THE ARITHMETIC OF $1/F$ NOISE IN A PHASE LOCKED LOOP

Michel Planat and Edouard Henry

Laboratoire de Physique et Métrologie du CNRS,
associé á l'Université de Franche-Comté,
32 avenue de l'Observatoire, 25044 Besançon Cedex, France
e-mail: planat@lpmo.edu

Abstract

Frequency countings close to a phase locked zone in an electronic receiver show a $1/f$ power spectral density. The noise scaling versus the frequency deviation and the open loop gain is found from Adler's model of the phase locked loop. This fully agrees with experiments performed at 5 MHz on a receiver with a Schottky diode mixer and a low pass filter. The $1/f$ amplitude and frequency noise due to the whole set of (sub)harmonics is explained from a nonlinear mapping, with a coupling coefficient related to the structure of prime numbers. The number theoretical approach of $1/f$ noise and the link to the generalized Riemann hypothesis is explained.

Keywords: oscillators, phase locking, $1/f$ noise, number theory

1 INTRODUCTION

The nature of the ubiquitous $1/f$ noise found in most electronic circuits still remains uncertain [1, 2]. Despite our lack of a universal accepted model we have at one's disposal a genuine generator of $1/f$ noise: the phase locked loop (PLL) used in a FM radio receiver [3]. In the time and frequency field, it is used to register minute frequency fluctuations of an oscillator under test (RF) versus the frequency of a local oscillator (LO). The measure of frequency fluctuations is in general taken to be the Allan variance [4, 5], that is the mean squared value $\sigma^2(\tau)$ of the relative frequency deviation between adjacent samples in the time series, counted over an integration time τ. The loop includes a wide band mixer, a low pass filter and an amplifier; the local oscillator is voltage controlled.

Looking at the whole (sub)harmonic structure we established that the open loop receiver realizes the continued fraction expansion of the frequency ratio $\nu = \omega/\omega_0$ between the input oscillators [6, 7]. Thus arithmetic enters in a natural way into the frequency countings at the output of the detector. New effects were also observed in the closed loop: i) the phase locked zone equals twice the open loop gain, ii) close to the phase locked zone, the white frequency noise with Allan deviation proportional to $\tau^{-1/2}$ and the flicker pedestal of Allan deviation σ transform into $1/f$ frequency noise of constant Allan deviation $\tilde{\sigma}$, iii) the level $\tilde{\sigma}$ of the $1/f$ noise approximately scales with the open loop gain K and the beat frequency $\tilde{\omega}_{\mathrm{LF}}$ as $\tilde{\sigma} = \sigma K/\tilde{\omega}_{\mathrm{LF}}$. We could conclude that, either the PLL set-up behaves as a microscope of the flicker floor σ, or the $1/f$ noise is a nonlinear dynamical property of the PLL [8]. We looked at a possible low dimensional structure of the time series an found a stable embedding dimension lower or equal to four [9, 10]. But the dynamical model of $1/f$ noise still remained elusive [3]. In contrast $1/f$ noise may well be a property of prime number distributions [6, 11].

In this paper it is shown that Adlers's model [12] of the PLL correctly predicts the main features of the fundamental beat note as well as the magnification of $1/f$ noise close to base band. To explain the origin of $1/f$ noise we derive a more general model accounting for the (sub)harmonics, which is based on an Arnold map [14] with a digital coupling coefficient related to prime number theory.

2 EXPERIMENTS

For the experiments we used a quartz crystal (LO) oscillator with a constant frequency $f_0 = 2\pi\omega_0 = 5.0206$ MHz, a (RF) synthesizer of variable frequency $f = 2\pi\omega$, a balanced mixer with Schottky diodes and a filter with a low pass frequency equal to 10 kHz. The PLL was obtained by applying the beat note to the voltage controlled RF oscillator. To a first approximation the phase shift $\theta(t)$ between the two oscillators can be described from Adler's model [12]

$$\dot{\theta}(t) + K \sin\theta(t) = \omega_{\mathrm{LF}}, \qquad (1)$$

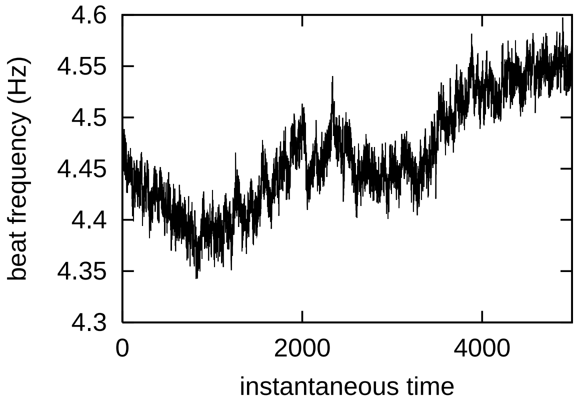

Figure 1: Beat frequency close to the phase locked zone 1/1 of a PLL ($f_0 = 5$ MHz). The power spectrum of these data has a pure $1/f$ dependance and the Allan deviation is flat with $\tilde{\sigma} \simeq 0.0095$.

where $\omega_{\rm LF} = \omega(t) - \omega_0$ is the frequency shift of input oscillators. We are interested in the resulting beat note $\tilde{\omega}_{\rm LF}$ close to phase locking that one derives from (1)

$$\tilde{\omega}_{\rm LF} = (\omega_{\rm LF}^2 - K^2)^{1/2}. \quad (2)$$

Using differentiation with respect to the frequency shift $\omega_{\rm LF}$ one easily obtains [13]

$$\delta\tilde{\omega}_{\rm LF} = \delta\omega_{\rm LF}(1 + K^2/\tilde{\omega}_{\rm LF}^2)^{1/2}. \quad (3)$$

Relation (2) is defined outside the mode locked zone $|\omega_{\rm LF}| > K$ of width $2K$; close to it, if the effective beat note $\tilde{\omega}_{\rm LF} \le K$, the square root term is about $K/\tilde{\omega}_{\rm LF}$. If one identifies $\delta\omega_{\rm LF}/\tilde{\omega}_{\rm LF}$ with the pedestal Allan deviation σ and $\delta\tilde{\omega}_{\rm LF}/\tilde{\omega}_{\rm LF}$ with the magnified Allan deviation $\tilde{\sigma}$ we get the announced result. In

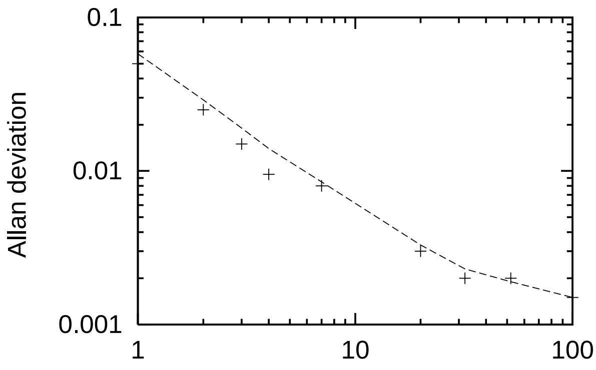

Figure 2: Allan deviation $\tilde{\sigma} = \delta\tilde{\omega}_{\rm LF}/\tilde{\omega}_{\rm LF}$ versus beat frequency $\tilde{\omega}_{\rm LF}$ at constant gain $K = 40$ Hz. A good fit of (3) is obtained (dotted line).

Fig. 1 we counted the instantantaneous beat signal at $\tilde{\omega}_{\rm LF} \simeq 4.5$ Hz with an open loop gain 40 Hz. The time series was found to produce an almost constant Allan deviation $\tilde{\sigma}$ irrespective of the integration time τ, which is characteristic of a $1/f$ noise of power spectral density $S(f) = \tilde{\sigma}/(2\ln 2f)$. Varying $\tilde{\omega}_{\rm LF}$ from 1 Hz to 100 Hz at constant gain we still found $1/f$ noise with a good fit of relation (3) as shown in Fig. 2. Similarly we varied the gain K from 2 Hz to 2 kHz and found the expected linear dependance of the flicker floor as soon as $K \ge \tilde{\omega}_{\rm LF}$. It is now clear that the model based on Adler's equation correctly predicts the amplitude of $1/f$ noise but fails to discover its origin.

3 ARITHMETIC OF OPEN LOOP BEAT FREQUENCY MEASUREMENTS

Low frequency noise of electronic oscillators is usefully interpreted in terms of arithmetic[6]: this is because the measurement of the frequency $f(t)$ of an oscillator under test compares the one f_0 of a reference oscillator thanks to a nonlinear mixing set-up and a filter. The beat frequency

$$f_B = |p_i f_0 - q_i f(t)|\,, \quad \text{with } \mathrm{p_i} \text{ and } \mathrm{q_i} \text{ integers}, \quad (4)$$

follows from the continued fraction expansion of the frequency ratio $\nu = \frac{f}{f_0} = [a_0; a_1, a_2, ...a_i, a, ...] = a_0 + 1/\{a_1 + 1/\{a_2 + ... + 1/\{a_i + 1/\{a...\}\}\}\} = \frac{p_i(a)}{q_i(a)} \simeq \frac{p_i}{q_i}$ of the input oscillators. Here $a_{\min} \le a \le a_{\max}$, with $a_{\min} = \lfloor \frac{f_0}{f_c q_i} \rfloor$, $a_{\max} = \lfloor \frac{f_0}{f_d q_i} \rfloor$ and f_c and f_d are the low and high frequency cut-off of the filter. Since $a \gg 1$ in typical measurements, the beat note is well approximated by the convergent p_i/q_i used in (4) which restricts to the partial quotient a_i in the expansion. Fig. 3 shows a schematic of the resulting intermodulation spectrum [7].

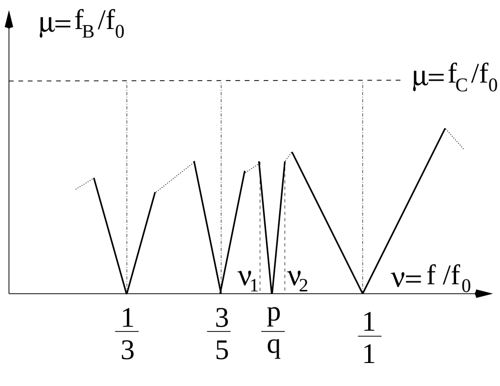

Figure 3: The intermodulation spectrum at the output of the mixer+filter set-up.

In some bad circumstances [6] the partial quotients after a don't play any role and frequency jumps occurs randomly at definite values of a leading to a large white frequency noise arising from the detection set-up instead of the oscillator under test. This can be compared to the measurement of time from a moon-sun calendar. Early calendars have been devised from the motion of moon and sun as observed from the earth. The continued fraction expansion of the ratio ν between the sun year and the moon year is

$$\nu = \frac{365.242191}{29.530589} = [12; 2, 1, 2, 1, 1, 17, \ldots]. \quad (5)$$

The first approximation $\nu = 12$ (with 354 days) can be corrected by adding one month every two years, the second one (with 369 days) may be corrected by adding one month every three years and so on. Fluctuation of the integer a in the frequency measurement set-up has the same aim to correct the measurement versus time.

4 ARITHMETIC OF PHASE LOCKING AND $1/F$ FREQUENCY NOISE

If one accounts for the whole set of harmonics, the differential equation for the phase shift $\dot{\theta}(t; q_i, p_i)$ at the harmonic (p_i, q_i) of angular beat frequency $\omega_{\mathrm{LF}} = |p_i\omega_0 - q_i\omega(t)|$, can be written as [7]

$$\begin{aligned}&\dot{\theta}(t; q_i, p_i) + q_i\ H(P) \sum_{r_i, s_i} K(r_i, s_i)\\ &\times \sin(\frac{s_i}{q_i}\ \theta(t; q_i, p_i) - \frac{\omega_0 t}{q_i}(q_i r_i - p_i s_i) + \theta_0(r_i, s_i))\\ &= \omega_{\mathrm{LF}}(p_i, q_i). \quad (6)\end{aligned}$$

Here the index i in p_i and q_i indicates the continued fraction expansion $\nu = p_i/q_i$ at the level of approximation i imposed by the resolution constraints in the experiment [6]. The notation $K(r_i, s_i)$ means the effective gain at harmonic r_i/s_i and $H(P)$, where the operator $P = d/dt$ is the open loop transfer function. Solving (6) is a difficult task. It is enough here to observe that the reference signal at frequency ω_0 acts as a periodic perturbation of the standard Adler's model of the PLL. If one neglects harmonic interactions, (6) simplifies to the standard Arnold map model [7]

$$\theta_{n+1} = \theta_n + 2\pi\Omega - c\ \sin\theta_n, \quad (7)$$

where $\Omega = \omega/\omega_0$ is the bare frequency ratio and $c = K/\omega_0$. Such a nonlinear map is studied by introducing the winding number $\nu = \lim_{n\to\infty}(\theta_n - \theta_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$, the curve ν versus Ω is a devil's staircase (as shown in Fig. 3) with steps attached to rational values of $\Omega = p_i/q_i$ and with width increasing with the coupling coefficient c. The phase locking zones may overlap if $c > 1$ leading to chaos from quasiperiodicity [14].

To appreciate the impact of harmonics on the coupling coefficient one should observe that each harmonic of denominator q_i leads to the same fluctuating frequency $\delta\omega_{\mathrm{LF}} = q_i\delta\omega(t)$. There are $\varphi(q_i)$ of them, where $\varphi(q_i)$ is Euler totient function, that is the number of integers less or equal to q_i and prime to it; the average coupling coefficient is thus expected to be $1/\varphi(q_i)$ [15]. We developed a more refined model based on the properties of primes by defining the coupling coefficient as $c^* = c\Lambda(n; q_i, p_i)$ with

$$\Lambda(n; q_i, p_i) = \begin{cases} \ln b & \text{if } n = b^k, b \text{ a prime and } \mathrm{n \equiv p_i \bmod (q_i)} \\ 0 & \text{otherwise.}\end{cases} \quad (8)$$

This means a non zero coupling to harmonics at times $n = p_i + q_i\ l$, l integer whenever n is a power of a prime; the coupling at the fundamental mode is the so-called von Mangoldt function $\Lambda(n)$[16].

According to the generalized Riemann hypothesis one gets the average[17]

$$c^*_{\mathrm{av}}/c = \frac{1}{t}\sum_{n=1}^{t}\Lambda(n; q_i, p_i) = \frac{1}{\varphi(q_i)} + \varepsilon(t), \quad (9)$$

with $\varepsilon(t) = 0(t^{-1/2} ln^2(t))$ which is a good approximation as long as $q_i < \sqrt{t}$. A better estimate may also be obtained at larger q_i[17].

For $p_i/q_i = 1/1$ the fluctuating term may be expressed in terms of the zeros of the Riemann zeta function $\zeta(s)$. The trivial ones at $s = -2l$, l integer, connect to Bernouilli polynomials; according to Riemann hypothesis (still unproved), they are infinitely many non trivial zeros at the critical line $s = 1/2$ which are randomly distributed. In the general case of the p_i/q_i harmonic, $\zeta(s)$ generalizes to a Dirichlet series. One finds numerically that the power spectral density of the fluctuating term looks like a $1/f$ noise [6].

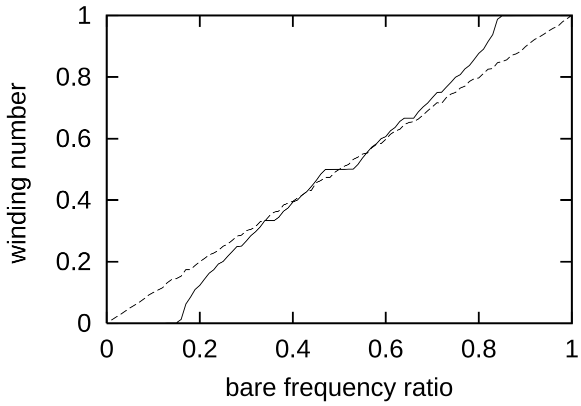

Figure 4: Phase locking steps for the Arnold map (plain line), and lack of phase locking steps for the Arnold–von Mangoldt map (dotted line); coupling coefficient $c = 1$.

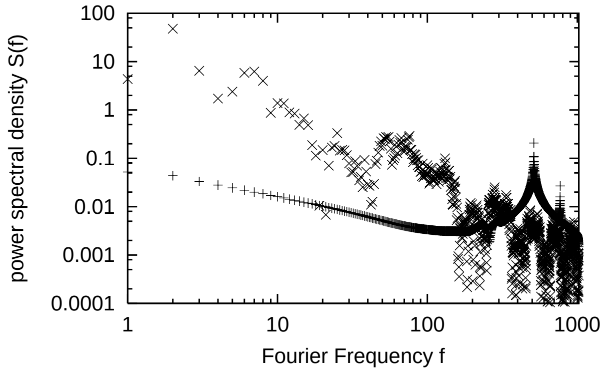

Figure 5: $1/f$ noise of the winding number in the Arnold–von Mangoldt map ($\times$) in comparison to the Arnold map ($+$); coupling coefficient $c = 0.2$, frequency ratio $\Omega = 7/10$.

We have studied numerically phase locking properties of mapping (7) with c^*_{av} as the coupling coefficient in place of the standard one c. The more drastic effect is to prevent the phase locking of the oscillators. Fig. 3 shows that the phase locking steps are removed. In addition the $1/f$ noise in the power spectral density of $\varepsilon(t)$ converts to $1/f$ noise in the fluctuation of the frequency ratio ν as shown in Fig. 4.

In conclusion the present theory relates $1/f$ noise to a phase locking between two oscillators. The level of experimental $1/f$ frequency noise follows Adler's model of the PLL. Assuming that the coupling coefficient of (sub)harmonics is related to prime numbers, a new model of $1/f$ amplitude and frequency fluctuations is proposed. For another link between number theory and physics see http://www.maths.ex.ac.uk/~mwatkins/zeta/physics.htm.

5 NUMBER THEORY OF $1/F$ NOISE

5.1 The $1/f$ Noise Term and Riemann Hypothesis

We remind the relationship between the fluctuating term $\varepsilon(t)$ and the theory of the Riemann zeta function[16].

Riemann zeta function $\zeta(s)$ is defined from the Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{b \text{ prime}} \frac{1}{1 - \frac{1}{b^s}} \quad \text{where } \Re(s) > 1. \quad (10)$$

Riemann's great achievment in 1859 was his ability to complete the formula to the whole complex plane of the parameter s. By logarithmic derivation (10) can be rewritten as

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$$

$$= \int_1^{\infty} t^{-s} d\psi(t)$$

$$= s \int_1^{\infty} t^{-s-1} \psi(t) dt, \quad (11)$$

with $\Re(s) > 1$ and the von Mangoldt function $\Lambda(n) = \ln(b)$ if $n = b^k$, b a prime and 0 otherwise. Function $\psi(t) = \sum_{n \le t} \Lambda(n)$ is the summatory function.

The inverse transform

$$\psi(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} t^s \frac{ds}{s} \quad \text{with c} = \Re(\text{s}) > 1, \quad (12)$$

allows an estimate of $\psi(t)$ if one knows the singularities of $\zeta(s)$. The pole of $-\frac{\zeta'(s)}{\zeta(s)}$ at $s = 1$ contributes t; the pole $1/s$ at $s = 0$ contributes $-\frac{\zeta'(0)}{\zeta(0)} = -\ln(2\pi)$ and the zeros ϱ contribute $-\frac{t^\varrho}{\varrho}$. One gets

$$\psi(t) = t(1 + \varepsilon(t)),$$
$$t\varepsilon(t) = -\ln(2\pi) - \frac{1}{2}\ln(1 - t^{-2}) - \sum_\varrho \frac{t^\varrho}{\varrho}. (13)$$

The second term in $\varepsilon(t)$ is due to the trivial zeros of $\zeta(s)$ which are located at $s = -2l$ (l a positive integer). The third term is due to the remaining zeros of $\zeta(s)$. Billions of them have been computed; all are found to be located on the line $s = \frac{1}{2}$. Riemann hypothesis is the (unsolved) conjecture that all non trivial zeros belong to the critical line. These zeros are very irregularly spaced and are responsible for the very irregular shape of the error term as shown in Fig. 4. It was

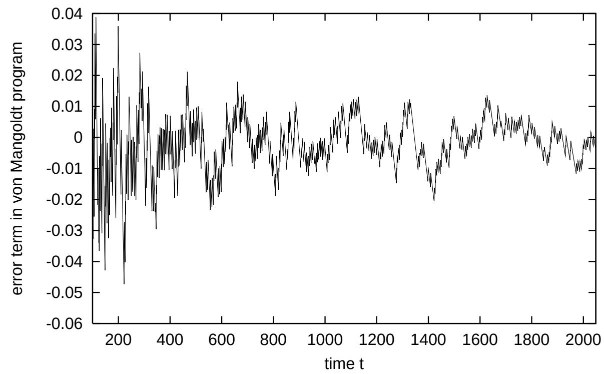

Figure 6: The error term $\varepsilon(t)$ in the summatory function $\psi(t)$.

shown numerically[6] that the power spectral density of $\varepsilon(t)$ has a $1/f$ dependance on the Fourier frequency f.

The fluctuating term $\sum_\varrho \frac{t^\varrho}{\varrho}$, where $\varrho = \frac{1}{2} + iy$, can be bounded if one knows the number $N(y)$ of zeros between 0 and y. Since $N(y) < y\ln(y)$, assuming Riemann hypothesis, this implies the von Koch estimate $\varepsilon(t) = O(t^{-1/2}\ln^2(t))$ of the error term[17].

5.2 Harmonic Interactions, $1/f$ Noise and the Generalized Riemann Hypothesis

Euler's identity (10) can be generalized to the Dirichlet L-series

$$L(s,\kappa) = \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \prod_{b \text{ prime}} \frac{1}{1-\kappa(b)b^{-s}}, \ \Re(s) > 1$$
$$\text{with } \kappa(\mathrm{n}) = \kappa(\mathrm{n}) \bmod(\mathrm{q}) \ \text{ for } (\mathrm{n},\mathrm{q}) = 1$$
$$\text{and } 0 \ \text{ otherwise.} \qquad (14)$$

In (14) the notation $(n,q) = 1$ means that n and q are coprimes[1]. The Dirichlet character $\kappa(n)$ is thus a multiplicative function.

Using $\Lambda(n;q,p)$ in place of $\Lambda(n)$, (11)-(13) can be generalized to the summatory function $\psi(t;q,p) = \sum_{n \le t} \Lambda(n;q,p)$, where $n \equiv p \bmod(\mathrm{q})$, with the result

$$\psi(t;q,p) = t(1+\varepsilon(t;q,p)),$$
$$\text{with } \mathrm{t}\varepsilon(\mathrm{t};\mathrm{q},\mathrm{p}) =$$
$$-\frac{L'(0,\kappa)}{L(0,\kappa)} + \sum_{m\ge1} \frac{t^{1-2m}}{2m-1} - \sum_{\varrho} \frac{t^{\varrho}}{\varrho}. \qquad (15)$$

The error term is shown in Fig. 5. As above its power spectral density approximates a $1/f$ law. The fluctuating term can be bounded assuming the generalized Riemann hypothesis that all non trivial zeros of L-functions belongs to the critical line $s = \frac{1}{2}$. The same (poor) estimate[17] $\varepsilon(t;q,p) = O(t^{-1/2}) \ln^2(t))$ follows.

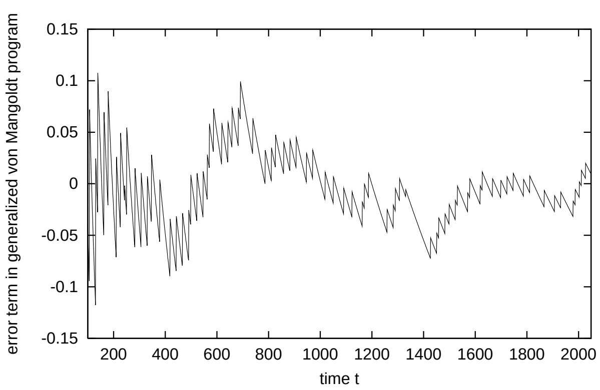

Figure 7: The error term $\varepsilon(t;q,p)$ in the summatory function $\psi(t;q,p)$; $p/q = 3/8$.

[1] In this section we use the notation p, q instead of p_i, q_i

References

[1] M. B. Weissman, Rev. Mod. Phys., **60**, 537 (1988).

[2] P. H. Handel, in *in Wiley Encyclopedia of Electrical and Electronic Engineering*, ed. John G. Webster (John Wiley and Sons, N. Y.), **14**, 428 (1999).

[3] S. Dos Santos and M. Planat, in *Fractals and Beyond, Complexity and Fractals in the Sciences*, ed. M.N. Novak (World Scientific, Singapore), 297 (1998).

[4] D. Wolf, *Noise in Physical Systems* (Springer, New York, 1978).

[5] J. A. Barnes, IEEE Trans. Instrum. Meas., IM-20, 105 (1971).

[6] M. Planat, *Fluctuation and Noise Letters*, **1**, R65 (2001)

[7] M. Planat, *Noise, Oscillators and Algebraic Randomness*, *Lecture Notes in Physics*, **550**, Springer, Berlin (2000).

[8] S. Dos Santos and M. Planat, IEEE Trans. on UFFC, **47**, 1147 (2000).

[9] C. Eckert and M. Planat, Phys. Rev E, **54**, 6093 (1996).

[10] M. Planat and C. Eckert, IEEE Trans. on UFFC, **47**, 1173 (2000).

[11] M. Wolf, Physica A, *241*, 493 (1997)

[12] R. Adler, Proc. IRE, **34**, 351, (1946), (reprinted in Proc. IEEE, **61**, 1380 (1973))

[13] This corrects formula (13) in [3] and formula (10) in [8] in agreement with our new experiments.

[14] P. Cvitanovic, in *From Number Theory to Physics*, ed. M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson (Springer Verlag, Berlin), 631 (1992)

[15] This should be corrected to account for the symmetry of the mixer. In the balanced phase bridge we used in the experiments, only harmonics with odd values of p_i and q_i contribute significantly.

[16] H.M. Edwards, *Riemann's Zeta Function.* Academic Press, N.Y. (1974).

[17] H. Davenport, *Multiplicative Number Theory.* Springer Verlag, N.Y. (1980)