0.1 AMBIGUITY FUNCTIONS[1]

0.1.1 The radar/sonar problem

Let us consider the typical radar/sonar problem in which the detection of a target (and the estimation of its relative range d and velocity v with respect to the emitter/receiver) is achieved from the analysis of the returning echo $r(t)$ associated to a given emitted waveform $x(t)$. Assuming a perfect reflection in the echo formation process and a constant radial velocity between the emitter/receiver and the target, $r(t)$ can be modelled as an attenuated replica of $x(t)$, up to a (range encoding) round trip delay, a (velocity encoding) modification due to the Doppler effect and some observation noise. On the basis of various criteria (maximum likelihood, Neyman-Pearson strategy, maximum contrast,...), it is known [14] that a basic ingredient for solving the detection problem is a measure of (linear) similarity, in the L^2-sense of a correlation, between the signal to detect and the actual echo ("matched filter" principle). Given the assumed model, it is therefore natural to compare the received echo with a battery of templates $(\mathbf{T}_{d',v'}x)(t)$, where $\mathbf{T}_{d',v'}$ stands for the range-velocity transformation attached to the candidate pair (d', v'), so that estimates of d and v can be inferred from:

$$(\hat{d}, \hat{v}) := \arg \max_{(d',v')} |\langle r, \mathbf{T}_{d',v'}x\rangle| . \tag{0.1.1}$$

As far as the deterministic part of the above inner product is concerned, the ideal situation would be to deliver zero values for all range-velocity pairs, except for $(d', v') \equiv (d, v)$. This, however, proves not to be achievable (as justified below), leading to a joint determination of range and velocity which is intrinsically ambiguous: this is the reason why a quantity of the type $\langle x, \mathbf{T}_{d,v}x\rangle$ is loosely referred to as an *ambiguity function* (AF).

0.1.2 Definitions of ambiguity functions

In order to be more specific in defining AF's, care has to be taken to physical considerations about the Doppler effect, which accounts for a time stretching of the returning echo.

Narrowband ambiguity functions. In the general case, the Doppler factor expresses as $\eta := (c + v)/(c - v)$, where c stands for the celerity of the propagating waves in the considered medium. In the radar case, the celerity of electromagnetic waves is $c = 3 \times 10^8$ m/s and, even if we assume a relative target velocity as large as $v = 3,600$ km/h, we end up with a Doppler factor such that $\eta - 1 = 6.66 \times 10^{-6} \ll 1$, thus justifying the approximation $\eta \approx 1 + 2v/c$. It follows that, if the emitted signal

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is of the form $x(t) := \tilde{x}(t)\,\exp\{i2\pi f_0 t\}$, where $\tilde{x}(t)$ is a complex envelope that is narrowband with respect to the carrier f_0, the deterministic part $r_d(t)$ of the returning echo $r(t)$ admits the approximation $r_d(t) \propto x(t-\tau)\,\exp\{i2\pi\nu t\}$, with τ a round trip delay such that $\tau := 2d/v$ and ν a Doppler *shift* such that $\nu := 2\pi f_0 v/c$. The corresponding inner product

$$\langle x, r_d\rangle \propto \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, e^{-i2\pi\nu t}\, dt \tag{0.1.2}$$

is therefore proportional to a quantity referred to as the *narrowband* AF of $x(t)$. Whereas this formulation is the one initially introduced by Woodward [16], it often proves useful to rather adopt the following *symmetrized* definition:

$$A_x(\nu,\tau) := \int_{-\infty}^{+\infty} x\left(t+\tau/2\right)\, x^*\left(t-\tau/2\right)\, e^{i2\pi\nu t}\, dt. \tag{0.1.3}$$

Wideband ambiguity functions. The above definitions (0.1.2)–(0.1.3) are based on approximations that may prove not to be relevant in contexts different from radar. This is especially the case in airborne sonar, where the celerity of acoustic waves is $c = 340$ m/s, thus leading to $\eta \approx 1.2$ for relative radial velocities $v \approx 100$ km/h. A similar situation can also be observed (although to a smaller extent) in underwater sonar, where the sound celerity in water $c = 1500$ m/s and typical relative velocities $v \approx 2.6$ m/s lead to $\eta \approx 1.034$. In such cases, the previous approximation of a Doppler shift is no longer valid for wideband signals, and the more general form

$$\tilde{A}_x(\eta,\tau) := \sqrt{\eta}\int_{-\infty}^{+\infty} x(t)\, x^*(\eta(t-\tau))\, dt \tag{0.1.4}$$

has to be preferred as a definition of a *wideband* AF [6].

Such a wideband definition naturally reduces to the narrowband one when the analyzed signal is narrowband.

Ambiguity functions and time-frequency distributions. AF's can be viewed as two-variable generalizations of correlation functions. In this respect, they are dual of energy distributions. In particular, it directly follows from the definition (0.1.3) that

$$\iint_{-\infty}^{+\infty} A_x(\nu,\tau)\, e^{-i2\pi(\nu t+\tau f)}\, d\nu\, d\tau = W_x(t,f), \tag{0.1.5}$$

where $W_x(t,f)$ is the Wigner-Ville distribution (WVD)[2]. More generally, the whole Cohen's class of quadratic time-frequency distributions can be obtained as the 2D Fourier transform of weighted (narrowband) AF's $g(\nu,\tau)\, A_x(\nu,\tau)$ (see, e.g., [4]).

[2] It is worth noting that the symmetrized AF (0.1.3) has in fact been pionneered by J. Ville [15] as a form of "time-frequency characteristic function".

Similarly, a properly symmetrized version of the wideband ambiguity function (0.1.4) can be shown [2] to be in Mellin-Fourier duality with a wideband time-frequency distribution, referred to as *Altes' Q-distribution* $Q_x(t,f)$. More precisely, if we let

$$\hat{A}_x(\eta,\tau) := \int_{-\infty}^{+\infty} x(\eta^{-1/2}(t+\tau/2))\, x^*(\eta^{+1/2}(t-\tau/2))\, dt = \tilde{A}_x(\eta,\eta^{-1/2}\tau), \quad (0.1.6)$$

we have [2]

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \hat{A}_x(\eta,\tau)\, e^{-i2\pi f\tau}\, \eta^{i2\pi t-1}\, d\eta\, d\tau = Q_x(t,f), \quad (0.1.7)$$

with the warping equivalence $Q_x(t,f) \equiv W_{\check{x}}(t,\log f)$, if $\check{X}(f) := X(e^f)$.

Another interesting connection can be pointed out between AF's and linear time-frequency (or time-scale) representations. In fact, the right-hand side of (0.1.2) can be viewed as the *short-time Fourier transform* of $x(t)$, with window $h(t) := x(t)$; in the same respect, (0.1.4) is nothing but the *wavelet transform* of $x(t)$, with wavelet $\psi(t) := x(t)$ (and scale $a := 1/\eta$). In both cases, the AF is exactly identical to the *reproducing kernel* of the corresponding linear transform [4].

0.1.3 Properties of narrowband ambiguity functions

Invariances and covariances. Whereas members of Cohen's class are *covariant* with respect to time and frequency shifts, the squared modulus of the AF (a quantity referred to as the *ambiguity surface* (AS)) is *invariant* to such transformations (i.e., $|A_y(\nu,\tau)|^2 = |A_x(\nu,\tau)|^2$ for any shifted version $y(t) := x(t-\theta)\exp\{i2\pi\xi t\}$ of a given signal $x(t)$). In a similar way, the AF inherits—by Fourier duality—from a number of properties satisfied by the WVD, such as covariance with respect to dilations, rotations or chirp modulations [4] [8].

Cross-sections. As it has been mentioned, the narrowband AF can be seen as a *correlation function* with respect to time and frequency shifts. As such, it is hermitian symmetric: $A_x(-\nu,-\tau) = A_x^*(\nu,\tau)$, and it satisfies the inequality

$$|A_x(\nu,\tau)| \le |A_x(0,0)| = \|x\|_2^2. \quad (0.1.8)$$

Although this interpretation cannot be pushed too far (in particular, the AF is not a non-negative definite quantity, since its 2D Fourier transform—namely, the WVD—can attain negative values), cross-sections of the AF are meaningful 1D correlation functions, since we have:

$$A_x(0,\tau) = \int_{-\infty}^{+\infty} x\,(t+\tau/2)\; x^*\,(t-\tau/2)\; dt \quad (0.1.9)$$

and

$$A_x(\nu,0) = \int_{-\infty}^{+\infty} X\,(f+\nu/2)\; X^*\,(f-\nu/2)\; df. \quad (0.1.10)$$

This idea of a time-frequency correlation function (which is illustrated in Figure 1) is instrumental in the design of reduced interference distributions within Cohen's class [5].

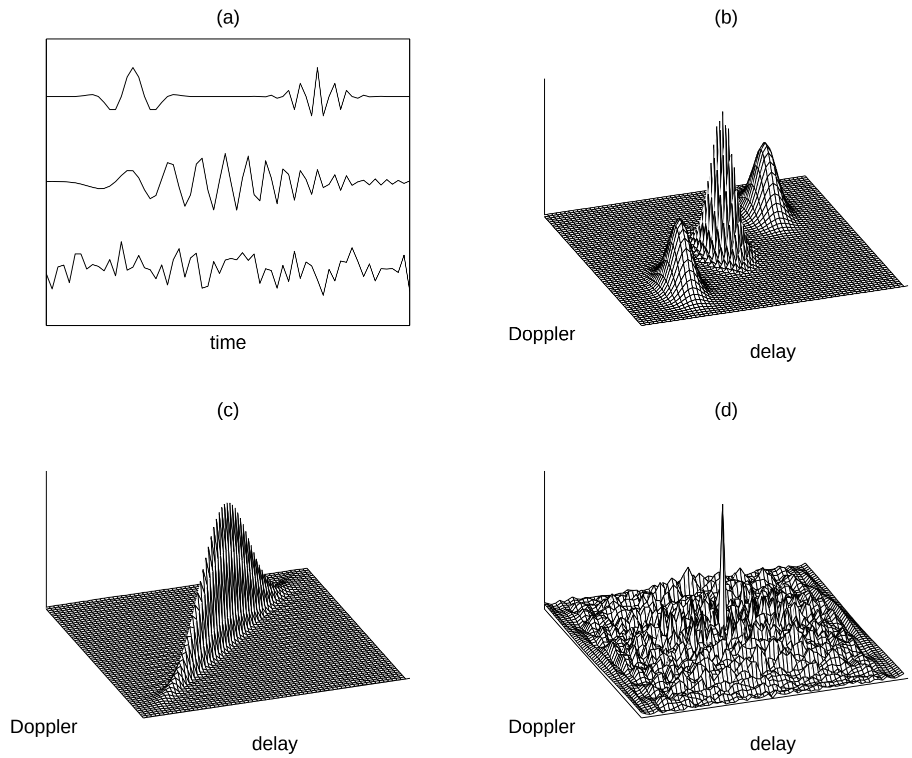

Figure 1: **Ambiguity functions as time-frequency correlation functions.** Subplots (b) to (d) display the ambiguity surfaces attached respectively to the three signals plotted in (a), namely (from top to bottom): two Gabor logons, a linear chirp and a sample of white Gaussian noise. All three surfaces share the common property of attaining their maximum value at the origin of the plane, with values off the origin that reveal correlations in the signal structure, with respect to both time and frequency shifts (referred to as delay and Doppler).

Volume invariance and self-transformation. Using Parseval's relation and Moyal's formula [4], we readily get that, for any two signals x and y,

$$\iint_{-\infty}^{+\infty} A_x(\nu,\tau)\, A_y^*(\nu,\tau)\, d\nu\, d\tau = \left| \int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt \right|^2 . \tag{0.1.11}$$

Setting $y \equiv x$ in this equation, it follows that

$$\iint_{-\infty}^{+\infty} |A_x(\nu,\tau)|^2 \, d\nu \, d\tau = \|x\|_2^4, \qquad (0.1.12)$$

i.e., that the AS has an *invariant volume* that is only fixed by the signal's energy. More remarkably, (0.1.12) is just a special case of Siebert's *self-transformation* property [11]:

$$\iint_{-\infty}^{+\infty} |A_x(\nu,\tau)|^2 \, e^{i2\pi(\nu t+\tau f)} \, d\nu \, d\tau = |A_x(f,t)|^2, \qquad (0.1.13)$$

from which it can be inferred that an AF is a highly structured function.

Uncertainty principles. If we combine the "correlation" inequality (0.1.8) and the "volume invariance" property (0.1.12), it is clear that an AS cannot be perfectly concentrated at the origin of the plane. This limitation, that is sometimes referred to as the *radar uncertainty relation* [13], admits a more precise L^p-norm formulation ($p > 2$) as follows [7]:

$$I_x(p) := \iint_{-\infty}^{+\infty} |A_x(\nu,\tau)|^p \, d\nu \, d\tau \leq \frac{2}{p} \, \|x\|_2^{2p}, \qquad (0.1.14)$$

with equality if and only if x is a linear chirp with a Gaussian envelope. A similar result holds on the basis of an *entropic* measure of sharpness, leading to the inequality

$$S_x := -\iint_{-\infty}^{+\infty} |A_x(\nu,\tau)|^2 \, \log |A_x(\nu,\tau)|^2 \, d\nu \, d\tau \geq 1 \qquad (0.1.15)$$

for unit energy signals.

The common interpretation of those inequalities is that an AS cannot be zero everywhere except at the origin of the plane. In the case where all of the AS is supposed to be concentrated around the origin, it has necessarily to extend over a domain (whose area is non-zero) which defines the joint accuracy of any delay-Doppler measurement [10] [14]. However, AS's which are more sharply peaked at the origin can be found, provided that non-zero values are accepted somewhere off the origin in the plane: for unit energy signals, AS's with null values except in $(0,0)$ can be obtained on convex domains whose *clear area* cannot however be greater than 4 [9]. An example is given in Figure 2.

Delay/Doppler estimation. The best achievable performance in joint estimation of delay and Doppler is bounded. The actual Cramér-Rao bounds on variances and covariances can be derived from the Fisher information matrix of the problem, whose terms can themselves be expressed as partial derivatives of the AS, in the case of additive white Gaussian noise [14]. Since the AS is basically the maximum likelihood estimator for delay and Doppler, and since this estimator can be shown to

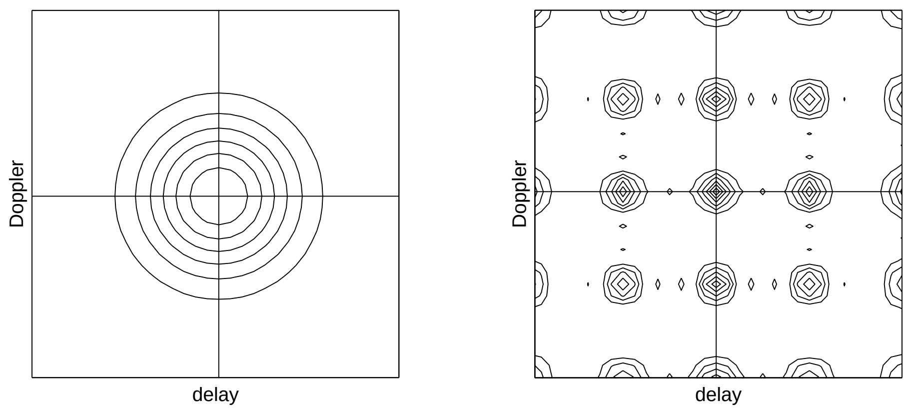

Figure 2: **Sharpening the central peak of an ambiguity surface.** Whereas a single Gaussian pulse has an ambiguity surface whose central peak cannot have an effective area $\mathcal{A}$ smaller than a limit fixed by the "radar uncertainty principle" (left), a signal defined by the superposition of a number of replicae of such a pulse, periodically shifted in time and frequency, may guarantee a sharper central peak (right). This has however to be paid at the price of auxiliary peaks off the origin, with a "clear area" of the order of $\mathcal{A}$.

be asymptotically efficient, it thus follows that the AS geometry is a direct indicator of the expected accuracy in the estimation. Roughly speaking, variances in delay and Doppler estimation are given by the effective widths of the central peak of the AS.

Signal design. In active problems, in which the emitted signal can be freely chosen (up to a certain extent), an important issue is to design waveforms with a prescribed AF (or AS), so that some desired performance can be guaranteed. From a purely theoretical point of view, a signal is entirely determined (up to a pure phase term) by its AF, since we can invert the definition (0.1.3) according to:

$$x(t) = \frac{1}{x^*(0)} \int_{-\infty}^{+\infty} A_x(\nu, \tau)\, e^{-i\pi\nu\tau}\, d\nu. \tag{0.1.16}$$

Unfortunately, as it has been said before, an AF is a highly structured function and an arbitrary 2D function has in general no reason to be admissible, i.e., to be the actual AF of some signal. Different approaches have been proposed to overcome this limitation. One can first think of looking for the signal $\hat{x}(t)$ whose AF approaches at best a given time-frequency function $F(\nu, \tau)$, according, e.g., to a L^2-distance [12]:

$$\hat{x}(t) = \arg\min_x \iint_{-\infty}^{+\infty} |A_x(\nu, \tau) - F(\nu, \tau)|^2\, d\nu\, d\tau. \tag{0.1.17}$$

One can also rely on the physical interpretation of the AF as a time-frequency correlation function and promote waveforms with adapted time-frequency characteristics. In this respect, a large bandwidth (resp., a long duration) is required for an accurate estimation of delay (resp. Doppler). The simultaneous consideration of these two design principles advocates the use of *chirp* signals with a large bandwidth-duration product [10].

0.1.4 Remarks on wideband ambiguity functions

In many respects, properties of the wideband AF can be seen as natural generalizations of the narrowband case (although some properties, such as volume invariance, may no longer be satisfied), reducing to them in the narrowband limit.

In parallel with what has been previously mentioned in the narrowband case, the best achievable performance in the joint estimation of delay and Doppler can be expressed, in the wideband case, in terms of geometrical properties of the wideband AS [3].

A companion problem is that of *Doppler tolerance*, which consists in obtaining an unbiased estimate of delay in the presence of any unknown Doppler [1]. Doppler acting as a stretching on the emitted signal, the condition for no bias can be translated into the fact that the effective time-frequency structure of the emitted waveform is invariant under stretching. It turns out that the hyperbola is the only curve of the plane which is invariant under dilation/compression transformations: in terms of chirps, assumed to be conveniently described on the plane by a time-frequency skeleton, this justifies [4] the use of *logarithmic* phases, i.e., of hyperbolic chirps resembling those commonly observed in natural sonar systems (bats) [1].

Finally, it must be pointed out that computing an AF proves more involved in the wideband case than in the narrowband case. Efficient solutions, based on the Mellin transform, have been proposed in [3].

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