On the Existence of Discrete Wigner Distributions

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***Abstract*—Among the myriad of time-frequency distributions, the Wigner distribution stands alone in satisfying many desirable mathematical properties. Attempts to extend definitions of the Wigner distribution to discrete signals have not been completely successful. In this letter, we propose an alternative definition for the Wigner distribution, which has a clear extension to discrete signals. Under this definition, we show that the Wigner distribution does not exist for certain classes of discrete signals.**

I. Introduction

In signal processing, one is often interested in four different types of signals characterized by being either continuous or discrete, and by being either aperiodic or periodic.[1] The four types of signals are listed in Table I along with the signal properties in the time domain and the corresponding Fourier transform. The Wigner (or Wigner–Ville) distribution was originally defined for type I signals and is usually presented in the following form [1]–[5]:

$$W_x^{\mathrm{I}}(t,\omega) = \int x\Big(t+\frac{\tau}{2}\Big)x^*\Big(t-\frac{\tau}{2}\Big)e^{-j\omega\tau}\,d\tau. \tag{1}$$

The Wigner distribution satisfies many desirable properties proposed for time-frequency distributions [4], [5].

Since there are four types of Fourier transforms, it is reasonable to assume that there could potentially be four types of Wigner distributions. To avoid confusion, the original Wigner distribution will be referred to as the type I Wigner distribution and the potential, discrete Wigner distributions will be referred to as the type II, III, and IV Wigner distributions. It is straightforward to compute samples of the type I Wigner distribution in the time-frequency plane [6], [7], but this is not necessarily the same as computing a type II Wigner distribution. For comparison, note that the type II spectrogram is not a sampled version of the type I spectrogram [8].

II. Definitions of the Wigner Distribution

The Wigner distribution was originally defined [1], [2] as in (1) with no obvious means for extending the definition to the three types of discrete signals presented above. Here, we briefly present three methods that have been used to create discrete Wigner distributions and indicate their shortcomings. We then present a fourth method that extends easily to the three types of discrete signals.

TABLE I
Four Types of Signals

Type	Time Domain Properties	Spectrum
type I	continuous and aperiodic	Fourier transform
type II	discrete and aperiodic	discrete-time Fourier transform
type III	continuous and periodic	Fourier series
type IV	discrete and periodic	discrete Fourier transform

TABLE II
Six Properties for Time-Frequency Distributions

(i)	quadratic	$T_x(t,\omega) = \iint x(t_1)\,x^*(t_2)\,K(t,\omega;t_1,t_2)\,dt_1\,dt_2$
(ii)	shift covariant	$y(t) = x(t-t_0)\,e^{j\omega_0 t} \rightarrow T_y(t,\omega) = T_x(t-t_0,\omega-\omega_0)$
(iii)	Moyal formula	$\left\lvert\int x(t)\,y^*(t)\,dt\right\rvert^2 = \iint T_x(t,\omega)\,T_y^*(t,\omega)\,dt\,d\omega$
(iv)	modulations	$z(t) = x(t)\,y(t) \rightarrow T_z(t,\omega) = T_x(t,\omega) *_\omega T_y(t,\omega)$
(v)	convolutions	$z(t) = x(t) * y(t) \rightarrow T_z(t,\omega) = T_x(t,\omega) *_t T_y(t,\omega)$
(vi)	real	$T_x(t,\omega) = T_x^*(t,\omega)$

Claasen and Mecklenbräuker [6] used discretization methods for computing a Wigner distribution from an oversampled signal. While they are computing samples of a type I Wigner distribution, they are not computing a type II Wigner distribution since it does not satisfy properties corresponding to type II signals.

Richman *et al.* used a group theoretic definition to create type IV Wigner distributions [9]. They define two type IV Wigner distributions, one for signals with an even length period and another for signals with an odd length period. The difference is due to the fact that the element 2^{-1} exists for the odd case and not for the even case. Their even length distribution is qualitatively very different from their odd length distribution (the former is qualitatively similar to the type IV Margenau–Hill distribution [8]). Their even length distribution also does not satisfy as many properties as their odd length distribution (as will be shown below).

McLaughlin *et al.* [10] and Narayanan *et al.* [11] attempted to apply operator theory to create a type IV Wigner distribution, but they were unable to extend the theory to type IV signals.

Our definition is based on the following result from [4].

Theorem: The Wigner distribution is the only time-frequency distribution for type I signals that satisfies the six properties in Table II.

Manuscript received April 7, 1998. This work was supported in part by the Office of Naval Research under ONR Contracts N00014-90-J-1654 and N00014-97-1-0072. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. D. L. Jones.

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Publisher Item Identifier S 1070-9908(99)09284-6.

[1] We will use $x(\cdot)$ to denote signals of all types and let the context indicate the type.

Proof: Properties (i) and (ii) limit us to distributions in the Cohen class [4], [5]

$$C_x^{\mathrm{I}}(t,\omega) = \iint R_x^{\mathrm{I}}(s,\tau)\phi(t-s,\tau)e^{-j\omega\tau}\,ds\,d\tau$$
$$= \frac{1}{2\pi}\iint A_x^{\mathrm{I}}(\theta,\tau)\psi(\theta,\tau)e^{-j\omega\tau}e^{j\theta t}\,d\theta\,d\tau$$

where $R_x^{\mathrm{I}}(t,\tau) = x(t+\tau)x^*(t)$, $A_x^{\mathrm{I}}(\theta,\tau) = \int R_x^{\mathrm{I}}(t,\tau)e^{-j\theta t}\,dt$, and $\psi(\theta,\tau) = \int \phi(t,\tau)e^{-j\theta t}\,dt$. The kernel thatcorresponds to the Wigner distribution is $\phi(t,\tau) = \delta(t+\frac{\tau}{2})$. The Cohen class is not usually presented in this form, but the above form is easier to apply to the three types of discrete signals. Properties (iii) and (iv) constrain the kernel, respectively, to be of the form $|\psi(\theta,\tau)|^2 = 1$ and $\psi(\theta,\tau) = e^{\theta f(\tau)}$ for a complex function $f(\cdot)$. Together, they constrain the kernel to be of the form $\psi(\theta,\tau) = e^{-j\theta g(\tau)}$ or $\phi(t,\tau) = \delta(t+g(\tau))$ for a real function $g(\cdot)$. Adding property (v) further constrains the kernel such that $\phi(t,\tau) = \delta(t+c\tau)$ for a real constant c. Finally, adding property (vi) requires that $c = \frac{1}{2}$. □

III. Type II Wigner Distribution

The above six properties are easily converted to type II signals and here we apply the above theorem to attempt to define a type II Wigner distribution. Properties (i) and (ii) restrict us to the type II Cohen class [8]

$$C_x^{\mathrm{II}}(n,\omega) = \sum_m \sum_p R_x^{\mathrm{II}}(p,m)\phi(n-p,m)e^{-j\omega m}$$
$$= \int_0^{2\pi} \sum_m A_x^{\mathrm{II}}(\theta,m)\psi(\theta,m)e^{-j\omega m}e^{j\theta n}\,d\theta$$

where R_x^{II}, A_x^{II}, and ψ are defined analogous to the above. Properties (iii) and (iv) constrain the kernel to be of the form $\phi(n,m) = \delta(n+f(m))$ for some function $f(\cdot)$. Adding property (v) requires that $f(m) = cm$ for $c \in Z$. However, for all $c \in Z$, the distribution will not be real. Thus, the type II Wigner distribution does not exist under this definition.

There are type II distributions that satisfy all but one of the above properties. The type II Page and the type II Rihaczek distributions [8] satisfy all of the properties except for (v) and (vi), respectively.

IV. Type IV Wigner Distribution

Here, we apply the above theorem to attempt to define a type IV Wigner distribution. Properties (i) and (ii) restrict us to the type IV Cohen class [8]

$$C_x^{\mathrm{IV}}(n,k) = \sum_{m,p=0}^{N-1} R_x^{\mathrm{IV}}(p,m)\phi(n-p,m)e^{-j2\pi km/N}$$
$$= \sum_{v,m=0}^{N-1} A_x^{\mathrm{IV}}(v,m)\psi(v,m)e^{-j2\pi km/N}e^{j2\pi vn/N}$$

where R_x^{IV}, A_x^{IV}, and ψ are defined analogous to the above, and all functions are periodic. Properties (iii) and (iv) constrain the kernel to be of the form $\phi(n,m) = \delta(n+f(m))$ for some

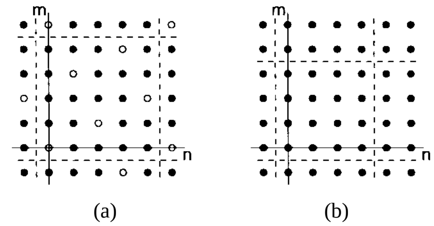

Fig. 1. Examples of the kernel function for even and odd length type IV signals. The solid lines denote the axes and the dashed lines denote one period. (a) Type IV: odd length. (b) Type IV: even length.

function $f(\cdot)$. Adding property (v) requires that $f(m) = cm$ for $c = \{0,\cdots,N-1\}$ and (c,N) are relatively prime (see Fig. 1).

For N odd, there exists exactly one value of c that gives a real distribution. This value is $c = 2^{-1}$ and equals $\frac{N+1}{2}$ for odd N. The resulting distribution is the one originally defined by Richman *et al.* [9] and corresponds to the kernel function $\phi(n,m) = \delta(n+2^{-1}m)$. In Fig. 1(a), we show an example for $N = 5$ where the open circles correspond to a value of one and the filled circles correspond to a value of zero. We will call this distribution the type IV Wigner distribution (for signals with an odd length period).

For N even, there are no values of c that give a real distribution. Thus, the type IV Wigner distribution does not exist under this definition for signals with an even length period. The distribution defined by Richman *et al.* [9] for even length signals does not satisfy properties (iv) and (v).

V. Conclusion

The classical Wigner distribution was originally defined [1], [2] as presented in (1), although one can put forth innumerable definitions of the Wigner distribution, which are equivalent to the original. However, previous definitions [6], [9]–[11] have not been able to be applied to all types of discrete signals.

In this letter, we propose an alternative definition for the Wigner distribution, based on fundamental properties, that is easily applied to discrete signals. Under this definition, we show that the Wigner distribution exists only for type I signals and for type IV signals with an odd length period. The former corresponds to the classical definition, and the latter corresponds to the definition given by Richman *et al.* [9].

One could argue that another set of properties might provide a different answer than obtained here. However, since the chosen properties are fundamental in nature and have a clear extension to discrete signals, we argue that a distribution that does not satisfy them should not be considered a Wigner distribution. A Matlab software package that computes all distributions mentioned in this letter is available at http://mdsp.bu.edu/jeffo.

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