On Phase-Magnitude Relationships in the Short-Time Fourier Transform

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***Abstract*—A complete evaluation of first-order, second-order and mixed derivatives is proposed for both the (log-)magnitude and the phase of a given Short-Time Fourier Transform (STFT), leading to equivalent expressions based on additional STFTs with specific windows. Consequences are drawn in terms of phase-magnitude relationships, resulting in new formulations of time-frequency techniques such as reassignment, as well as new insights in the structure of admissible STFTs in some special cases.**

***Index Terms*—Time-frequency, Short-time Fourier transform, phase, magnitude, reassignment.**

I. Introduction

The Short-Time Fourier Transform (STFT) and the associated spectrogram are widely used techniques to analyze a signal jointly in time and frequency. In order to circumvent some of the limitations of such approaches that are due to the necessary choice of a short-time window, techniques such as reassignment have been proposed, based on some (explicit or implicit) use of the STFT phase information that is usually discarded when using the (magnitude-based) spectrogram. However, A valid STFT is not any 2D complex function of time and frequency: it is therefore the purpose of this Letter to explore further some of the links that exist between its phase and magnitude. A number of results are already available in the literature. We propose here to extend and interpret them in a more exhaustive way. More precisely, the Letter is organized as follows: in Section II, we explicitly evaluate first-order, second-order and mixed derivatives of both the log-magnitude and the phase of a given STFT, and we derive implicit companion expressions based on STFTs with specific windows. Section III restricts the discussion to the important case of Gaussian windows for which strongly constrained relationships between phase and magnitude are established. Finally, Section IV is dedicated to interpretations of such results in terms of admissibility, analysis (reassignment), processing ("transient vs. tone" detection) and structure of a STFT around its zeros. Some open questions are outlined in the Conclusion.

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II. STFT derivatives

Given an analysis window $h(t)$, the Short-Time Fourier Transform (STFT) of a signal $x(t)$ is a complex-valued function that can be defined as

$$\mathrm{F}_x^h(t,\omega) = e^{\jmath\omega t/2}\int_{-\infty}^{+\infty} x(u)\, h^*(t-u)\, e^{-\jmath\omega u}\, du \qquad (1)$$

$$= \mathrm{M}_x^h(t,\omega)\, e^{\jmath\Phi_x^h(t,\omega)}, \qquad (2)$$

where $\mathrm{M}_x^h(t,\omega)$ and $\Phi_x^h(t,\omega)$ respectively stand for its magnitude and its phase. Following the convention used in [1], the above definition includes a pure phase term whose origin comes from the use of the Weyl operator and whose main purpose is to end up with more symmetrical expressions in the following developments. Without this term, the phase of the STFT is equal to $\varphi_x^h(t,\omega) = \Phi_x^h(t,\omega) - \omega t/2$, and expressions involving this phase can easily be deduced from the results presented below.

In the spirit of the so-called "dynamic signal" introduced in [2], it can be convenient to switch from the STFT to its complex logarithm[1], and to consider the derivatives of both real and imaginary parts of this quantity. Doing so, it is easy to establish that

$$\frac{\partial}{\partial t}\log(\mathrm{M}_x^h(t,\omega)) \;=\; \mathrm{Re}\left(\frac{\mathrm{F}_x^{Dh}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right) \qquad (3)$$

$$\frac{\partial\Phi_x^h}{\partial t}(t,\omega) \;=\; \mathrm{Im}\left(\frac{\mathrm{F}_x^{Dh}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right) + \frac{\omega}{2}, \qquad (4)$$

where $Dh(t) = \frac{dh}{dt}(t)$. Introducing in a similar way the window $Th(t) = t\,h(t)$, we immediately get the companion expressions:

$$\frac{\partial}{\partial\omega}\log(\mathrm{M}_x^h(t,\omega)) \;=\; -\mathrm{Im}\left(\frac{\mathrm{F}_x^{Th}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right) \qquad (5)$$

$$\frac{\partial\Phi_x^h}{\partial\omega}(t,\omega) \;=\; \mathrm{Re}\left(\frac{\mathrm{F}_x^{Th}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right) - \frac{t}{2}. \qquad (6)$$

It should be noticed that the introduction of the windows $Dh(t)$ and $Th(t)$ was instrumental in the reformulation (discussed in [3], [4]) of the reassignment operators initially introduced in [5]. In this context however, only the expressions of the (first-order) phase derivatives were obtained and effectively used.

Now, if we interest ourselves to derivatives of higher order [6], we can define the new window $D^2h(t) = \frac{d^2h}{dt^2}(t)$ and

[1]Throughout this paper, the signal is supposed to be properly normalized so that $\mathrm{M}_x^h(t,\omega)$ is an adimensional quantity.

establish that

$$Z_{tt}(t,\omega) = \frac{1}{\mathrm{F}_x^h}\frac{\partial^2 \mathrm{F}_x^h}{\partial t^2} - \left(\frac{1}{\mathrm{F}_x^h}\frac{\partial \mathrm{F}_x^h}{\partial t}\right)^2 \quad (7)$$

$$= \frac{\mathrm{F}_x^{D^2h}(t,\omega)}{\mathrm{F}_x^h(t,\omega)} - \left(\frac{\mathrm{F}_x^{Dh}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right)^2, \quad (8)$$

from which we deduce that

$$\frac{\partial^2}{\partial t^2}\log(\mathrm{M}_x^h(t,\omega)) = \mathrm{Re}\left(Z_{tt}(t,\omega)\right) \quad (9)$$

$$\frac{\partial^2 \Phi_x^h}{\partial t^2}(t,\omega) = \mathrm{Im}\left(Z_{tt}(t,\omega)\right). \quad (10)$$

Introducing similarly the window $T^2h(t) = t^2\, h(t)$, we get

$$Z_{\omega\omega}(t,\omega) = \frac{1}{\mathrm{F}_x^h}\frac{\partial^2 \mathrm{F}_x^h}{\partial \omega^2} - \left(\frac{1}{\mathrm{F}_x^h}\frac{\partial \mathrm{F}_x^h}{\partial \omega}\right)^2 \quad (11)$$

$$= -\frac{\mathrm{F}_x^{T^2h}(t,\omega)}{\mathrm{F}_x^h(t,\omega)} + \left(\frac{\mathrm{F}_x^{Th}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\right)^2 \quad (12)$$

and, therefore,

$$\frac{\partial^2}{\partial \omega^2}\log(\mathrm{M}_x^h(t,\omega)) = \mathrm{Re}\left(Z_{\omega\omega}(t,\omega)\right) \quad (13)$$

$$\frac{\partial^2 \Phi_x^h}{\partial \omega^2}(t,\omega) = \mathrm{Im}\left(Z_{\omega\omega}(t,\omega)\right). \quad (14)$$

Finally, mixed partial derivatives can be obtained as well. This requires the introduction of the mixed window defined as $TDh(t) = t\,\frac{dh}{dt}(t)$, leading to

$$Z_{t\omega}(t,\omega) = -\jmath\frac{1}{\mathrm{F}_x^h}\frac{\partial^2 \mathrm{F}_x^h}{\partial t\partial \omega} + \jmath\left(\frac{1}{\mathrm{F}_x^h}\frac{\partial \mathrm{F}_x^h}{\partial t}\right)\left(\frac{1}{\mathrm{F}_x^h}\frac{\partial \mathrm{F}_x^h}{\partial \omega}\right) \quad (15)$$

$$= \frac{\mathrm{F}_x^{TDh}(t,\omega)}{\mathrm{F}_x^h(t,\omega)} - \frac{\mathrm{F}_x^{Th}(t,\omega)}{\mathrm{F}_x^h(t,\omega)}\frac{\mathrm{F}_x^{Dh}(t,\omega)}{\mathrm{F}_x^h(t,\omega)} \quad (16)$$

and

$$\frac{\partial^2}{\partial t\partial \omega}\log(\mathrm{M}_x^h(t,\omega)) = -\mathrm{Im}\left(Z_{t\omega}(t,\omega)\right) \quad (17)$$

$$\frac{\partial^2 \Phi_x^h}{\partial t\partial \omega}(t,\omega) = \mathrm{Re}\left(Z_{t\omega}(t,\omega)\right). \quad (18)$$

It has to be noted that Eqs. (10), (14) and (18) appear in [7]. These results are supplemented here by Eqs. (9), (13) and (17) giving the second-order partial derivatives of the STFT log-magnitude.

III. The Gaussian case

While the results obtained in the previous section are quite general and hold true for any admissible window, it turns out that much simpler expressions can be obtained in the Gaussian case, leading eventually to direct explicit relationships connecting the magnitude and the phase of the corresponding STFTs. In practical applications, such a window is of particular interest, since it provides an optimal time-frequency resolution in the Heisenberg-Gabor sense. More precisely, if the short-time window is chosen as a unit-energy Gaussian function with a given time-width λ,

$$h(t) = \lambda^{-1/2}\pi^{-1/4}e^{-t^2/(2\lambda^2)} \quad (19)$$

it directly follows its definition that

$$Dh(t) = -\lambda^{-2}\,Th(t) \quad (20)$$

$$D^2h(t) = -\lambda^{-2}\,h(t) + \lambda^{-4}\,T^2h(t) \quad (21)$$

$$TDh(t) = -\lambda^{-2}\,T^2h(t). \quad (22)$$

By linearity, the identities shown above for the windows carry over to the associated STFTs, allowing for the combination of the four equations (3) to (6) into one system of only two equations, namely:

$$\frac{\partial \Phi_x^h}{\partial t}(t,\omega) = \lambda^{-2}\,\frac{\partial}{\partial \omega}\log(\mathrm{M}_x^h(t,\omega)) + \frac{\omega}{2} \quad (23)$$

$$\frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) = -\lambda^{2}\,\frac{\partial}{\partial t}\log(\mathrm{M}_x^h(t,\omega)) - \frac{t}{2}. \quad (24)$$

This means that, in the case of a Gaussian window, the phase and the log-magnitude of a STFT strongly depend on each other (via their partial derivatives). Differentiating the expressions above with respect to both t and ω leads to the four equations

$$\frac{\partial^2 \Phi_x^h}{\partial t^2}(t,\omega) = \lambda^{-2}\,\frac{\partial^2}{\partial t\partial \omega}\log(\mathrm{M}_x^h(t,\omega)) \quad (25)$$

$$\frac{\partial^2 \Phi_x^h}{\partial \omega^2}(t,\omega) = -\lambda^{2}\,\frac{\partial^2}{\partial t\partial \omega}\log(\mathrm{M}_x^h(t,\omega)) \quad (26)$$

$$\frac{\partial^2}{\partial t^2}\log(\mathrm{M}_x^h(t,\omega)) = -\lambda^{-2}\left(\frac{\partial^2 \Phi_x^h}{\partial t\partial \omega}(t,\omega) + \frac{1}{2}\right) \quad (27)$$

$$\frac{\partial^2}{\partial \omega^2}\log(\mathrm{M}_x^h(t,\omega)) = \lambda^{2}\left(\frac{\partial^2 \Phi_x^h}{\partial t\partial \omega}(t,\omega) - \frac{1}{2}\right). \quad (28)$$

Combining them pairwise, we finally end up with the two remarkable identities

$$\lambda^2\,\frac{\partial^2}{\partial t^2}\log(\mathrm{M}_x^h(t,\omega)) + \lambda^{-2}\,\frac{\partial^2}{\partial \omega^2}\log(\mathrm{M}_x^h(t,\omega)) = -1 \quad (29)$$

$$\lambda^2\,\frac{\partial^2 \Phi_x^h}{\partial t^2}(t,\omega) + \lambda^{-2}\,\frac{\partial^2 \Phi_x^h}{\partial \omega^2}(t,\omega) = 0. \quad (30)$$

IV. Interpretations

A. STFT admissibility

Up to a rescaling governed by the time-width λ of the Gaussian window, equations (29) and (30) show that, for a complex-valued function of time and frequency to be an admissible STFT, its log-magnitude and phase must satisfy a necessary condition that takes the form of a Poisson/Laplace-like equation. Indeed, in the "circular" case where $\lambda = 1$, we obtain exactly

$$\Delta\log(\mathrm{M}_x^h(t,\omega)) = -1 \quad \text{and} \quad \Delta\Phi_x^h(t,\omega) = 0. \quad (31)$$

Whereas Eqs. (23) and (24) explicitate how the magnitude and the phase of a valid STFT are coupled, Eq. (31) (and, more generally, Eqs. (29) and (30)) show that such quantities cannot be arbitrarily specified by themselves either.

B. Spectrogram reassignment

One major use of STFT phase derivatives is the spectrogram reassignment. Whereas the method initially relied directly on the phase to compute the time and frequency reassigned locations as [5]

$$\hat{t}_x(t,\omega) \;=\; \frac{t}{2} - \frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) \qquad (32)$$

$$\hat{\omega}_x(t,\omega) \;=\; \frac{\omega}{2} + \frac{\partial \Phi_x^h}{\partial t}(t,\omega), \qquad (33)$$

the re-interpretation proposed in [3] has led to an implicit evaluation that avoids the explicit computation of phase derivatives, exploiting the equivalent expressions given in (4) and (6). What results from the derivations conducted above is that a third possibility is offered in the Gaussian case, according to which the reassignment process can be carried out on the basis of only knowledge of the STFT magnitude:

$$\hat{t}_x(t,\omega) - t \;=\; \lambda^2 \frac{\partial}{\partial t} \log(\mathrm{M}_x^h(t,\omega)) \qquad (34)$$

$$\hat{\omega}_x(t,\omega) - \omega \;=\; \lambda^{-2} \frac{\partial}{\partial \omega} \log(\mathrm{M}_x^h(t,\omega)). \qquad (35)$$

Reassignment operators derived from the STFT magnitude, such as those already studied in [8], are therefore strictly equivalent to the traditional operators in the Gaussian case.

Eqs. (34) and (35) also show that the reassignment vector, defined as $\mathbf{r}_x^h(t,\omega) = (\hat{t}_x(t,\omega) - t, \hat{\omega}_x(t,\omega) - \omega)^t$ satisfies

$$\mathbf{r}_x^h(t,\omega) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \nabla \log(\mathrm{M}_x^h(t,\omega)), \qquad (36)$$

thus generalizing a result given in [1] in the case where $\lambda = 1$, and which was the starting point of the so-called "differential reassignment". To be more precise about this point, we can go back to Eqs. (23) and (24) with $\lambda = 1$ and establish that

$$\frac{\partial \Phi_x^h}{\partial t}\left(\frac{\partial}{\partial t}\log \mathrm{M}_x^h + \frac{t}{2}\right) + \frac{\partial \Phi_x^h}{\partial \omega}\left(\frac{\partial}{\partial \omega}\log \mathrm{M}_x^h + \frac{\omega}{2}\right) = 0. \qquad (37)$$

Recalling that, thanks to the Bargmann factorization [9], any STFT with a "circular" (i.e., $\lambda = 1$) Gaussian window admits the decomposition

$$F_x^h(t,\omega) = \mathcal{F}_x^h(t,\omega)\, e^{-(t^2+\omega^2)/4}, \qquad (38)$$

the equation above (37) can be equivalently rewritten as

$$\nabla \Phi_x^h . \nabla \log(\mathcal{M}_x^h) = 0, \qquad (39)$$

$$\text{with } \mathcal{M}_x^h(t,\omega) = |\mathcal{F}_x^h(t,\omega)| = \mathrm{M}_x^h(t,\omega)\, e^{(t^2+\omega^2)/4}. \qquad (40)$$

It thus appears that isocontours of the phase and of this modified log-magnitude are orthogonal, which is another way of characterizing the analyticity of $\mathcal{F}_x^h(t,\omega)$ when considered — with an abuse of notation — as a function of the complex variable $z = \omega + \jmath t$. Indeed, it can be directly observed that, in the case where $\lambda = 1$, Eqs. (23) and (24) can be re-expressed as Cauchy conditions [10] on the pair $(\log(\mathcal{M}_x^h(t,\omega)), \Phi_x^h(t,\omega))$:

$$\frac{\partial \Phi_x^h}{\partial t}(t,\omega) \;=\; \frac{\partial}{\partial \omega}\log(\mathcal{M}_x^h(t,\omega)) \qquad (41)$$

$$\frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) \;=\; -\frac{\partial}{\partial t}\log(\mathcal{M}_x^h(t,\omega)). \qquad (42)$$

Those relations between partial derivatives give an explicit form to the strong coupling that exists, in the Gaussian case, between phase and magnitude. In the context of reassignment, this complements the final outcome of the process (getting a squeezed distribution) by the way it is achieved (moving values along trajectories pointing towards local maxima).

In this respect, it is worth mentioning that a new variant of reassignment has recently been introduced [11]. It is referred to as "Levenberg-Marquardt reassignment", in reference to an adjustable root finding algorithm. This process consists in moving each value of the spectrogram one step towards the nearest ridge [12] of the signal, from the point (t,ω) where it is computed to the point $(\tilde{t}_x, \tilde{\omega}_x)$ deduced from the second-order single and mixed derivatives of the STFT phase

$$\begin{pmatrix} \tilde{t}_x(t,\omega) \\ \tilde{\omega}_x(t,\omega) \end{pmatrix} = \begin{pmatrix} t \\ \omega \end{pmatrix} - \left(\nabla^t R_x^h(t,\omega) + \mu\, I_2\right)^{-1} R_x^h(t,\omega) \qquad (43)$$

$$R_x^h(t,\omega) = -\mathbf{r}_x^h(t,\omega) = \begin{pmatrix} \frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) + \frac{t}{2} \\ -\frac{\partial \Phi_x^h}{\partial t}(t,\omega) + \frac{\omega}{2} \end{pmatrix} \qquad (44)$$

$$\nabla^t R_x^h(t,\omega) = \begin{pmatrix} \frac{\partial R_x^h}{\partial t}(t,\omega) & \frac{\partial R_x^h}{\partial \omega}(t,\omega) \end{pmatrix} \qquad (45)$$

$$= \begin{pmatrix} \frac{\partial^2 \Phi_x^h}{\partial t \partial \omega}(t,\omega) + \frac{1}{2} & \frac{\partial^2 \Phi_x^h}{\partial \omega^2}(t,\omega) \\ -\frac{\partial^2 \Phi_x^h}{\partial t^2}(t,\omega) & -\frac{\partial^2 \Phi_x^h}{\partial t \partial \omega}(t,\omega) + \frac{1}{2} \end{pmatrix} \qquad (46)$$

where $\mu \in I\!R_+$ and I_2 is the two-dimensional identity matrix. The extra parameter μ allows to choose the degree of concentration of the resulting distribution. For example, a reduced squeezing may be desiderable to increase robustness of reassigned distributions to random noise. In the Gaussian case, Eqs. (25)-(28) allow to deduce $(\tilde{t}_x, \tilde{\omega}_x)$ from the STFT magnitude:

$$R_x^h(t,\omega) = -\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \nabla \log(\mathrm{M}_x^h(t,\omega)) \qquad (47)$$

$$\nabla^t R_x^h(t,\omega) = -\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \nabla\nabla^t \log(\mathrm{M}_x^h(t,\omega)) \qquad (48)$$

$$\begin{pmatrix} \tilde{t}_x \\ \tilde{\omega}_x \end{pmatrix} - \begin{pmatrix} t \\ \omega \end{pmatrix} = -\left(\nabla\nabla^t \log(\mathrm{M}_x^h) - \mu\Lambda\right)^{-1} \nabla \log(\mathrm{M}_x^h)$$

$$\text{with } \quad \Lambda = \begin{pmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^2 \end{pmatrix} \qquad (49)$$

These additional results show that in the Gaussian case, the Levenberg-Marquardt reassignment moves the spectrogram values towards the maxima of the log-magnitude.

C. Transients and tones

Using the mixed derivative of the phase has been advocated by some authors as a way of discriminating between short, pulse-like transients and long, stationary-like tones [7], [6]. This can be easily formalized within the present framework by reorganizing Eqs. (27) and (28) so as to get the two equivalent expressions:

$$\frac{\partial^2 \Phi_x^h}{\partial t \partial \omega}(t,\omega) = -\lambda^2 \frac{\partial^2}{\partial t^2} \log(\mathrm{M}_x^h(t,\omega)) - \frac{1}{2} \quad (50)$$

$$= \lambda^{-2} \frac{\partial^2}{\partial \omega^2} \log(\mathrm{M}_x^h(t,\omega)) + \frac{1}{2}. \quad (51)$$

It is clear from Eqs. (50) and (51) that the mixed derivative of the phase estimates the local curvature of the STFT log-magnitude along the time and frequency axes, and hence is an indicator of the time and frequency variations of the signal content. Considering idealized impulses ($x(t) = \delta(t)$) and pure tones ($x(t) = e^{\jmath\omega_0 t}$) as limiting examples, a direct evaluation of either those equations leads to a value of the mixed derivative tending to $+1/2$ and $-1/2$, respectively. The main interest of this result is that it holds true regardless of the time-width λ of the STFT (Gaussian) window.

D. STFT phase structure around zeros of the magnitude

In a recent work [13], a recurring pattern of the phase derivative of the STFT has been observed around the zeros of the transform. It consists in a singularity with a positive and a negative peak at those points. Whereas this observation has been supported by numerical simulations and some analytic treatments in [13], a simple justification can be derived in the Gaussian case, from phase-magnitude relationships arguments. Indeed, it follows from Eqs. (41)-(42) that the phase derivatives around zeros of the STFT are directly controlled by the derivatives of the log-magnitude at those points. Zeros of $\mathrm{F}_x^h(t,\omega)$, $\mathcal{F}_x^h(t,\omega)$, $\mathrm{M}_x^h(t,\omega)$ and $\mathcal{M}_x^h(t,\omega)$ are identical and, following the argument outlined in [14], the analyticity of $\mathcal{F}_x^h(z) = \mathcal{F}_x^h(t,\omega)$ guarantees a regular behavior of $\mathcal{M}_x^h(t,\omega)$ around any of its zeros $z_n = \omega_n + \jmath t_n$. More precisely, the analytic function $\mathcal{F}_x^h(z)$ that enters the Bargmann representation (38) being an entire function of order at most 2, it admits (up to some possible multiple zeros at the origin) a Weierstrass-Hadamard factorization given by [15]

$$\mathcal{F}_x^h(z) = e^{Q(z)} \prod_n (1 - \tilde{z}_n) \exp\left(\tilde{z}_n + \tilde{z}_n^2/2\right), \quad (52)$$

where $Q(z)$ is a quadratic polynomial and $\tilde{z}_n = z/z_n$. It follows from this factorization that, in the vicinity of its zeros, the vanishing of the considered STFT is such that

$$\mathcal{M}_x^h(t,\omega) \propto |1 - \tilde{z}_n| \propto \left[(\omega - \omega_n)^2 + (t - t_n)^2\right]^{1/2}. \quad (53)$$

When combined with the logarithms involved in (41)-(42), this leads therefore to a universal form for the local divergence of the associated phase derivatives, namely

$$\left.\frac{\partial \Phi_x^h}{\partial t}(t_n,\omega)\right|_{\omega\sim\omega_n} \sim (\omega - \omega_n)^{-1} \quad (54)$$

$$\left.\frac{\partial \Phi_x^h}{\partial \omega}(t,\omega_n)\right|_{t\sim t_n} \sim (t_n - t)^{-1}. \quad (55)$$

More generally, according to Eqs. (41)-(42), the complete structure of phase derivatives around zeros of the magnitude can be generically approximated by plugging (53) within the gradient of the log-magnitude. Among other things, this allows to show that the mixed derivative goes to $+\infty$ when z goes to z_n.

V. Conclusion

A complete evaluation of first-order, second-order and mixed derivatives has been proposed for both the phase and the (log-)magnitude of a given STFT, leading to equivalent expressions based on additional STFTs with specific windows. This led to a number of phase-magnitude relationships that may offer new ways of addressing time-frequency analysis (e.g., reassignment) or processing (e.g., transient vs. tone detection) problems. Some issues are however still left open and would need further study. For instance, generalizations of the results discussed in Sections IV-A and IV-D—that are known to hold beyond the Gaussian case [13]—would be interesting. At a more fundamental level, it would be worth investigating under which additional conditions the necessary conditions (23)-(24) or (29)-(30) would be sufficient for making of a 2D complex function of time and frequency an admissible STFT.

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