Since the WVS provides us with a time-dependent spectrum, it is possible to deduce from it an average spectrum over a time interval of length T. This operation may be expressed as

$$S_x(\omega;T) = \frac{1}{T}\int_0^T W_x(t,\omega)\,dt. \tag{16}$$

Applying this definition to FBM, we get from (6)

$$S_{B_H}(\omega;T) = \left[1 - 2^{1-2H}\cdot\frac{\sin 2\omega T}{2\omega T}\right]\cdot\frac{1}{|\omega|^{2H+1}}. \tag{17}$$

A first consequence of this is that, if T is chosen in a suitable way, the oscillations of (17) vanish, namely,

$$S_{B_H}\left(\omega; k\frac{\pi}{2\omega}\right) = \frac{1}{|\omega|^{2H+1}}, \qquad k = \cdots,-1,0,+1,\cdots. \tag{18}$$

From a practical point of view, the definition of (16) can be viewed as related to a measurement [12]; indeed, measurement at a given frequency requires an analysis time interval whose duration is related to the inverse of that frequency. Therefore, the second consequence of (18) is that, for any given frequency ω_0,

$$T \gg \frac{1}{\omega_0} \Rightarrow S_{B_H}(\omega_0;T) \cong \frac{1}{|\omega_0|^{2H+1}} \tag{19}$$

and, obviously,

$$S_{B_H}(\omega) = \lim_{T\to\infty} S_{B_H}(\omega;T) = \frac{1}{|\omega|^{2H+1}}. \tag{20}$$

The desired spectral behavior of FBM (power law of fractional order) is thus obtained as the result of a measurement process associated with a time-dependent spectrum.

D. Positivity

We note that

$$W_{B_H}(t,\omega) \geq 0 \Leftrightarrow H \geq 1/2. \tag{21}$$

This shows that FBM's with parameter $1/2 \leq H < 1$ are examples of nonstationary processes with a nonnegative WVS [3]. This gives us an alternative way of thinking about this range of values of the parameter H [11].

IV. Time–Scale Analysis of FBM

Due to the self-similarity properties of FBM's, a second approach is to analyze them by means of a time–scale method, aimed at examining their behavior relative to different observation scales. A simple way of performing such an analysis is to make use of the so-called *wavelet transform* [2], [4], [6] which, for a process $x(t)$, is expressed as

$$T_x(t,a) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} x(s)\,g\left(\frac{s-t}{a}\right)ds. \tag{22}$$

In (22), $a > 0$ is the scale parameter and $g(t)$ is an arbitrary (but localized) analyzing wavelet, normalized so that its Fourier transform $G(\omega)$ satisfies $G(0) = 0$ and

$$\int_0^{+\infty} |G(\omega)|^2 \cdot \frac{d\omega}{\omega} = 1. \tag{23}$$

Given a basic wavelet $g(t)$, (22) exhibits a multiscale analysis by filtering $x(t)$ with dilated and compressed versions of a unique observation system.

In the case of a zero-mean random process, the transform (22) is itself a zero-mean random process, and a quantity of interest is its second-order behavior with respect to time. Introducing

$$R_x(t,s;a) = E[T_x(t,a)T_x(s,a)] \tag{24}$$

and using the fact that $G(0) = 0$, we obtain from (3)

$$R_{B_H}(t,s;a) = -(V_H/2)\,a^{2H+1}\cdot\int_{-\infty}^{+\infty}\gamma_g\left(\tau - \frac{t-s}{a}\right)|\tau|^{2H}\,d\tau, \tag{25}$$

with

$$\gamma_g(\tau) = \int_{-\infty}^{+\infty} g(\theta)\,g(\theta-\tau)\,d\theta. \tag{26}$$

Equation (25) exhibits two features worth noting. The first is that, when analyzed relative to a given scale, FBM appears to be *stationary*. The second is that the relevant parameter in (25) is the ratio $(t-s)/a$, which is a new expression of self-similarity in the sense that second-order statistical properties, relative to a scaled time interval $b(t-s)$, $b > 0$, are identical (up to a scale-dependent factor) to those of the unchanged interval but observed relative to an accordingly modified scale a/b.

Since the covariance (25) is a function of t and s only through the difference $(t-s)$, it is possible to obtain from this expression a power spectral density by taking the Fourier transform. The result is

$$S'_{B_H}(\omega; a) = a|G(a\omega)|^2 \cdot \frac{1}{|\omega|^{2H+1}}, \tag{27}$$

which corresponds to the FBM spectrum, as seen through the filter of nominal scale a.

In fact, since (22) is related to $x(t)$ by a linear filter with impulse response

$$h_a(t) = \frac{1}{\sqrt{a}} g\left(-\frac{t}{a}\right), \tag{28}$$

it would have been possible to obtain the result of (27) by using the fact that, in this case,

$$S'_{B_H}(\omega; a) = \int_{-\infty}^{+\infty} W_{h_a}(t-s, \omega) W_{B_H}(s, \omega)\, ds. \tag{29}$$

Equation (29) results from the compatibility of the WVS with linear filtering and its reduction to power spectral density in the case of stationary processes [12]. It can be checked by direct computation that substitution of (6) and (28) into (29) leads to (27).

Since the spectral content of FBM is characterized relative to each scale by (27), it is at last possible to obtain a global spectrum by adding up the contributions pertaining to each scale. Taking into account the natural measure associated with the energy distribution of the wavelet transform in the scale direction [6], we end up with the result

$$S'_{B_H}(\omega) = \int_0^{+\infty} S'_{B_H}(\omega; a) \cdot \frac{da}{a^2} = \frac{1}{|\omega|^{2H+1}}, \tag{30}$$

which is again the desired behavior for a spectrum of FBM.

V. Conclusion

Two approaches have been proposed for the analysis of FBM's. The first one focused on their nonstationary character; by means of a time-frequency analysis, it was shown that their spectral behavior can be described in terms of average spectra. The second approach focused on self-similarity properties of FBM's; by means of a time-scale analysis, a decomposition in terms of self-similar stationary processes was effected. In both cases, the description of a global spectral behavior of FBM's relative to a measurement process (either a time average or a scale filter) was achieved, providing new insight into investigations of such processes.

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Optimum Linear Causal Coding Schemes for Gaussian Stochastic Processes in the Presence of Correlated Jamming

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Abstract —**The complete solution is obtained to the following problem. A Gaussian stochastic process $\{\theta_t,\ t \in [0, t_f]\}$ satisfying a certain stochastic differential equation is to be transmitted through a stochastic channel to a receiver under minimum mean-squared error distortion measure. The channel is to be used for exactly t_f seconds and, in addition to white Gaussian noise with a given energy level, the channel is corrupted by another source whose output may be correlated with the input to the channel and which satisfies a given power constraint. There is an input power constraint to the channel, and noiseless feedback is allowed between the receiver (decoder) and the transmitter (encoder). We determine the linear causal encoder and decoder structures that function optimally under the worst admissible noise inputs to the channel. The least favorable probability distribution for this unknown noise is found to be Gaussian and is correlated with the transmitted signal. Also included is a comparative study of these results with earlier ones that addressed a similar problem without a causality restriction imposed on the transmitter.**

I. Introduction and Problem Description

Consider the communication system depicted in Fig. 1. A stochastic process $\{\theta_t\}$, $t \in [0, t_f]$, which satisfies the equation

$$d\theta_t = a(t)\theta_t\, dt + b(t)\, dV_t; \qquad \theta_0 \sim N(0, \gamma), \tag{1}$$

where V_t is a standard Wiener process and is independent of θ_0, and $a(t)$ and $b(t)$ are uniformly bounded (i.e., for some $k > 0$, $|a(t)| \le k$, $|b(t)| \le k$), is to be transmitted through a continuous-time stochastic channel which is corrupted by additive white Gaussian noise and is also tapped by an intelligent jammer. The jammer sends a stochastic process (possibly correlated with the encoded message process) at a certain power level so as to maximize the interference to the channel. The transmission channel is to be used for exactly t_f seconds and has an instantaneous

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