From stationarity to self-similarity, and back: Variations on the Lamperti transformation

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Abstract. The Lamperti transformation defines a one-to-one correspondence between stationary processes on the real line and self-similar processes on the real half-line. Although dating back to 1962, this fundamental result has further received little attention until a recent past, and it is the purpose of this chapter to survey the Lamperti transformation and its (effective and/or potential) applications, with emphasis on variations which can be made on the initial formulation. After having recalled basics of the transform itself, some results from the literature will be reviewed, which can be broadly classified in two types. In a first category, classical concepts from stationary processes and linear filtering theory, such as linear time-invariant systems or ARMA modeling, can be given self-similar counterparts by a proper "lampertization" whereas, in a second category, problems such as spectral analysis or prediction of self-similar processes can be addressed with classical tools after stationarization by a converse "delampertization". Variations and new results will then be discussed by investigating consequences of the Lamperti transformation when applied to weakened forms of stationarity, and hence of self-similarity. Different forms of locally stationary processes will be considered this way, as well as cyclostationary processes for which "lampertization" will be shown to offer a suitable framework for defining a stochastic extension to the notion of *discrete scale invariance* which has recently been put forward as a central concept in many critical systems. Issues concerning the practical analysis (and synthesis) of such processes will be examined, with a possible use of Mellin-based tools operating directly in the space of scaling data.

1 Introduction

In a seminal paper published in 1962 [19], J.W. Lamperti introduced key concepts related to what is now referred to as *self-similar processes*. Among other important results, he first pointed out the one-to-one connection which exists between self-similar processes and stationary processes, *via* a transform which essentially consists in a proper warping of the time axis. This result has often been quoted in the literature (e.g., in [3], [30] or [34]), but rarely used and even discussed *per se*. Notable exceptions are the contributions of Burnecki *et al.* [8] who proved unicity, and of Nuzman and Poor [25,26] who explicitly (and extensively) took profit of it for linear estimation issues concerning fractional Brownian motion (fBm).

The transform that Lamperti initially pushed forward in 1962 has, since then, been rediscovered from time to time, under different forms. For instance, Gray and Zhang re-established in [17] a weakened form of Lamperti's theorem, upon which they based a discussion on specific classes of self-similar processes, referred to as *multiplicative stationary processes*. From a very close (yet independent) perspective, Yazici and Kashyap advocated in [37] the use of a transform—which indeed identifies to Lamperti's—for constructing related classes of self-similar processes referred to as *scale stationary processes*, a concept which had also been briefly discussed and commented in [12]. More recently, Vidács and Virtamo proposed in [35,36] an original ML estimation scheme for fBm parameters, which basically relies on a geometrical sampling of the data, i.e., on a pre-processing guaranteeing a stationarization in the spirit of the Lamperti approach.

Recognizing both the importance of the Lamperti transform and the sparsity of its coverage in the literature, the purpose of this text is to offer a guided tour of existing material in a unified form, and also to discuss new extensions. More precisely, the text is organized as follows. In Section 2, basics of stationarity and self-similarity are first recalled, and the Lamperti transform is introduced. The ability of this transform to put self-similar and stationary processes in a one-to-one correspondence is then proved, and a number of consequences are detailed, with respect to covariances, spectra, long-range dependence and scale-covariant generating systems for self-similar processes. Some examples and applications are dealt with in Section 3, including either stationary processes (random phase tones, Ornstein-Uhlenbeck, ARMA) and their self-similar counterparts, or self-similar processes (fractional Brownian motion, Euler-Cauchy) and their stationary counterparts. Section 4 is then devoted to variations on the original approach, obtained by applying the Lamperti transform to weakened forms of stationarity or self-similarity. Following a brief introduction of relevant concepts such as multiplicative harmonizability or scale-invariant Wigner spectra, special emphasis is put on the newly introduced notion of *stochastic discrete scale invariance* which is shown to be the Lamperti image of cyclostationarity.

2 The Lamperti transformation

2.1 Stationarity and self-similarity

The notion of stationarity is basic in the study of many stochastic processes. Heuristically, the idea of stationarity is equivalent to that of statistical invariance under time shifts, and this concept has proven most useful in many steady-state applications. From a different perspective, scale invariance (or self-similarity) is also ubiquitous in many natural and man-made phenomena (landscape texture, turbulence, network traffic, ...). The underlying idea is in this case that a function is scale invariant if it is identical to any of its rescaled versions, up to some suitable renormalization in amplitude.

To make these ideas more precise, let us first introduce two basic operations.

Definition 1 *Given some number $\tau \in \mathbb{R}$, the shift operator $\mathcal{S}_\tau$ operates on processes $\{Y(t), t \in \mathbb{R}\}$ according to:*

$$(\mathcal{S}_\tau Y)(t) := Y(t+\tau). \tag{1}$$

Definition 2 *Given some numbers $H > 0$ and $\lambda > 0$, the renormalized dilation operator $\mathcal{D}_{H,\lambda}$ operates on processes $\{X(t), t > 0\}$ according to:*

$$(\mathcal{D}_{H,\lambda} X)(t) := \lambda^{-H} X(\lambda t). \tag{2}$$

Using these operators in the context of stochastic processes, and introducing the notation "$\stackrel{d}{=}$" for equality of all finite-dimensional distributions, the definitions of stationarity and self-similarity follow as:

Definition 3 *A process $\{Y(t), t \in \mathbb{R}\}$ is said to be stationary if*

$$\{(\mathcal{S}_\tau Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\} \tag{3}$$

for any $\tau \in \mathbb{R}$.

Definition 4 *A process $\{X(t), t > 0\}$ is said to be self-similar of index H (or "H-ss") if*

$$\{(\mathcal{D}_{H,\lambda} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\} \tag{4}$$

for any $\lambda > 0$.

Such an equality holds in the usual sense for homogeneous functions proportional to $t^H, t > 0$, and it is useful to remark that, whenever $\{X(t), t > 0\}$ is H-ss, then the modulated process $\{X_{H'}(t), t > 0\}$ such that

$$X_{H'}(t) := t^{H'} X(t) \tag{5}$$

is $(H + H')$-ss.

Although Definition 4 puts no restriction on H, a limited range of values may result from specific constraints. In particular, mean-square continuity and non-degeneracy lead to the usually considered range $0 < H < 1$ [30].

2.2 The transform

Definition 5 *Given some number $H > 0$, the Lamperti transform $\mathcal{L}_H$ operates on processes $\{Y(t), t \in \mathbb{R}\}$ according to:*

$$(\mathcal{L}_H Y)(t) := t^H Y(\log t), t > 0, \tag{6}$$

and the corresponding inverse Lamperti transform $\mathcal{L}_H^{-1}$ operates on processes $\{X(t), t > 0\}$ according to:

$$(\mathcal{L}_H^{-1} X)(t) := e^{-Ht} X(e^t), t \in \mathbb{R}. \tag{7}$$

The Lamperti transform is invertible, which guarantees that $(\mathcal{L}_H^{-1}\mathcal{L}_H Y)(t) = Y(t)$ for any process $\{Y(t), t \in \mathbb{R}\}$, and $(\mathcal{L}_H\mathcal{L}_H^{-1}X)(t) = X(t)$ for any process $\{X(t), t > 0\}$. We can however remark that, given two different parameters H_1 and H_2, we only have

$$(\mathcal{L}_{H_2}^{-1}\mathcal{L}_{H_1} Y)(t) = e^{-(H_2-H_1)t}\, Y(t), \tag{8}$$

and, in a similar way, it is immediate to establish that

$$(\mathcal{L}_{H_2}\mathcal{L}_{H_1}^{-1} X)(t) = t^{H_2-H_1}\, X(t). \tag{9}$$

2.3 From stationarity to self-similarity, and back

Lemma 1 *The Lamperti transform (6)-(7) guarantees an equivalence between the shift operator (1) and the renormalized dilation operator (2) in the sense that, for any* $\lambda > 0$*:*

$$\mathcal{L}_H^{-1}\mathcal{D}_{H,\lambda}\mathcal{L}_H = \mathcal{S}_{\log\lambda}. \tag{10}$$

Proof — Assuming that $\{Y(t), t \in \mathbb{R}\}$ is stationary and using the Definitions 1, 2 and 5, we may write

$$\begin{aligned}
(\mathcal{L}_H^{-1}\mathcal{D}_{H,\lambda}\mathcal{L}_H Y)(t) &= (\mathcal{L}_H^{-1}\mathcal{D}_{H,\lambda})(t^H\, Y(\log t)) \\
&= \mathcal{L}_H^{-1}(\lambda^{-H}(\lambda t)^H Y(\log \lambda t)) \\
&= e^{-Ht}(s^H Y(\log\lambda s))_{s=e^t} \\
&= Y(t+\log\lambda) \\
&= (\mathcal{S}_{\log\lambda} Y)(t).
\end{aligned}$$

□

This observation is the key ingredient for establishing a one-to-one connection between self-similarity and stationarity. This fact is referred to as Lamperti's theorem and reads as follows [19] :

Theorem 1 *If* $\{Y(t), t \in \mathbb{R}\}$ *is stationary, its Lamperti transform* $\{(\mathcal{L}_H Y)(t), t > 0\}$ *is H-ss. Conversely, if* $\{X(t), t > 0\}$ *is H-ss, its inverse Lamperti transform* $\{(\mathcal{L}_H^{-1}X)(t), t \in \mathbb{R}\}$ *is stationary.*

Proof — Let $\{Y(t), t \in \mathbb{R}\}$ be a stationary process. Using Definition 3 and Lemma 1, we have for any $\lambda > 0$,

$$\{Y(t), t \in \mathbb{R}\} \stackrel{d}{=} \{(\mathcal{S}_{\log\lambda} Y)(t) = (\mathcal{L}_H^{-1}\mathcal{D}_{H,\lambda}\mathcal{L}_H Y)(t), t \in \mathbb{R}\} \tag{11}$$

and it follows from Definition 4 that the Lamperti transform $X(t) := (\mathcal{L}_H Y)(t)$ is H-ss since

$$\{X(t), t > 0\} \stackrel{d}{=} \{(\mathcal{D}_{H,\lambda} X)(t), t > 0\} \tag{12}$$

for any $\lambda > 0$.

Conversely, let $\{X(t), t > 0\}$ be a H-ss process. Using Definition 4 and Lemma 1, we have for any $\lambda > 0$,

$$\{X(t), t > 0\} \stackrel{d}{=} \{(\mathcal{D}_{H,\lambda}X)(t) = (\mathcal{L}_H \mathcal{S}_{\log \lambda} \mathcal{L}_H^{-1} X)(t), t > 0\} \tag{13}$$

and it follows from Definition 3 that the inverse Lamperti transform $Y(t) := (\mathcal{L}_H^{-1} X)(t)$ is stationary since

$$\{Y(t), t \in \mathbb{R}\} \stackrel{d}{=} \{(\mathcal{S}_{\log \lambda} Y)(t), t \in \mathbb{R}\} \tag{14}$$

for any $\lambda > 0$.

$\square$

The Lamperti transform establishes therefore a one-to-one connection between stationary and self-similar processes, and it is worth noting that it is in fact the unique transform to allow such a connection [8]. A graphical illustration of this one-to-one correspondence is given in Figure 1.

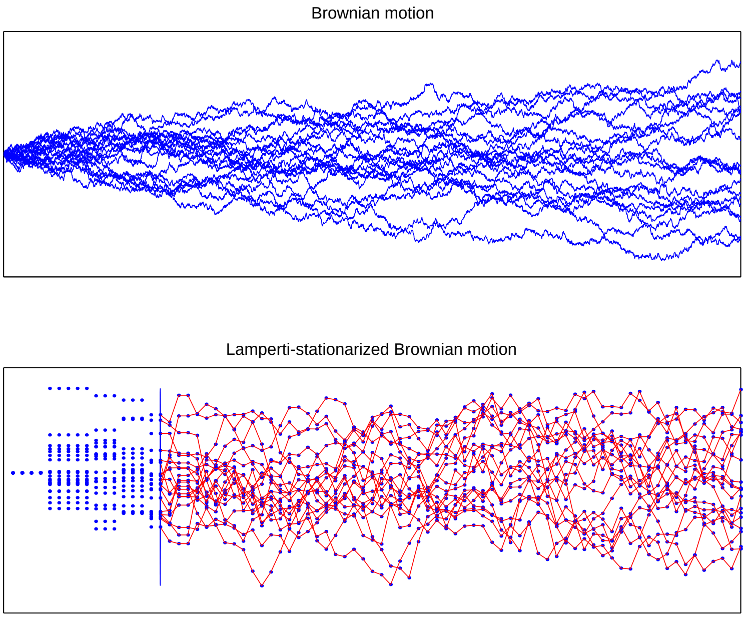

Fig. 1. A graphical illustration of Lamperti's theorem — Whereas sample paths of (nonstationary and self-similar) Brownian motion (top) reveal a time-dependence of variance as a square-root function of time, their "lampertized" versions (bottom) essentially lie within a band of constant width, in accordance with the stationarity properties induced by the inverse Lamperti transform.

Using (9), one can remark that, if $\{Y(t), t \in \mathbb{R}\}$ is stationary, the transformed process $\{(\mathcal{L}_{H_2}^{-1}\mathcal{L}_{H_1}Y)(t), t \in \mathbb{R}\}$ cannot be stationary, unless $H_1 = H_2$. In a similar way, making use of the remark on processes as in (5), the composition rule given in (8) shows that, if $\{X(t), t > 0\}$ is H-ss, the transformed process $(\mathcal{L}_{H_2}\mathcal{L}_{H_1}^{-1}X)(t)$ is $(H + H_2 - H_1)$-ss.

2.4 Consequences

Covariances and spectra As a direct consequence of Theorem 1, statistical properties of self-similar processes can be inferred from those of their Lamperti counterparts, and vice-versa. In particular, if we restrict to zero-mean second-order processes and if we introduce the notation $\mathbf{R}_X(t,s) := \mathbb{E}X(t)X(s)$, it is straightforward to establish that, for any process $\{X(t), t > 0\}$, the covariance function of its inverse Lamperti transform is given by:

$$\mathbf{R}_{\mathcal{L}_H^{-1}X}(t,s) = e^{-H(t+s)}\,\mathbf{R}_X(e^t, e^s) \tag{15}$$

for any $t, s \in \mathbb{R}$.

Conversely, for any process $\{Y(t), t \in \mathbb{R}\}$, the covariance function of its Lamperti transform reads

$$\mathbf{R}_{\mathcal{L}_H Y}(t,s) = (ts)^H\,\mathbf{R}_Y(\log t, \log s) \tag{16}$$

and, if $Y(t)$ happens to be stationary, we then have $\mathbf{R}_Y(t,s) = \boldsymbol{\gamma}_Y(t-s)$ (with $\boldsymbol{\gamma}_Y(.)$ a non-negative definite function), leading to:

$$\mathbf{R}_{\mathcal{L}_H Y}(t,s) = (ts)^H\,\boldsymbol{\gamma}_Y(\log(t/s)). \tag{17}$$

Two corollaries to Theorem 1 are therefore as follows:

Corollary 1 *Any second-order H-ss process $\{X(t), t > 0\}$ has necessarily a covariance function of the form*

$$\mathbf{R}_X(t,s) = (ts)^H\,c_H(t/s) \tag{18}$$

for any $t, s > 0$, with $c_H(\exp(\cdot))$ a non-negative definite function.

In the specific case where $H = 0$, we recover this way the class of "multiplicative stationary processes" introduced in [17], whereas the more general factorization given by (18) has been pointed out, e.g., in [12] and [37].

Corollary 2 *Given a second-order H-ss process $\{X(t), t > 0\}$, the power spectrum density of its stationary counterpart $(\mathcal{L}_H^{-1}X)(t)$ is the Mellin transform of the scale-covariant function c_H given in Eq. (18).*

Proof — Starting from (15) and using (18), it is immediate to establish that

$$\mathbf{R}_{\mathcal{L}_H^{-1}X}(t+\tau/2, t-\tau/2) = c_H(e^\tau),$$

from which it follows that the power spectrum density $\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f)$ of the inverse Lamperti transform of $X(t)$ is such that

$$\begin{aligned}\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f) &:= \int_{-\infty}^{+\infty} \mathbf{R}_{\mathcal{L}_H^{-1}X}(t+\tau/2, t-\tau/2)\, e^{-i2\pi f\tau}\, d\tau \\ &= \int_{-\infty}^{+\infty} c_H(e^{\tau})\, e^{-i2\pi f\tau}\, d\tau \\ &= \int_{0}^{+\infty} c_H(\theta)\, \theta^{-i2\pi f-1}\, d\theta \\ &= (\mathcal{M}c_H)(i2\pi f),\end{aligned}$$

with

$$(\mathcal{M}X)(s) := \int_0^{+\infty} X(t)\, t^{-s-1}\, dt \tag{19}$$

the Mellin transform [5].

□

Long-range dependence In the case of stationary processes, long-range dependence (LRD), or long-memory, is usually associated with a slow power-law decay of the correlation function [3] but, more generally, it may also be defined as follows:

Definition 6 *A second-order stationary process $\{Y(t), t \in \mathbb{R}\}$ is said to be long-range dependent if its normalized correlation function*

$$\tilde{\boldsymbol{\gamma}}_Y(\tau) := \boldsymbol{\gamma}_Y(\tau)/\boldsymbol{\gamma}_Y(0) \tag{20}$$

is not absolutely summable:

$$\int_0^{+\infty} |\tilde{\boldsymbol{\gamma}}_Y(\tau)|\, d\tau = \infty. \tag{21}$$

In the case of nonstationary processes, a generalization of this definition can be given as follows [1,23]:

Definition 7 *A second-order nonstationary process $\{X(t), t > 0\}$ is said to be LRD if its normalized covariance function*

$$\tilde{\mathbf{R}}_X(t,s) := \frac{\mathbf{R}_X(t,s)}{(\mathbf{R}_X(t,t)\,\mathbf{R}_X(s,s))^{1/2}} \tag{22}$$

is such that

$$\int_0^{+\infty} |\tilde{\mathbf{R}}_X(t, t+\tau)|\, d\tau = \infty \tag{23}$$

for any fixed t.

Starting from (17), we get that

$$\tilde{\mathbf{R}}_{\mathcal{L}_H Y}(t,s) = \tilde{\gamma}_Y(\log(t/s)), \tag{24}$$

and it follows from a direct calculation that a (nonstationary) H-ss process $\{X(t), t > 0\}$ will be LRD in the sense of Definition 7 if and only if its (stationary) Lamperti counterpart is such that

$$\int_1^\infty |\tilde{\gamma}_{\mathcal{L}_H^{-1} X}(\log \tau)| \, d\tau = \infty. \tag{25}$$

Conversely, a stationary process $\{Y(t), t \in \mathbb{R}\}$ will be LRD in the sense of Definition 6 if and only if its nonstationary (H-ss) Lamperti counterpart is such that

$$\int_1^\infty |\tilde{\mathbf{R}}_{\mathcal{L}_H Y}(t, \lambda t)| \, d\lambda/\lambda = \infty, \tag{26}$$

or, equivalently (since, from (24), we have $\tilde{\mathbf{R}}_{\mathcal{L}_H Y}(t, \lambda t) = \tilde{\mathbf{R}}_{\mathcal{L}_H Y}(t, t/\lambda)$ for any $\lambda > 0$),

$$\int_0^1 |\tilde{\mathbf{R}}_{\mathcal{L}_H Y}(\lambda t, t)| \, d\lambda/\lambda = \infty. \tag{27}$$

Scale-covariant systems In classical linear system theory, it is well-known that linear *filters* are those linear operators $\mathcal{H}$ which are shift-covariant, i.e., such that

$$\mathcal{H}\mathcal{S}_\tau = \mathcal{S}_\tau \mathcal{H} \tag{28}$$

for any $\tau \in \mathbb{R}$. By analogy, it is natural to introduce systems which preserve self-similarity, according to the following definition:

Definition 8 *A linear operator $\mathcal{G}$, acting on processes $\{X(t), t > 0\}$, is said to be scale-covariant if it commutes with any renormalized dilation, i.e., if*

$$\mathcal{G}\mathcal{D}_{H,\lambda} = \mathcal{D}_{H,\lambda}\mathcal{G} \tag{29}$$

for any $H > 0$ and any $\lambda > 0$.

Proposition 1 *If an operator $\mathcal{G}$ is scale-covariant, then it necessarily acts on processes $\{X(t), t > 0\}$ as a multiplicative convolution, according to*

$$(\mathcal{G}X)(t) = \int_0^{+\infty} g(t/s) \, X(s) \, ds/s. \tag{30}$$

Proof — Let $k(t,s)$ be the kernel of some operator $\mathcal{G}$ acting on processes $\{X(t), t > 0\}$. We then have, for any $t > 0$,

$$\begin{aligned}(\mathcal{G}\mathcal{D}_{H,\lambda}X)(t) &= \int_0^{+\infty} k(t,s) \, \lambda^{-H} \, X(\lambda s) \, ds \\ &= \lambda^{-H-1} \int_0^{+\infty} k(t, s/\lambda) \, X(s) \, ds\end{aligned}$$

and

$$(\mathcal{D}_{H,\lambda}\mathcal{G}X)(t) = \lambda^{-H} \int_0^{+\infty} k(\lambda t, s)\, X(s)\, ds.$$

It follows that imposing the scale-covariance of $\mathcal{G}$ for any process $X(t)$ (in the sense of Definition 8) amounts to equating the two above expressions, and thus to require that

$$k(t, s) = k(t/\lambda, s/\lambda)/\lambda \tag{31}$$

for any $t, s > 0$ and any $\lambda > 0$. In particular, the specific choice $\lambda = s$ leads to

$$k(t, s) = k(t/s, 1)/s =: g(t/s)/s, \tag{32}$$

which concludes the proof.

□

Corollary 3 *Scale-covariant operators preserve self-similarity.*

Proof — Let $(\mathcal{G}X)(t)$ be the ouput of a scale-covariant system whose input $\{X(t), t > 0\}$ is H-ss. We then have from (4) and (29):

$$\{(\mathcal{D}_{H,\lambda}\mathcal{G}X)(t) = (\mathcal{G}\mathcal{D}_{H,\lambda}X)(t), t > 0\} \stackrel{d}{=} \{(\mathcal{G}X)(t), t > 0\}, \tag{33}$$

thus guaranteeing that $\{(\mathcal{G}X)(t), t > 0\}$ is H-ss.

□

Corollary 4 *The Lamperti transform maps linear filters onto scale-covariant systems.*

Proof — The output $\{Z(t), t \in \mathbb{R}\}$ of a linear filter $\mathcal{H}$ of impulse response $h(.)$ is given by the convolution

$$Z(t) := (\mathcal{H}Y)(t) = \int_{-\infty}^{+\infty} h(t - s)\, Y(s)\, ds \tag{34}$$

for any input process $\{Y(t), t \in \mathbb{R}\}$. Using (5), we may write

$$\begin{aligned}
(\mathcal{L}_H Z)(t) &= t^H\, Z(\log t) \\
&= t^H \int_{-\infty}^{+\infty} h(\log t - s)\, Y(s)\, ds \\
&= t^H \int_0^{+\infty} h(\log(t/v))\, Y(\log v)\, dv/v \\
&= \int_0^{+\infty} (t/v)^H\, h(\log(t/v))\, (\mathcal{L}_H Y)(v)\, dv/v \\
&= \int_0^{+\infty} (\mathcal{L}_H h)(t/v)\, (\mathcal{L}_H Y)(v)\, dv/v
\end{aligned}$$

and it thus follows that, when "lampertized," the input-ouput relationship (34) is transformed into

$$(\mathcal{L}_H Z)(t) = \int_0^{+\infty} (\mathcal{L}_H h)(t/s)\,(\mathcal{L}_H Y)(s)\,ds/s, \tag{35}$$

taking on the form of a scale-covariant system, according to (30).

□

Fourier transforming (34) leads to a product form for the input-output relationship of linear filters in the frequency domain:

$$(\mathcal{F}Z)(f) = (\mathcal{F}h)(f)\,(\mathcal{F}X)(f), \tag{36}$$

with $\mathcal{F}$ the Fourier transform operator, defined by

$$(\mathcal{F}X)(f) := \int_{-\infty}^{+\infty} X(t)\,e^{-i2\pi ft}\,dt. \tag{37}$$

In a very similar way, Mellin transforming (35) leads to a product form too, as expressed by:

$$(\mathcal{M}\mathcal{L}_H Z)(s) = (\mathcal{M}\mathcal{L}_H h)(s)\,(\mathcal{M}\mathcal{L}_H Y)(s). \tag{38}$$

Continuing along this analogy, H-ss processes can be represented as the output of scale-covariant systems, as stationary processes are outputs of linear filters. More precisely, stationary processes $\{Y(t), t \in \mathbb{R}\}$ are known to admit the Cramér representation [28]

$$Y(t) = \int_{-\infty}^{+\infty} e^{i2\pi ft}\,d\xi(f), \tag{39}$$

with spectral increments $d\xi(f)$ such that

$$\mathbb{E}d\xi(f)\overline{d\xi(\nu)} = \delta(f-\nu)\,d\mathbf{S}_Y(f)\,d\nu, \tag{40}$$

and $d\mathbf{S}_Y(f) = \mathbf{\Gamma}_Y(f)\,df$ in case of absolute continuity with respect to the Lebesgue measure. Stationarity being preserved by linear filtering, stationary processes admit an equivalent representation as in (34):

$$Y(t) = \int_{-\infty}^{+\infty} h(t-s)\,dB(s), \tag{41}$$

with $\mathbb{E}dB(t)dB(s) = \sigma^2\,\delta(t-s)\,dt\,ds$, and therefore:

$$d\mathbf{S}_Y(f) = \sigma^2\,|(\mathcal{F}h)(f)|^2\,df. \tag{42}$$

Applying the Lamperti transformation to (41) ends up with the relation

$$(\mathcal{L}_H Y)(t) = \int_0^{+\infty} (\mathcal{L}_H h)(t/s)\,(\mathcal{L}_H dB)(s)/s. \tag{43}$$

Comparing with (35), this corresponds to the output of a linear scale-covariant system whose input is such that

$$\begin{aligned}\mathbb{E}(\mathcal{L}_H dB)(t)(\mathcal{L}_H dB)(s) &= \mathbb{E}t^H dB(\log t)\, s^H dB(\log s)\\ &= \sigma^2\, (ts)^H\, \delta(\log(t/s))\, dt\, ds\\ &= \sigma^2\, t^{2H+1}\, \delta(t-s)\, dt\, ds,\end{aligned}$$

and it follows that

Proposition 2 *Any H-ss process $\{X(t), t > 0\}$ can be represented as the output of a linear scale-covariant system of impulse response $g(.)$:*

$$X(t) = \int_0^{+\infty} g(t/s)\, dV(s)/s, \tag{44}$$

with

$$\mathbb{E}dV(t)dV(s) = \sigma^2\, t^{2H+1}\, \delta(t-s)\, dt\, ds. \tag{45}$$

Corollary 5 *Given the representation (44), the covariance function of a H-ss process $\{X(t), t > 0\}$ can be expressed as in Eq. (18), with:*

$$c_H(\lambda) = \sigma^2\, \lambda^{-H} \int_0^{+\infty} g(\theta)\, g(\lambda\theta)\, d\theta/\theta^{2H+1}. \tag{46}$$

Corollary 6 *Given the representation (44), the power spectrum density of the stationary counterpart $\{(\mathcal{L}_H^{-1}X)(t), t \in \mathbb{R}\}$ of a H-ss process $\{X(t), t > 0\}$ is given by*

$$\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f) = \sigma^2\, |(\mathcal{M}g)(H + i2\pi f)|^2. \tag{47}$$

One can remark that this result is in accordance with the fact that the Mellin transform of a function g can be equivalently expressed as the Fourier transform of its inverse Lamperti transform according to :

$$(\mathcal{F}\mathcal{L}_H^{-1}g)(f) = (\mathcal{M}g)(H + i2\pi f). \tag{48}$$

3 Examples and applications

Examples and applications of the Lamperti transformation can be broadly classified in two types. One can for instance be interested in "lampertizing" (according to (6)) some specific stationary processes $\{Y(t), t \in \mathbb{R}\}$ and constructing this way classes of specific self-similar processes. From a reversed perspective, one can use the inverse transform (7) for "delampertizing" self-similar processes and making them amenable to the large body of machineries aimed at stationary processes. In this case, some desired operation $\mathcal{T}$ on H-ss processes $\{X(t), t > 0\}$ can rather be handled via the commutative diagram

$$\begin{array}{ccc} X(t) & \xrightarrow{?} & (\mathcal{T}X)(t) \\ \mid & & \uparrow \\ \text{inverse Lamperti} & & \text{Lamperti} \\ \downarrow & & \mid \\ (\mathcal{L}_H^{-1}X)(t) & \longrightarrow & (\tilde{\mathcal{T}}\mathcal{L}_H^{-1}X)(t) \end{array} \tag{49}$$

according to which the overall operation is decomposed as

$$\mathcal{T} = \mathcal{L}_H \tilde{\mathcal{T}} \mathcal{L}_H^{-1}, \tag{50}$$

where the companion operation $\tilde{\mathcal{T}}$ acts on stationary processes.

3.1 Tones and chirps

Besides white noise, maybe the simplest example of a stationary process is

$$Y_0(t) := a \cos(2\pi f_0 t + \varphi), \tag{51}$$

with $a, f_0 > 0$ and $\varphi \in \mathcal{U}(0, 2\pi)$. "Lampertizing" such a random phase "tone," i.e., applying (6) to (51), leads to

$$X_0(t) := (\mathcal{L}_H Y_0)(t) = a\, t^H \cos(2\pi f_0 \log t + \varphi). \tag{52}$$

The transformed process takes therefore on the form of a (random phase) "chirp," in the sense of, e.g., [9,22]. One can remark that $X_0(t) = \mathrm{Re}\{a\, e^{i\varphi}\, m_s(t)\}$, with $s = H + i2\pi f_0$ and $m_s(t) := t^s$ the basic building block of the Mellin transform (see Figure 2).

3.2 Fractional Brownian motion

If we consider second-order processes $\{X(t), t > 0\}$ which are not only H-ss but also have stationary increments (or, "H-sssi" processes), it is well-known that their covariance function is necessarily of the form

$$\mathbf{R}_X(t, s) = \frac{\sigma^2}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right), \tag{53}$$

with $\sigma^2 := \mathbb{E}X^2(1)$.

If we further assume Gaussianity and if we restrict to $0 < H < 1$, we end up with the only family of *fractional Brownian motions* (fBm) $B_H(t)$ [20]. This offers an extension of ordinary Brownian motion $B(t) \equiv B_{1/2}(t)$, known to have

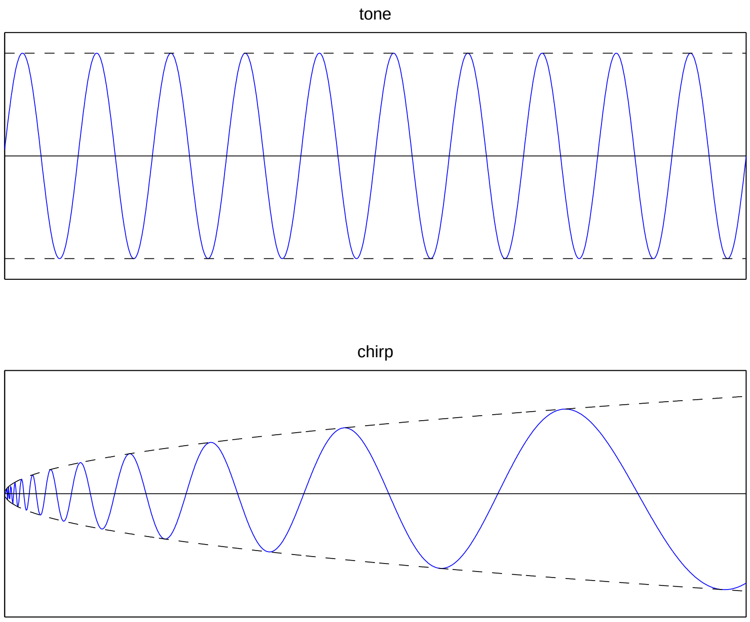

Fig. 2. Tones and chirps — The Lamperti transform of a pure tone (top) is a "chirp" (bottom) with a power-law amplitude modulation and a logarithmic frequency modulation. Said in other words, the Lamperti transform maps the Fourier basis onto a Mellin basis.

uncorrelated increments, to situations where increments may be correlated (negatively if $0 < H < 1/2$ and positively if $1/2 < H < 1$).

Since fBm is H-ss, its covariance function (53) can be factorized according to (18), with

$$c_H(\lambda) = \frac{\sigma^2}{2}\left[\lambda^H + \lambda^{-H}\left(1 - |1-\lambda|^{2H}\right)\right]. \tag{54}$$

By application of (15) to (53), the covariance function of the inverse Lamperti transform $\{Y_H(t) := (\mathcal{L}_H^{-1} B_H)(t), t \in \mathbb{R}\}$ expresses as

$$\mathbf{R}_{Y_H}(t,s) = e^{-H(t+s)}\,\frac{\sigma^2}{2}\,\left(e^{2Ht} + e^{2Hs} - |e^t - e^s|^{2H}\right), \tag{55}$$

and it is immediate to reorganize terms so that $\gamma_{Y_H}(\tau) := \mathbf{R}_{Y_H}(t, t+\tau)$ reads:

$$\gamma_{Y_H}(\tau) = \sigma^2\left(\cosh(H|\tau|) - 2^{2H-1}[\sinh(|\tau|/2)]^{2H}\right). \tag{56}$$

This stationary covariance function is plotted in Figure 3, as a function of the Hurst parameter H. If we let $H = 1/2$ in (56), we readily get

$$\gamma_{Y_{1/2}}(\tau) = \sigma^2\, e^{-|\tau|/2}, \tag{57}$$

in accordance with the known-fact that the (Ornstein-Uhlenbeck) process whose stationary covariance function is given by (57) is the Lamperti image of the ordinary Brownian motion [30]. The stationary counterpart of fBm appears therefore as a form of *generalized Ornstein-Uhlenbeck* (gOU) process.

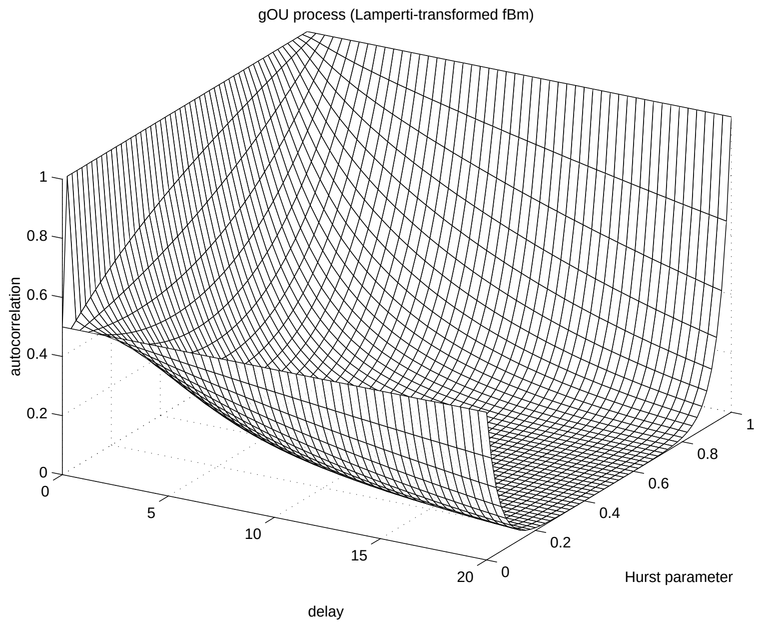

Fig. 3. Stationary covariance function of generalized Ornstein-Uhlenbeck processes — "Delampertizing" fractional Brownian motion (fBm) ends up with a stationary process, referred to as a *generalized Ornstein-Uhlenbeck* (gOU) process, whose covariance function is plotted here as a function of the Hurst parameter H. It is worth noting that this covariance decays exponentially fast for any $H \in (0, 1)$, indicating that gOU is always *short-range dependent*.

As a remark, it is worth noting that resorting to fBm increments rather than to the self-similar process itself guarantees stationarity (and, hence, eases further processing), but at the expense of facing *long-range dependence* (LRD) when $1/2 < H < 1$. In contrast, it follows from (56) that

$$\gamma_{Y_H}(\tau) \sim \frac{\sigma^2}{2} \left(e^{-H\tau} + 2He^{-(1-H)\tau}\right) \propto e^{-\min(H,1-H)\tau} \tag{58}$$

when $\tau \to \infty$, which means that the stationary counterpart of fBm is indeed *short-range dependent* for any $H \in (0, 1)$, since its correlation function decreases exponentially fast at infinity. This result is nevertheless consistent with the fact that, according to (25), fBm itself is LRD in the sense of Definition 7 for any

$H \in (0,1)$ since, for any τ_*,

$$\int_{\tau_*}^{\infty} |\tilde{\gamma}_{Y_H}(\tau)|\, e^{\tau}\, d\tau \sim \int_{\tau_*}^{\infty} e^{[1-\min(H,1-H)]\tau}\, d\tau = \infty. \tag{59}$$

As shown in [25,26], using the Lamperti transformation in the context of linear estimation of self-similar processes makes possible a number of manipulations (such as whitening or prediction) which otherwise prove much more difficult to handle. Indeed, it first follows from (58) that the power spectrum density of the (stationary) Lamperti counterpart of fBm reads

$$\mathbf{\Gamma}_{Y_H}(f) = \frac{\sigma^2}{H^2 + 4\pi^2 f^2} \left| \frac{\Gamma((1/2) + i2\pi f)}{\Gamma(H + i2\pi f)} \right|^2. \tag{60}$$

Given this quantity, it becomes possible to get its spectral factorization and to write $\mathbf{\Gamma}_{Y_H}(f) = |\Phi_+(f)|^2$, with $\Phi_+(f)$ the transfer function of a causal filter. (One can remark that, instead of the exact fBm, we could have considered its (Barnes-Allan [2]) version $\{\tilde{B}_H(t), t > 0\}$, with

$$\tilde{B}_H(t) := \frac{1}{\Gamma(H + 1/2)} \int_0^t (t-s)^{H-1/2}\, dB(s). \tag{61}$$

This corresponds to an H-ss process (with nonstationary increments) that admits a representation as in (44), with

$$g(\theta) = \frac{1}{\Gamma(H + 1/2)} (\theta - 1)^{H-1/2}\, u(\theta - 1), \tag{62}$$

(where $u(.)$ stands for the unit step function) and, as first established in [12], it follows from Corollary 6 that the Lamperti counterpart $\{\tilde{Y}_H(t), t \in \mathbb{R}\}$ of $\{\tilde{B}_H(t), t > 0\}$ has a power spectrum density which turns out to exactly identify with (60).)

Considering the above-mentioned factorization of (60) and using representations of H-ss processes as given by Proposition 2, it then becomes possible [25,26] to re-derive representations formulæ for fBm on a finite interval using a finite interval of ordinary Bm (and vice-versa), as well as to get explicit prediction formulæ for fBm (including a new one for the case $H < 1/2$).

3.3 Ornstein-Uhlenbeck processes

The Ornstein-Uhlenbeck process $\{Y_{1/2}(t), t \in \mathbb{R}\}$ is solution of the Langevin equation:

$$dY(t) + \alpha\, Y(t)\, dt = dB(t), \tag{63}$$

with $\alpha = 1/2$.

Lamperti transforming the general Langevin equation (63), and using appropriate differentiation rules (as justified in [26]), we get

$$\begin{aligned}(\mathcal{L}_H dY)(t) &= t^H\, dY(\log t)\\ &= t^{H+1}\, d(Y(\log t))\\ &= t^{H+1}\, d(t^{-H}\, X(t))\\ &= t^{H+1}\, [t^{-H}\, dX(t) - H\, t^{-H-1}\, X(t)\, dt]\\ &= t\, dX(t) - H\, X(t)\, dt,\end{aligned}$$

with $X(t) := (\mathcal{L}_H Y)(t)$. It thus follows that the H-ss process $\{X(t), t > 0\}$ is solution of

$$t\, dX(t) + (\alpha - H)\, X(t)\, dt = dV(t), \tag{64}$$

where $dV(t) := (\mathcal{L}_H dB)(t)$ is such that $\mathbb{E} dV(t) dV(s) = \sigma^2\, t^{2H+1}\, \delta(t-s)\, dt\, ds$, and is thus covariance-equivalent to $d\tilde{V}(t) := t^{H+1/2}\, dB(t)$.

Indeed, for a given $\alpha > 0$, Ornstein-Uhlenbeck processes admit the integral representation

$$Y_\alpha(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)}\, dB(s), \tag{65}$$

whose Lamperti transform reads

$$X_{\alpha,H}(t) := (\mathcal{L}_H Y_\alpha)(t) = t^{H-\alpha} \int_0^t s^\alpha\, dB(\log s). \tag{66}$$

Noting that $dB(\log t)$ is covariance-equivalent to $t^{-1/2} dB(t)$, we end up with

$$X_{\alpha,H}(t) = t^{H-\alpha} \int_0^t s^{\alpha-1/2}\, dB(s), \tag{67}$$

an expression which can be equivalently rewritten as

$$X_{\alpha,H}(t) = \int_0^{+\infty} [(t/s)^{H-\alpha}\, u(t/s-1)]\, [s^{H+1/2}\, dB(s)]/s. \tag{68}$$

We recognize in (67) the form resulting from the approach described in [24], whereas (68) enters the framework of the general representation (44), with the explicit identification $g(\theta) := \theta^{H-\alpha}\, u(\theta - 1)$ and $dV(t) := s^{H+1/2}\, dB(t)$.

Given $\alpha > 0$, the (Ornstein-Uhlenbeck) solution $Y_\alpha(t)$ of the general Langevin equation (63) is known to have a (stationary) covariance function $\gamma_{Y_\alpha}(\tau)$ which reads

$$\gamma_{Y_\alpha}(\tau) = \sigma^2\, e^{-\alpha|\tau|}, \tag{69}$$

thus generalizing (57). It readily follows from (17) that the Lamperti transform (66)-(68), solution of (64), admits the (nonstationary) covariance function

$$\begin{aligned}\mathbf{R}_{X_{\alpha,H}}(t,s) &= \sigma^2\, (ts)^H\, e^{-\alpha|\log(t/s)|}\\ &= \sigma^2\, (\min(t,s))^{H+\alpha}\, (\max(t,s))^{H-\alpha}.\end{aligned}$$

Letting $\alpha = H$ in the above expression, we get $\mathbf{R}_{X_{H,H}}(t,s) = (\min(t,s))^{2H}$, in trivial generalization of the ordinary Brownian situation, corresponding to $H = 1/2$. In the special case where $\alpha = 1/2$, it follows from the composition rule (9) that the solution $X_{1/2,H}(t)$ of (64) is given by

$$X_{1/2,H}(t) := (\mathcal{L}_H Y_{1/2})(t) = (\mathcal{L}_H \mathcal{L}_{1/2}^{-1} B)(t) = t^{H-1/2}\, B(t), \tag{70}$$

which, as expected, identifies to $B(t)$ too if $H = 1/2$.

In the general case of arbitrary α and H, $\{X_{\alpha,H}(t), t > 0\}$ has been put forward [24] as a versatile two-parameter model, in which H controls self-similarity whereas α may be related to long-range dependence. Indeed, we know from (69) that $\tilde{\gamma}_{Y_\alpha}(\tau) = e^{-\alpha|\tau|}$, and it follows from Definition 7 and (25) that $\{X_{\alpha,H}(t), t > 0\}$ will be LRD if $\alpha < 1$.

3.4 Euler-Cauchy processes

Whereas it is known that Brownian motion is not differentiable in the classical sense, the Langevin equation is usually written as the stochastic (first-order) differential equation

$$\frac{dY}{dt}(t) + \alpha\, Y(t) = W(t), \tag{71}$$

where the "white noise" $W(t)$ (such that $\mathbb{E}W(t)\,W(s) = \sigma^2\,\delta(t-s)$) plays formally the role of a "derivative" for Brownian motion.

The interpretation of (71) is that $Y(t)$ is the output of a first-order linear system whose input is white noise. As such, it may constitute a building block for more complicated systems (with elementary sub-systems in cascade and/or in parallel [18,23]), and it can also be generalized to higher orders, as in ARMA(p,q) processes of the form

$$\sum_{n=0}^{p} \alpha_n\, Y^{(n)}(t) = \sum_{n=0}^{q} \beta_n\, W^{(n)}(t), \tag{72}$$

with the notation $Y^{(n)}(t) := (d^n Y/dt^n)(t)$.

Such (stationary) processes have (self-similar) Lamperti counterparts that are solutions of a generalization of (64) [37].

Lemma 2 *Let $\{Y(t), t \in \mathbb{R}\}$ be a stationary process, with $\{X(t) := (\mathcal{L}_H Y)(t), t > 0\}$ its Lamperti transform. Given a set of coefficients $\{\alpha_n, n = 0, \ldots N\}$, one can find another set of coefficients $\{\alpha'_n, n = 0, \ldots N\}$ such that the Lamperti transform of the linear process*

$$Z(t) = \sum_{n=0}^{N} \alpha_n\, Y^{(n)}(t) \tag{73}$$

takes on the form

$$(\mathcal{L}_H Z)(t) = \sum_{n=0}^{N} \alpha'_n\, t^n\, X^{(n)}(t). \tag{74}$$

Proof — From the definition and the linearity of the Lamperti transform, we may write:

$$\begin{aligned}(\mathcal{L}_H Z)(t) &= \sum_{n=0}^{N} \alpha_n \, (\mathcal{L}_H Y^{(n)})(t) \\ &= \sum_{n=0}^{N} \alpha_n \, t^H \, Y^{(n)}(\log t).\end{aligned}$$

Iterating the differentiation rule

$$Y^{(1)}(t) = t \, \frac{d}{dt}(Y(\log t)),$$

there exist coefficients $\gamma_j(n)$, functionnally dependent on the α_n's, such that the quantity $Y^{(n)}(\log t)$ admits an expansion of the form

$$Y^{(n)}(\log t) = \sum_{j=0}^{N} \gamma_j(n) \, t^j \, \frac{d^j}{dt^j}(Y(\log t)).$$

After a suitable re-organization of terms, we have therefore

$$(\mathcal{L}_H Z)(t) = \sum_{n=0}^{N} \delta_n \, t^{H+n} \, \frac{d^n}{dt^n}(t^{-H} X(t)), \tag{75}$$

with $X(t) = (\mathcal{L}_H Y)(t)$ and

$$\delta_n := \alpha_n \sum_{j=n}^{N} \gamma_n(j).$$

The above expression (75) can be simplified further by remarking that

$$\frac{d}{dt}(t^{-H} X(t)) = -H \, t^{-H-1} \, X(t) + t^{-H} \, X^{(1)}(t),$$

thanks to which there exist coefficients $\mu_k(n)$, functionnally dependent on the δ_n's, such that

$$\frac{d^n}{dt^n}(t^{-H} X(t)) = \sum_{k=0}^{n} \mu_k(n) \, t^{-H+k-n} \, X^{(k)}(t),$$

thus leading to the claimed result (74), with

$$\alpha'_n := \delta_n \sum_{k=0}^{n} \mu_k(n).$$

□

It follows from this Lemma that [37]

Proposition 3 *The stationary ARMA process (72) has an H-ss Lamperti counterpart, referred to as an Euler-Cauchy process, which is solution of an equation of the form*

$$\sum_{n=0}^{p} \alpha'_n \, t^n \, X^{(n)}(t) = \sum_{n=0}^{q} \beta'_n \, t^n \, \tilde{W}^{(n)}(t), \tag{76}$$

with $\tilde{W}(t) = t^{H+1/2}\, W(t)$ *and* $t > 0$.

Proof — The proof, which follows directly from the application of Lemma 2 to both sides of (72), is completed by noting that $(\mathcal{L}_H W)(t) = t^H \, W(\log t)$ has for covariance function

$$\begin{aligned} \mathbb{E}(\mathcal{L}_H W)(t)\,(\mathcal{L}_H W)(s) &= \sigma^2 \, (ts)^H \, \delta(\log(t/s)) \\ &= \sigma^2 \, t^{2H} \, \delta(t/s - 1) \\ &= \sigma^2 \, t^{2H+1} \, \delta(t - s) \\ &= \mathbb{E}\tilde{W}(t)\, \tilde{W}(s), \end{aligned}$$

with $\tilde{W}(t) = t^{H+1/2}\, W(t)$.

□

4 Variations

Given Lamperti's theorem, it is easy to develop variations on the same theme by relaxing in some way the strict notion of scale invariance, or of stationarity.

4.1 Nonstationary tools

Multiplicative harmonizability In the case of nonstationary processes $\{Y(t), t \in \mathbb{R}\}$, a Cramér representation of the type (39) stills holds, but with non orthogonal increments:

$$\mathbb{E} d\xi(f)\overline{d\xi(\nu)} = d^2\mathbf{\Phi}_Y(f, \nu), \tag{77}$$

i.e., with spectral masses which are not located along the only diagonal of the frequency-frequency plane. Provided that Loève's condition

$$\iint_{-\infty}^{+\infty} |d^2\mathbf{\Phi}_Y(f, \nu)| < \infty \tag{78}$$

is satisfied, the corresponding nonstationary processes are referred to as *harmonizable*, and such that

$$\mathbf{R}_Y(t, s) = \iint_{-\infty}^{+\infty} e^{i2\pi(ft - s\nu)} \, d^2\mathbf{\Phi}_Y(f, \nu). \tag{79}$$

A companion concept of *multiplicative harmonizability* can be introduced in the case of processes $\{X(t), t > 0\}$ deviating from exact self-similarity [6]. This readily follows from the "lampertization" of (39) which, together with (78), leads to

$$(\mathcal{L}_H Y)(t) = \int_{-\infty}^{+\infty} t^{H+i2\pi\sigma}\, d\xi(\sigma), \tag{80}$$

whereas the restriction of this general expression to the special case of independent spectral increments leads to the representation considered, e.g., in [12,17,37]. Provided that (78) holds, multiplicatively harmonizable processes $\{Y(t), t > 0\}$ have a (nonstationary) covariance function such that

$$\mathbf{R}_Y(t,s) = \iint_{-\infty}^{+\infty} t^{H+i2\pi f}\, s^{H-i2\pi\nu}\, d^2\mathbf{\Phi}_Y(f,\nu). \tag{81}$$

Time-dependent spectra In the general nonstationary case, (multiplicatively) harmonizable processes have a second-order structure which is described by a two-dimensional function, either in the time-time plane (covariance function) or in the frequency-frequency plane (spectral distribution function). These two equivalent descriptions can be supplemented by *mixed* time-frequency representations interpreted as time-dependent spectra. Starting from (79) and assuming further that me may write $d^2\mathbf{\Phi}_Y(f,\nu) = \tilde{\mathbf{\Phi}}_Y(f,\nu)\, df\, d\nu$, a proper symmetrization of the covariance function, followed by a partial Fourier transform, leads to:

$$\int_{-\infty}^{+\infty} \mathbf{R}_Y(t+\tau/2, t-\tau/2)\, e^{-i2\pi f\tau}\, d\tau = \int_{-\infty}^{+\infty} \tilde{\mathbf{\Phi}}_Y(f+\nu/2, f-\nu/2)\, e^{i2\pi t\nu}\, d\nu. \tag{82}$$

Both sides of the above equation equivalently define the so-called *Wigner-Ville spectrum* (WVS) [14], thereafter labelled $\mathbf{W}_Y(t,f)$.

By construction, the WVS is a nonstationary extension of the classical power spectrum density, and it reduces to the latter in the stationary case: if we have $\mathbf{R}_Y(t,s) = \gamma_Y(t-s)$ or, equivalently, $\tilde{\mathbf{\Phi}}_Y(f,\nu) = \delta(f-\nu)\,\mathbf{\Gamma}_Y(f)$, we simply get

$$\mathbf{W}_Y(t,f) = \int_{-\infty}^{+\infty} \gamma_Y(\tau)\, e^{-i2\pi f\tau}\, d\tau = \mathbf{\Gamma}_Y(f) \tag{83}$$

for all t's. Among the many other properties of the WVS [14], one can cite those related to marginalizations, according to which:

$$\int_{-\infty}^{+\infty} \mathbf{W}_Y(t,f)\, df = \mathbf{R}_Y(t,t) \tag{84}$$

and

$$\int_{-\infty}^{+\infty} \mathbf{W}_Y(t,f)\, dt = \tilde{\mathbf{\Phi}}_Y(f,f). \tag{85}$$

Conventional mixed representations of nonstationary processes are based on Fourier transforms, but alternative forms based on Mellin transforms can also be considered, which prove especially useful in the case of self-similar processes. According to the interconnection (48) which exists between the Fourier, Mellin and Lamperti transforms, and using the notation $\mathbf{R}_{Y,t}(\tau) := \mathbb{E}Y(t+\tau/2)Y(t-\tau/2)$, we have

$$\begin{aligned}\mathbf{W}_Y(t,f) &= (\mathcal{F}\mathbf{R}_{Y,t})(f)\\ &= (\mathcal{M}\mathcal{L}_H\mathbf{R}_{Y,t})(H+i2\pi f)\\ &= \int_0^{+\infty} \tau^H\,\mathbf{R}_Y(t+\log\tau^{+1/2}, t-\log\tau^{-1/2})\,\tau^{-H-i2\pi f-1}\,d\tau,\end{aligned}$$

whence

$$\begin{aligned}\mathbf{W}_Y(\log t,f) &= \int_0^{+\infty} \mathbf{R}_Y(\log t+\log\tau^{+1/2}, \log t-\log\tau^{-1/2})\,\tau^{-i2\pi f-1}\,d\tau\\ &= \int_0^{+\infty} \mathbb{E}Y(\log(t\tau^{+1/2}))Y(\log(t\tau^{-1/2}))\,\tau^{-i2\pi f-1}\,d\tau\\ &= t^{-2H}\,\underline{\mathbf{W}}_{\mathcal{L}_H Y}(t,f), \qquad (86)\end{aligned}$$

with

$$\underline{\mathbf{W}}_X(t,f) := \int_0^{+\infty} \mathbf{R}_X(t\tau^{+1/2}, t\tau^{-1/2})\,\tau^{-i2\pi f-1}\,d\tau. \qquad (87)$$

The above quantity $\underline{\mathbf{W}}_X(t,f)$ is referred to as a *scale-invariant Wigner spectrum* [12], since we have, for any H-ss process $\{X(t), t>0\}$ and any $k>0$:

$$\begin{aligned}\underline{\mathbf{W}}_{\mathcal{D}_{H,k}X}(t,f) &= t^{2H}\,\mathbf{W}_{\mathcal{L}_H^{-1}\mathcal{D}_{H,k}X}(\log t,f)\\ &= t^{2H}\,\mathbf{W}_{\mathcal{S}_{\log k}\mathcal{L}_H^{-1}X}(\log t,f)\\ &= t^{2H}\,\mathbf{W}_{\mathcal{L}_H^{-1}X}(\log(kt),f)\\ &= k^{-2H}\,\underline{\mathbf{W}}_X(kt,f).\end{aligned}$$

Proposition 4 *In the case of H-ss processes $\{X(t), t>0\}$, the scale-invariant Wigner spectrum is a separable function of its two variables which can be factorized as:*

$$\underline{\mathbf{W}}_X(t,f) = t^{2H}\,\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f). \qquad (88)$$

Proof — We know from (86) that

$$\underline{\mathbf{W}}_X(t,f) = t^{2H}\,\mathbf{W}_{\mathcal{L}_H^{-1}X}(\log t,f). \qquad (89)$$

If $\{X(t), t>0\}$ is H-ss, its inverse Lamperti transform $\{(\mathcal{L}_H^{-1}X)(t), t\in\mathbb{R}\}$ is stationary by construction, and Eq. (83) guarantees therefore that the WVS of the latter reduces to its power spectrum density for any t.

$\square$

4.2 From global to local

Locally stationary processes Rather than resorting to processes that are exactly (second-order) stationary, one can make use of the weakened model

$$\mathbf{R}_Y(t,s) = m_Y\left(\frac{t+s}{2}\right)\gamma_Y(t-s), \tag{90}$$

with $m_Y(t) \geq 0$ and $\gamma_Y(.)$ a non-negative definite function. This corresponds to a class of nonstationary processes referred to as *locally stationary* [31], since their symmetrized covariance function is given by

$$\mathbf{R}_Y(t+\tau/2, t-\tau/2) = m_Y(t)\,\gamma_Y(\tau), \tag{91}$$

i.e., as an ordinary stationary covariance function that is allowed to fluctuate as a function of the local time t. From an equivalent perspective, the WVS of a locally stationary process expresses simply as a modulation in time of an ordinary power spectrum, since it factorizes according to:

$$\mathbf{W}_Y(t,f) = m_Y(t)\,\mathbf{\Gamma}_Y(f).$$

When properly "lampertized," locally stationary processes are therefore such that:

$$\mathbf{R}_{\mathcal{L}_H Y}(t,s) = m_Y(\log\sqrt{ts})\,(ts)^H\,\gamma_Y(\log(t/s))$$

and

$$\underline{\mathbf{W}}_{\mathcal{L}_H Y}(t,f) = m_Y(t)\,t^{2H}\,\mathbf{\Gamma}_Y(f).$$

thus generalizing the forms given in (17) and (88), respectively.

Locally self-similar processes Another possible variation is to accommodate for deviations from strict self-similarity, as it may be the case with *locally self-similar processes*, i.e., those processes whose scaling properties are governed by a time-dependent function $H(t)$ in place of a unique constant Hurst exponent H. When dealing with second-order Gaussian processes, a useful framework for such a situation has been developed [1,27], referred to as *multifractional Brownian motion* (mBm). Such processes admit the harmonizable representation

$$B_{H(t)}(t) = \int_{-\infty}^{+\infty} \frac{e^{i2\pi ft}-1}{|f|^{H(t)+1/2}}\,d\xi(f),$$

with $d\xi(f)$ the Wiener spectral measure and $H : [0,\infty) \to [a,b] \subset (0,1)$ any Hölder function of exponent $\beta > 0$. It has been shown [1] that the covariance function of such processes generalizes that of fBm according to

$$\mathbf{R}_{B_{H(t)}}(t,s) = \frac{\sigma^2}{2}\left(t^{h(t,s)} + s^{h(t,s)} - |t-s|^{h(t,s)}\right), \tag{92}$$

with

$$h(t,s) := H(t) + H(s).$$

Using (16) and proceeding as in Section 3.2, it is easy to show that

$$\mathbf{R}_{Y_H}\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)=\sigma^2 e^{(\eta(t,\tau)-2H)t}\left(\cosh[\eta(t,\tau)|\tau|/2]-[2\sinh(|\tau|/2)]^{\eta(t,\tau)}/2\right), \tag{93}$$

with

$$\eta(t,\tau):=h(\exp(t+\tau/2),\exp(t-\tau/2)),$$

and where $\{Y_H(t),t\in\mathbb{R}\}$ stands for the inverse Lamperti transform of mBm, computed with some fixed exponent $H\in(0,1)$:

$$Y_H(t):=(\mathcal{L}_H^{-1}B_{H(t)})(t).$$

If we formally consider the case where $H(t):=H+\alpha\log t$, we have $\eta(t,\tau)=2(H+\alpha t)$ and the process

$$\tilde{Y}(t):=(\mathcal{L}_{H(e^t)}^{-1}B_{H(t)})(t)=e^{-(H+\alpha t)t}\,B_{H(e^t)}(e^t)$$

turns out to be such that

$$\mathbf{R}_{\tilde{Y}}\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)=\sigma^2 e^{-\alpha\tau^2/2}\left(\cosh[(H+\alpha t)|\tau|]-[2\sinh(|\tau|/2)]^{2(H+\alpha t)}/2\right). \tag{94}$$

Comparing with (56), it appears that the above covariance is identical to that of a gOU process, with H replaced, *mutatis mutandis*, by $H+\alpha t$. The interpretation of this result is that, when $H(t)$ admits locally the logarithmic approximation $H(t):=H+\alpha\log t$, "lampertizing" a mBm with the time-dependent exponent $H+\alpha t$ ends up with a process which can be approximated by a (tangential) locally stationary process of gOU type, locally controlled by the same exponent.

4.3 Discrete scale invariance

According to Definition 4, self-similarity usually refers to an invariance with respect to *any* dilation factor λ. In some situations however, this may be a too strong requirement (or assumption) for capturing scaling properties which are only observed for some preferred dilation factors (think of the triadic Cantor set [11], for which exact replication can only be achieved for scale factors $\{\lambda=3^k,k\in\mathbb{Z}\}$, or of the Mellin "chirps" of the form (52) for which scale invariance applies for $\{\lambda=(\exp 1/f_0)^k,k\in\mathbb{Z}\}$ only).

Such a situation, which is referred to as *discrete scale invariance* (DSI), has in fact been recently put forward as a central concept in the study of many critical systems [32], and it has received much attention in a deterministic context. The purpose of this Section is to show that the Lamperti transform may be instrumental in the definition and the analysis of processes which are DSI in a stochastic sense [6,7].

Definitions

Definition 9 *A process $\{Y(t), t \in \mathbb{R}\}$ is said to be periodically correlated of period T_0 (or "T_0-cyclostationary") if*

$$\{(\mathcal{S}_{T_0} Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}. \tag{95}$$

Definition 10 *A process $\{X(t), t > 0\}$ is said to possess a discrete scale invariance of index H and of scaling factor $\lambda_0 > 0$ (or to be "(H, λ_0)-DSI") if*

$$\{(\mathcal{D}_{H,\lambda_0} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}. \tag{96}$$

It naturally follows from these two definitions that T_0-cyclostationary processes are also T-cyclostationary for any $T = kT_0, k \in \mathbb{Z}$, and that (H, λ_0)-DSI processes are also (H, λ)-DSI for any $\lambda = \lambda_0^k, k \in \mathbb{Z}$.

Given Definition 9, second-order T_0-cyclostationary processes $\{Y(t), t \in \mathbb{R}\}$ have a covariance function $\mathbf{R}_Y(t, t+\tau)$ which is periodic in t of period T_0, and which can thus be decomposed in a Fourier series according to

$$\mathbf{R}_Y(t, t+\tau) = \sum_{n=-\infty}^{\infty} \mathbf{C}_n(\tau)\, e^{i2\pi nt/T_0}. \tag{97}$$

One deduces from this representation that the spectral distribution function of T_0-cyclostationary processes takes on the form:

$$\begin{aligned}
\tilde{\mathbf{\Phi}}_Y(f, \nu) &= \iint_{-\infty}^{+\infty} \mathbf{R}_Y(t, s)\, e^{-i2\pi(ft - \nu s)}\, dt\, ds \\
&= \iint_{-\infty}^{+\infty} \mathbf{R}_Y(t, t+\tau)\, e^{-i2\pi((f-\nu)t - \nu\tau)}\, dt\, d\tau \\
&= \sum_{n=-\infty}^{\infty} \mathbf{c}_n(\nu)\, \delta(\nu - (f - n/T_0)),
\end{aligned}$$

with

$$\mathbf{c}_n(\nu) := \int_{-\infty}^{+\infty} \mathbf{C}_n(\tau)\, e^{-i2\pi\nu\tau}\, d\tau. \tag{98}$$

In contrast with the stationary case for which $\tilde{\mathbf{\Phi}}_Y(f, \nu)$ is entirely concentrated along the main diagonal $\nu = f$ of the frequency-frequency plane, the spectral distribution function of cyclostationary processes is also non-zero along all the equally spaced parallel lines defined by $\nu = f - n/T_0, n \in \mathbb{Z}$.

More on the theory of cyclostationary processes can be found, e.g., in [16].

Characterization and analysis It has been stated in Theorem 1 that the Lamperti transformation establishes a one-to-one correspondence between stationary and self-similar processes. An extension of this result is given by the following Theorem:

Theorem 2 *If $\{Y(t), t \in \mathbb{R}\}$ is T_0-cyclostationary, then its Lamperti transform $\{(\mathcal{L}_H Y)(t), t > 0\}$ is (H, e^{T_0})-DSI. Conversely, if $\{X(t), t > 0\}$ is (H, e^{T_0})-DSI, its inverse Lamperti transform $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ is T_0-cyclostationary.*

Proof — Let $\{Y(t), t \in \mathbb{R}\}$ be a T_0-cyclostationary process. From Definition 9, we have $\{Y(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t + T_0), t \in \mathbb{R}\}$ and, using (6), we may write

$$\begin{aligned}(\mathcal{L}_H Y)(e^{T_0} t) &= (e^{T_0} t)^H \, Y(\log t + T_0) \\ &\stackrel{d}{=} e^{H T_0} \, t^H \, Y(\log t) \\ &= (e^{T_0})^H \, (\mathcal{L}_H Y)(t),\end{aligned}$$

thus proving that $\{(\mathcal{L}_H Y)(t), t > 0\}$ is (H, e^{T_0})-DSI.

Conversely, let $\{X(t), t > 0\}$ be a (H, e^{T_0})-DSI process. From Definition 10, we have $\{X(e^{T_0} t), t > 0\} \stackrel{d}{=} \{e^{H T_0} \, X(t), t > 0\}$ and, using (7), we may write

$$\begin{aligned}(\mathcal{L}_H^{-1} X)(t + T_0) &= e^{-Ht} \, e^{-H T_0} \, X(e^{T_0} \, e^t) \\ &\stackrel{d}{=} e^{-Ht} \, X(e^t) \\ &= (\mathcal{L}_H^{-1} X)(t),\end{aligned}$$

thus proving that $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ is T_0-cyclostationary.

□

Since DSI processes result from a "lampertization" of cyclostationary processes, the form of their covariance function can readily be deduced from the general correspondence (16) when applied to the specific form (97). We get this way that (H, λ_0)-DSI processes $\{X(t), t > 0\}$ have a covariance function such that:

$$\mathbf{R}_X(t, kt) = (kt)^H \sum_{n=-\infty}^{\infty} \mathbf{C}_n(\log k) \, t^{H + i2\pi n/T_0}, \tag{99}$$

with $T_0 = \log \lambda_0$. Plugging this expression into (87), we also get an expansion for the corresponding scale-invariant Wigner spectrum:

$$\underline{\mathbf{W}}_X(t, f) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n(f - n/2T_0) \, t^{2H + i2\pi n/T_0}.$$

While such representations might suggest to make use of Mellin-based tools for analyzing DSI processes by working directly in the observation space, Theorem 2 offers another possibility of action by first "delampertizing" the observed scaling data so as to make them amenable to more conventional cyclostationary techniques (see, e.g., [15,29]). This is in fact the procedure followed in [6,7], where the existence of stochastic DSI is unveiled by marginalizing an estimated cyclic spectrum computed on the "delampertized" data.

Examples *Weierstrass-Mandelbrot* — Let us consider the process

$$X(t) = \sum_{n=-\infty}^{\infty} \lambda^{-Hn} G(\lambda^n t) e^{i\varphi_n}, \tag{100}$$

with $\lambda > 1, 0 < H < 1, \varphi_n \in \mathcal{U}(0, 2\pi)$ i.i.d. random phases and $G(.)$ a 2π-periodic function. We get this way a generalization of the (randomized) Weierstrass-Mandelbrot function [4], the latter corresponding to the specific choice:

$$G(t) = 1 - e^{it}. \tag{101}$$

It is immediate to check that

$$\begin{aligned}(\mathcal{D}_{H,\lambda}X)(t) &= \lambda^{-H} \sum_{n=-\infty}^{\infty} \lambda^{-Hn} G(\lambda^n \lambda t) e^{i\varphi_n} \\ &= \sum_{n=-\infty}^{\infty} \lambda^{-Hn} G(\lambda^n t) e^{i\varphi_{n-1}} \\ &\stackrel{d}{=} X(t),\end{aligned}$$

thus guaranteeing that $\{X(t), t > 0\}$ is (H, λ)-DSI. In a similar way, Lamperti transforming (100) leads to

$$\begin{aligned}(\mathcal{L}_H^{-1}X)(t) &= e^{-Ht} \sum_{n=-\infty}^{\infty} \lambda^{-Hn} G(\lambda^n e^t) e^{i\varphi_n} \\ &= \sum_{n=-\infty}^{\infty} e^{-H(t+n\log\lambda)} G(e^{t+n\log\lambda}) e^{i\varphi_n} \\ &= \sum_{n=-\infty}^{\infty} (\mathcal{L}_H^{-1}G)(t + n\log\lambda) e^{i\varphi_n},\end{aligned}$$

from which we deduce that

$$\begin{aligned}(\mathcal{S}_{\log\lambda}\mathcal{L}_H^{-1}X)(t) &= \sum_{n=-\infty}^{\infty} (\mathcal{L}_H^{-1}G)(t + \log\lambda + n\log\lambda) e^{i\varphi_n} \\ &= \sum_{n=-\infty}^{\infty} (\mathcal{L}_H^{-1}G)(t + n\log\lambda) e^{i\varphi_{n-1}} \\ &\stackrel{d}{=} (\mathcal{L}_H^{-1}X)(t),\end{aligned}$$

evidencing therefore that the "delampertized" process $\{(\mathcal{L}_H^{-1}X)(t), t \in \mathbb{R}\}$ becomes $\log\lambda$-cyclostationary, as expected from Theorem 2. In the case where the phases would not be randomly chosen, but all set to the same given value (say, 0), the "delampertized" version of (100) would simply takes the form of a periodic function [33] (see Figure 4).

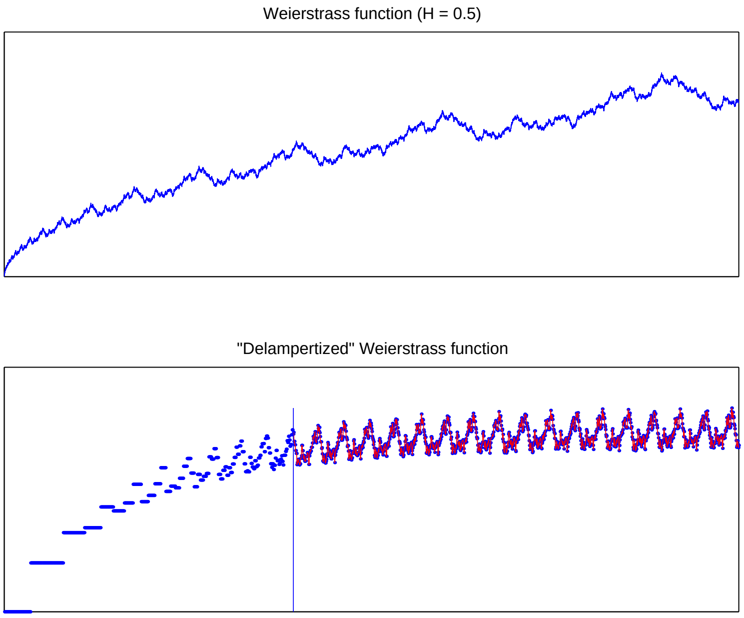

Fig. 4. Discrete scale invariance and cyclostationarity — The Weierstrass function (top) is not scale-invariant for all dilation factors, as evidenced by log-periodic fluctuations. "Delampertizing" this function ends up in a periodic function (bottom).

As a DSI process, $\{X(t), t > 0\}$ is necessarily nonstationary. However, introducing the notation

$$\Delta Z(t, \tau) := Z(t + \tau) - Z(t)$$

for the increment process of a given process $Z(t)$, we readily get from the definition (100) that

$$\Delta X(t, \tau) = \sum_{n=-\infty}^{\infty} \lambda^{-Hn}\, \Delta G(\lambda^n t, \lambda^n \tau)\, e^{i\varphi_n},$$

and it follows that the variance of this increment process expresses as

$$\begin{aligned}\mathbb{E}|\Delta X(t, \tau)|^2 &= \sum_{n,m=-\infty}^{\infty} \lambda^{-H(n+m)}\, \Delta G(\lambda^n t, \lambda^n \tau)\, \overline{\Delta G(\lambda^m t, \lambda^m \tau)} \\ &\qquad\qquad \times \iint_0^{2\pi} e^{i(\varphi_n - \varphi_m)}\, \frac{d\varphi_n}{2\pi}\, \frac{d\varphi_m}{2\pi} \\ &= \sum_{n=-\infty}^{\infty} \lambda^{-2Hn}\, |\Delta G(\lambda^n t, \lambda^n \tau)|^2 .\end{aligned}$$

In the specific case of the usual Weierstrass-Mandelbrot process defined through (101), it is interesting to note that

$$|\Delta G(\lambda^n t, \lambda^n \tau)|^2 = 2\ (1 - \cos \lambda^n \tau)\,,$$

evidencing the fact that the increment process $\Delta X(t, \tau)$ has a variance which does not depend on time t [4].

Parametric models — It has been shown in Section 3.4 that continuous-time H-ss processes can be obtained from Euler-Cauchy systems driven by some appropriately modulated white noise. Since such systems result from the lampertization of classical ARMA systems, it follows that varying their coefficients in a log-periodic way in time will generate DSI processes, in exactly the same way as cyclostationary processes can be obtained as the output of a (nonstationary) ARMA system with periodic time-varying coefficients.

The problem of getting corresponding models in discrete-time would need a specific discussion, and it will not be addressed here. Referring to [6,7] for some further details and illustrations, we will only mention that two preliminary approaches have been considered so far, both based on the idea of introducing a log-periodicity in the coefficients of a discrete-time model. In the first approach, the discretization is obtained by integrating the evolution of a continuous-time Euler-Cauchy system between two time instants, leading to an approximate form of DSI. In the second approach, a fractional difference operator (discretized, e.g., as in [38]) is introduced, and it is cascaded with a nonstationary AR filter whose coefficients are log-periodic.

5 Conclusion

The Lamperti transform is a simple way of connecting the two key concepts of stationarity and self-similarity, which are both known to be ubiquitously encountered in many applications. As such, it has been shown to offer simpler alternative viewpoints on some known problems, while providing new insights in their understanding. From a more innovative perspective, it has also been advocated as a new starting point for the analysis, modelling and processing of situations which depart from strict stationarity and/or self-similarity.

The purpose of this text was to collect and develop a number of general results related to the Lamperti transform and to support its revival but, of course, much work is still to be done in different directions. One can think of a number of further natural (e.g., multidimensional) extensions, as well as of the need for efficient algorithmic tools, based in particular on genuinely discrete-time formulations of the transform.

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