Time-frequency and chirps

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ABSTRACT

Chirps (i.e., transient AM-FM waveforms) are ubiquitous in nature and man-made systems, and they may serve as a paradigm for many nonstationary deterministic signals. The time-frequency plane is a natural representation space for chirps, and we will here review a number of questions related to chirps, as addressed from a time-frequency perspective. Global and local approaches will be described for matching and/or adapting representations to chirps. As a corollary, joint time-frequency descriptions of chirps will be shown to allow for effective definitions of "instantaneous frequencies" via localized trajectories on the plane. A number of applications will be mentioned, ranging from bioacoustics to turbulence and gravitational waves.

Keywords: time-frequency, chirps, localization, instantaneous frequency.

1. INTRODUCTION

Fourier analysis treats time and frequency in an exclusive fashion: time *or* frequency. Considering one of these two variables as being possibly dependent on the other (e.g., frequency as a function of time) is however a point of view which is clearly supported by intuition and by the everyday experience of music, whistles...in brief, of frequency modulations. Simple (transient) signals whose description is of this type are loosely referred to as "chirps", in reference to Webster's definition[66]:

DEFINITION 1. *Chirp, n. A short, sharp note, as of a bird or insect. "The chirp of flitting bird," – Bryant.*

Whereas intuition pleads in favor of heuristic descriptions of chirps as "gliding tones", their mathematical representation deserves some specific treatment, able to wedding time and frequency. The purpose of this paper is to discuss a number of issues related to chirps, as seen from a time-frequency perspective.

We will first list, in Section 2.1, many different instances where chirps can be naturally observed, and we will discuss various mathematical ways of modelling chirps that may support physical intuition. Signal decompositions based on chirp(let)s will then be briefly mentioned in Section 3, mostly as a motivation for the core of the paper, which is to discuss the rationale behind time-frequency representations matched to specific chirps. Basics of (energy) time-frequency distributions will be recalled in Section 4. Thinking of the time-frequency plane as a mathematical musical score, chirps are expected to localize on trajectories interpreted as pitch histories: this issue will be explored in Section 5, and it will be argued that effective (time-frequency) definitions of "instantaneous frequencies" can be obtained from distributions with suitable localization properties.

2. CHIRPS

2.1. Chirps everywhere

Chirps are ubiquitous in nature and man-made systems. Let us first enumerate a few examples where the notion of "chirp" naturally emerges.

Audio signals — Chirps are naturally encountered in many audio signals, ranging from bird songs and music ("glissando") to animal vocalization (frogs, whales) and speech. The so-called "sinusoidal models"[44] are a typical attempt to representing audio signals as a superposition of chirp-like components.

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Radar and sonar systems — Chirp signals are also commonly observed in natural sonar systems. Most species of bats make use of an ultrasound system based on chirps whose parameters can be shown to directly control echolocation performance.[47] Such a situation closely resembles that of man-made radar and sonar systems, where chirps are of common use too.[54]

Wave physics — Low-frequency chirps (such as, e.g., PC1 oscillations[39]) can be observed in the ionosphere as "whistling atmospherics".[62] Many time-varying oscillatory systems give birth to chirp-like behaviors: a beautiful (and well-documented) instance of a chirp is provided by the gravitational waves expected to be radiated by massive astrophysical objects such as coalescing binaries.[17,57] Another example is provided by breaking waves on a seashore, that have a wavelength modulated by the underwater profile of the ground, hence giving rise to 2D chirps. From a different perspective, non-harmonic waves propagating in a dispersive medium are naturally chirped by a warping mechanism.[58]

Mechanics and vibrations — A paradigm for a chirp is in fact the note played by a diapason (or a chord, or a pipe) with a time-varying length. Apart from music, such a phenomenon can be observed, e.g., in vibration signals recorded on car engines, due to the time-varying volume of the gas ignition chamber.[14]

Spirals in turbulence — One of the many pictures of turbulence is that of a collection of spiraling coherent structures (vortices)[46]: when advected by a mean flow and measured at a given point in space, spatio-temporal sections of such objects are "seen" as chirps.

Biology and medicine — Other forms of coherent structurations of waves as chirps arise in biomedical signals, e.g., in EEG (epileptic seizure[11]) or uterine EMG (pregnancy contractions[24]).

Critical phenomena — In a number of critical phenomena,[60] it has been evidenced that universal singular behaviors (typically, power-law divergences) are decorated by a chirp component related to accelerating oscillations (e.g., accumulation of precursors in the case of earthquakes, speculative bubbles in the case of financial crashes,[38] ...).

Special functions — Finally, chirps have also been shown to exist in purely mathematical objects such as Weierstrass[7,59] or Riemann[36] functions, not to mention[27] the chirp structure of (compactly supported and minimum phase) Daubechies wavelets[22] of large order.

2.2. Chirp models

Radio and AM-FM — In each of the above examples, the considered signals admit a decomposition in two (time-varying) terms of amplitude and phase (typically, something of the form $x(t) = a(t) \cos \varphi(t)$), so that the "chirping" nature of the observation stems from nonlinearities in the phase. Such a decomposition may appear as natural, but it is not necessarily so, depending on whether we adopt a point of view of analysis or of synthesis. In the synthesis case, we can say that the two signals $a(t)$ and $\varphi(t)$ pre-exist prior their combination as $x(t)$: the simplest example is given by radio broadcasting systems (AM or FM) in which a message is modulating a waveform, allowing for its recovering from the mixture via a matched demodulation. In passive (or "blind") situations where a modulated signal $x(t)$ is observed with no side information about its production, the situation is quite different since demodulation would amount two identify two unknowns (the amplitude $a(t)$ and the phase $\varphi(t)$) on the basis of one equation only (the observation $x(t)$).[53] In the monochromatic case $x(t) = a \cos 2\pi f_0 t$, no ambiguity exists in the decomposition and in the physical interpretation of f_0 as frequency. Formalizing the idea of an "instantaneous frequency" amounts therefore to generalizing the concept of a monochromatic wave by allowing the amplitude a and the frequency f_0 to become time-dependent, so that $x(t)$ takes on the desired form $a(t) \cos \varphi(t)$, with $a(t) > 0$ and $\varphi(t)$ nonlinear in t.

Analytic signals — The non-unicity of such a representation[41,53] has received different solutions and is at the heart of the time-frequency problem. Following Gabor[32] and Ville,[65] it is generally accepted to get rid of arbitrariness by considering a real-valued signal $x(t) \in \mathbb{R}$ as the real part of the complex-valued signal $z_x(t) := x(t) + i(\mathbf{H}x)(t)$, where $\mathbf{H}$ stands for the Hilbert transform. The rationale for introducing such an "analytic signal" $z_x(t)$ is that, when applied to a monochromatic wave, it simply reduces to a complex exponential, thus representing a "stationary" signal by a rotating vector whose modulus is constant and whose rotation is uniform. In the general case of an arbitrary signal, this leads naturally to define an instantaneous amplitude and an instantaneous frequency according to $a_x(t) := |z_x(t)|$ and $f_x(t) := (1/2\pi)\,(d/dt)\arg z_x(t)$, respectively. It is worth noting that, whereas the concept of an instantaneous frequency could seem to be attached to that of locality, its definition in Gabor-Ville's sense is highly non-local. This point of view (which could be referred to as "think local, act global") is due to the infinite support and the slow decay of the Hilbert filter, whose impulse response reads $h(t) = \text{p.v.}\{1/\pi t\}$.

Other ways of defining an instantaneous frequency can be imagined,[64] but none proved to be significantly better. Moreover, whatever the chosen definition, the principle of using a one-dimensional curve of the time-frequency plane has a natural limitation as soon as the analyzed signal is multicomponent, i.e., such that different frequency contributions are allowed to be simultaneously present. In such a case, the representation can at best give some "average" description*, and in no way the multivalued functions which would be necessary for a physically meaningful interpretation. Overcoming this limitation motivates the introduction of truely mixed (i.e., two-dimensional) representations, that will be discussed further in Section 4.

Modelling chirps — Keeping in mind the above remarks, we will adopt for the modelling of chirps the following, yet loose, definition (which, a priori, does not assume analyticity):

DEFINITION 2. *Chirps are signals of the form*

$$x(t) = a(t)\,\exp\{i\varphi(t)\}, \tag{1}$$

where $a(t)$ is some positive, low-pass and smooth amplitude function whose evolution is slow as compared to the oscillations of the phase $\varphi(t)$.

"Slow evolution" conditions on $a(t)$ and $\varphi(t)$ are usually[25,40,63] based on the two quantities $\epsilon_1(t) := \dot{a}(t)/a(t)\dot{\varphi}(t)$ and $\epsilon_2(t) := \ddot{\varphi}(t)/\dot{\varphi}^2(t)$, and read

$$\sup_t |\epsilon_1(t)| \ll 1\,; \;\; \sup_t |\epsilon_2(t)| \ll 1. \tag{2}$$

The first condition guarantees that, over a (local) pseudo-period $T(t) = 2\pi/|\dot{\varphi}(t)|$, the amplitude $a(t)$ experiences almost no relative change, whereas the second condition imposes that $T(t)$ itself is slowly-varying, thus giving sense to the notion of a pseudo-period.

Stationary phase approximations of chirp spectra — Although the definition of a chirp is usually given in the time domain (as in (1)), some applications may call for a companion description in the frequency domain.[17,25,56] In this respect, it is customary[63] to make use of a stationary phase approximation, assuming more or less explicitly that the conditions given in (2) support the effectiveness of the approach.

The argument of the stationary phase principle can be phrased as follows. Let I be an integral of the form

$$I = \int_\Omega b(t)\, e^{i\psi(t)}\, dt, \tag{3}$$

where both $b(t) > 0$ and $\psi(t)$ are C^1, whereas $\text{supp}\{\psi(t)\}$ is restricted to some interval $\Omega \subset \mathbb{R}$ over which $b(t)$ is integrable. If $b(t)$ is slowly-varying as compared to the oscillations controlled by $\psi(t)$, positive and negative values of the integrand tend to cancel each other, with the consequence that the main contribution to I only comes from the vicinity of those points where the derivative of the phase is zero. Assuming that $\psi(t)$ has one and only one non-degenerate stationary point t_s (i.e., that we have $\dot{\psi}(t_s) = 0$ and $\ddot{\psi}(t_s) \neq 0$), we can make the change of variables $u^2 := 2[\psi(t) - \psi(t_s)]/\ddot{\psi}(t_s)$, so as to rewrite (3) in the canonical form

$$I = e^{i\psi(t_s)} \int_{\Omega'} g(u)\, e^{i\beta u^2}\, du, \tag{4}$$

with $g(u) := b(t(u))(du/dt)^{-1}$ and $\beta := \ddot{\psi}(t_s)/2$. Using a Taylor expansion for the exponential in the right-hand side of (4), we are thus led[34] to decomposing (3) as $I =: I_a + R$, with

$$I_a = \sqrt{\frac{2\pi}{|\ddot{\psi}(t_s)|}}\, b(t_s)\, e^{i\psi(t_s)}\, e^{i(\text{sgn}\,\ddot{\psi}(t_s))\pi/4} \tag{5}$$

*It is worth noting that this idea of "average" frequency cannot be followed up stricto sensu: for instance, the instantaneous frequency of a signal composed of two tones f_1 and f_2 generally contains contributions outside the interval $[f_1, f_2]$, whereas the signal itself is strictly limited to this frequency band.[21,41]

the stationary phase approximation of I.

The quality of this approximation depends on the magnitude of the remainder R. Extending an approach developed in[34] allows for bounding explicitly the relative error $Q = |R/I_a|$ as

$$Q \leq Q_m = \frac{5/4}{|\beta|\, g(t_s)} \sup_{u\in\Omega'} (|\ddot{g}(u)|), \tag{6}$$

and the stationary phase approximation is therefore valid if $Q_m \ll 1$. Given the model (3), an explicit evaluation of this quantity leads to a fairly complicated function which is explicitly given in[15,17] and which depends non-linearly on $b(t)$, $\psi(t)$ and some of their derivatives up to third order.

This (sufficient) criterion can be readily applied to the problem of evaluating the spectrum of a chirp (1) with a monotonic instantaneous frequency by setting $b(t) := a(t)$ and $\psi(t) := \varphi(t) - 2\pi f t$ (with the stationary point t_s thus defined by $\dot{\varphi}(t_s) = 2\pi f$). What turns out is that the corresponding error is not only controlled by the terms $\epsilon_1(t)$ and $\epsilon_2(t)$ of eq.(2), but also by additional terms depending on more complicated combinations of $a(t)$, $\varphi(t)$ and some of their higher-order derivatives.[16] Provided that conditions are satisfied so as to validate (5), a by-product of the stationary phase approximation is that the group delay (defined as $t_x(f) := -(\partial\Psi/\partial f)(f)/2\pi$, with $\Psi(f)$ the phase function of the spectrum) coincides with the reciprocal function of the instantaneous frequency of the signal,[28] i.e., that $f_x(t_x(f)) \equiv f$.

Going back to the general model (3), it is worth investigating the case where there is no stationary point. In such a situation where $\dot{\psi}(t) \neq 0$ for all t's, (3) can be rewritten as

$$I = \int_\Omega \frac{b(t)}{i\dot{\psi}(t)}\, i\dot{\psi}(t)\, e^{i\psi(t)}\, dt,$$

and an integration by parts leads to $I/\|b\|_1 \leq \|\dot{b}(t)/b(t)\dot{\psi}(t)\|_\infty + \|\ddot{\psi}(t)/\dot{\psi}^2(t)\|_\infty$ if we further assume that $b(t) \in L^1(\Omega)$ and $b(\partial\Omega) = 0$. As compared to the situation where the oscillations of the phase would be infinitely slowed down, this means that the magnitude of (3) is in this case bounded from above by a quantity whose decay to zero is controlled by chirp-like conditions. Moreover, in the case where I corresponds to the Fourier transform of the chirp (1) (i.e. when $b(t) = a(t)$ and $\psi(t) = \varphi(t) - 2\pi f t$) and if we furthermore assume that $\dot{\varphi}(t) > 0$ for any $t \in \Omega$, we can conclude that the frequency domain for which no stationary point exists is the half-line of negative frequencies. Since we have in this case $\ddot{\psi}(t) = \ddot{\varphi}(t)$ and $\dot{\psi}(t) \geq \dot{\varphi}(t)$ when $f < 0$, we are ensured that $\|\dot{b}(t)/b(t)\dot{\psi}(t)\|_\infty \leq \|\dot{a}(t)/a(t)\dot{\varphi}(t)\|_\infty$ and $\|\ddot{\psi}(t)/\dot{\psi}^2(t)\|_\infty \leq \|\ddot{\varphi}(t)/\dot{\varphi}^2(t)\|_\infty$. It appears therefore that the heuristic conditions (2) are sufficient for guaranteeing the quasi-analyticity of the exponential model (1)—in the sense that spectral contributions at negative frequencies are almost zero—, with the consequence that the quantity $\dot{\varphi}(t)/2\pi$ can be effectively interpreted as the instantaneous frequency of the chirp.

Linear chirps — The simplest, and most commonly used, example of a chirp is the "linear" chirp, defined by:

DEFINITION 3. *A linear chirp is a chirp (1), in which* $a(t) \propto \exp\{-\pi\gamma t^2\}$ *and* $\varphi(t) = 2\pi(\alpha t^2/2 + \beta t)$, *with* α *and* $\beta \in \mathbb{R}$ *and* $\gamma \geq 0$.

Strictly speaking, such a linear chirp can never be analytic, and it is improperly that the quantity $\dot{\varphi}(t)/2\pi = \alpha t + \beta$ is often referred to as its "instantaneous frequency". Defined this way, linear chirps constitute however an interesting class of signals, since quasi-analyticity can obtained under the narrowband condition $\sqrt{\gamma + \alpha^2/\gamma} \ll \beta$. Moreover, in the purely FM case $\gamma = 0$, the exact spectrum actually coincides with its stationary phase approximation whereas, in the general case, the quality of the approximation is controlled by the time-bandwidth product α/γ.

Power-law chirps, hyperbolic chirps and oscillating singularities — Another particularly important class of chirps is that of "power-law" (and "hyperbolic") chirps, defined by[16,25]:

DEFINITION 4. *A power-law chirp is a chirp (1), in which* $a(t) \propto |t|^{-\alpha}$ *and* $\varphi(t) = 2\pi d|t|^\beta$, *with* $\alpha, d \in \mathbb{R}$ *and* $\beta \neq 0$.

DEFINITION 5. *A hyperbolic chirp is the natural extension of a power-law chirp when* $\beta \to 0$, *characterized by a logarithmic phase of the form* $\varphi(t) = 2\pi d \log|t|$, *with* $d \in \mathbb{R}$.

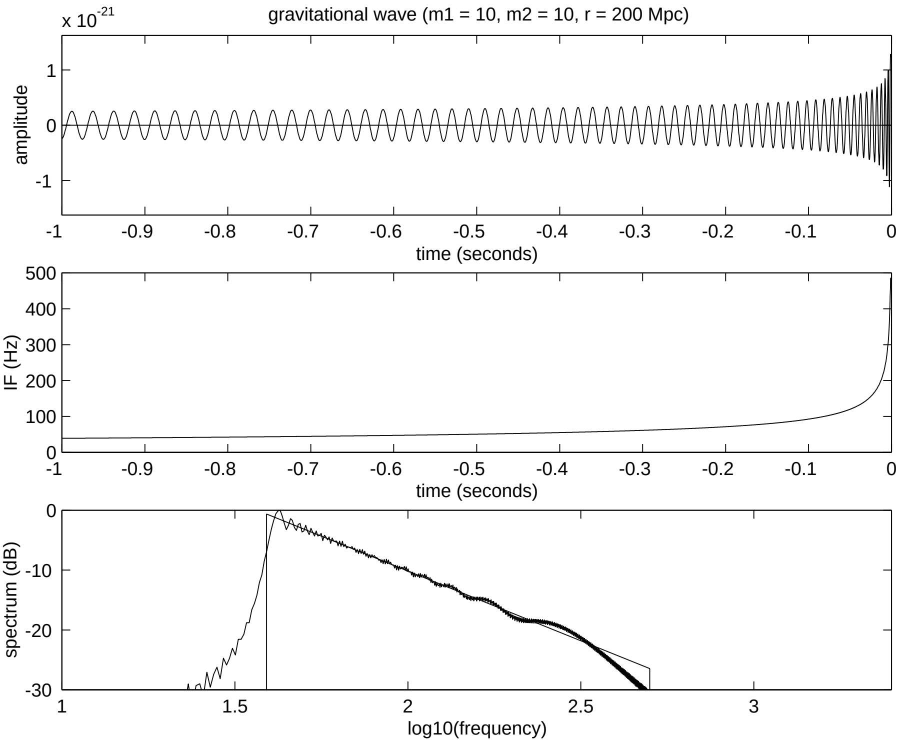

Figure 1. *Gravitational waves expected to be radiated by coalescing binaries are modelled (for $t < 0$) by power-law chirps with $\alpha = 1/4$ and $\beta = 5/8$. An example of such a waveform is given in the top row, in the case of two objects of identical masses $m1 = m2 = 10$ (in solar masses units), during the final stage preceding the coalescence time $t = 0$. The corresponding instantaneous frequency is given in the middle row, whereas the bottom row superimposes the actual spectrum and its stationary phase approximation, supposed to be linear in the chosen log-log plot.*

It follows from those definitions that qualitatively different types of waveforms can be obtained, depending on the values of the parameters α and β. First, considering $a(t)$ as the amplitude of the chirp, we can observe that $a(0) = 0$ (resp. $+\infty$) if $\alpha < 0$ (resp. > 0). Second, identifying $\dot{\varphi}(t)/2\pi = d\beta|t|^{\beta-1}$ with the "instantaneous frequency" of the chirp leads to a power-law divergence in 0 for all β's such that $\beta < 1$. This will however correspond to an "indefinitely oscillating" signal in 0 only if we have the stronger condition $\beta \leq 0$.[45] In fact, within the range $0 < \beta < 1$, the phase does present a well-defined value in $t = 0$, namely $\varphi(0) = 0$, thus connecting the singular behaviour of its derivative with a non-oscillating singularity of the waveform in 0.

As far as the spectrum of power-law chirps is concerned, it can be shown[16] that both stationary phase criteria, derived from either the heuristic conditions (2) or the refined analysis sketched in Section 2, share the same frequency dependence $\epsilon = C\,(\beta d/f^\beta)^{1/(\beta-1)}$, with the only difference that the pre-factor C reads $C = (1/2\pi)\max(|\alpha|, |\beta - 1|)$ in the first case[25] and $C = (5/48\pi)|12\alpha^2 - 12\alpha + 12\alpha\beta + 2\beta^2 - 5\beta + 2|/|\beta - 1|$ in the second one.[16] Depending on which of these quantities is greater, we can therefore evidence, for any given d, pairs (α, β) such that the stationary phase approximation still remains valid whereas the heuristic conditions (2) are violated or, on the contrary, such that the approximation breaks down whereas the same conditions are satisfied.[16]

Power-law chirps have been introduced either as suitable models for gravitational waves (in the case of coalescing binaries, a Newtonian approximation leads to $\alpha = 1/4$ and $\beta = 5/8$, see Figure 1),[25,17] or as powerful refinements to isolated singularities for which a Hölder-type characterization is not sufficient.[1]

3. CHIRPS AS SIGNAL BUILDING BLOCKS

The usual Fourier transform (FT) can formally be written as $(\mathcal{F}x)(f) := \langle x, e_f \rangle$, with $e_f(t) := \exp\{i2\pi ft\}$, so that the overall signal can be recovered as

$$x(t) = \int_{-\infty}^{+\infty} \langle x, e_f \rangle \, e_f(t) \, df,$$

i.e., as a (suitably weighted) superposition of pure tones. When switching from pure tones to chirps, the "stationary" structure attached to the linear phase $2\pi ft$ is replaced by a time-varying one which connects time and frequency by means of a one-dimensional curve, namely $\dot{\varphi}(t)$: in some sense, the frequency structure of a chirp can be viewed as that of a warped monochromatic wave. This naturally suggests the use of chirp-based substitutes to the ordinary Fourier analysis, that may explicitly take into account a possible time evolution of spectral properties.

3.1. Modified Fourier transforms

One way of modifying the monochromatic waves of Fourier analysis is to only allow for a variation of frequency as a function of time, while not introducing any idea of localization in time.

Fractional Fourier transform — A first instance of such a modified FT is given by the "fractional Fourier transform" (FrFT) of angle $\phi \in (-\pi/2, +\pi/2]$, defined as[51] $(\mathcal{F}_\phi x)(\xi) := \langle x, y_\phi \rangle$, with

$$y_\phi(t) := \sqrt{1 - i\cot\phi}\, \exp\{-i\pi(\xi^2 \cot\phi - 2\xi t \csc\phi + t^2 \cot\phi)\}. \tag{7}$$

As functions of t, the elementary waveforms $y_\phi(t)$ onto which $x(t)$ is projected in order to compute its FrFT happen to be linear FM signals whose "instantaneous frequencies" read $f_{y_\phi}(t) = \xi \csc\phi - t\cot\phi$. In the specific case where $\phi = \pi/2$, one can check that $f_{y_{\pi/2}}(t) = \xi$ and that $(\mathcal{F}_{\pi/2}x)(\xi) = (\mathcal{F}x)(\xi)$, thus recovering the ordinary FT with ξ interpreted as the usual frequency variable. In all other cases, the FrFT offers a convenient framework for analyzing, decomposing (or modifying) signals in terms of linear FM contributions which can be thought of as monochromatic waves whose instantaneous frequency law, which was initially constant (and, hence, "horizontal" in the time-frequency plane), has been chirped by a rotation.

Mellin transform — Another modified FT is the "Mellin transform" (MT). Restricting, for a sake of simplicity, to causal signals, the MT can be defined as[9]

$$(\mathcal{M}x)(s) := \int_0^{+\infty} x(t)\, t^{-i2\pi s - \alpha}\, dt,$$

where α is some free parameter. Defining $\tilde{x}(t) := e^{(1-\alpha)t}\, x(e^t)$, it is easy to check that $(\mathcal{M}x)(s) = (\mathcal{F}\tilde{x})(s)$, i.e., that a MT is nothing but the FT of an exponentially warped signal. From another perspective, computing a MT amounts to project a signal onto a family of elementary signals of the form $t^{-\alpha}\exp\{i2\pi s \log t\}, t > 0$. Such signals can be seen as (causal) hyperbolic chirps, in the sense of Def. 5. Given that the "instantaneous frequency" law of these chirps is $f_x(t) = s/t$, the Mellin parameter s can therefore be interpreted as a hyperbolic chirp rate.

3.2. Chirplet decompositions

A different way of modifying the monochromatic waves of Fourier analysis is to introduce the idea of some form of localization in time for the waves onto which the analyzed signal is projected.

Gabor, wavelet and warped bases — This point of view leads traditionnally to ordinary Gabor or wavelet decompositions/bases,[13,22,42] but it can also be suitably modified so as to accommodate for elementary waveforms tiling the plane in some non-rectangular way. Elements of such warped bases[4] are in fact nothing but chirps (or even "chirplets", in the sense that are indeed elementary signal building blocks), that can be tailored to specific time-varying structures. The efficiency of such warped bases is essentially of a computational nature, since they allow for compact representations whose coefficients can be obtained via fast algorithms. Their drawback lies however in their poor analysis capabilities, since they are highly dependent on the discrete nature of the tiling of the plane.

Chirplets — In order to overcome the above limitation, chirplets should better be parameterized in some (almost) continuous way. Defining a chirplet requires however (at least) four parameters. For instance, in the simplest case of a linear chirplet, one can modify Definition 3 so as to have

$$x_{t_0,f_0,\alpha,\gamma}(t) \propto \exp\{-\pi(\gamma+i\alpha)(t-t_0)^2 + i2\pi f_0(t-t_0)\}, \tag{8}$$

where t_0 and f_0 stand for the central locations of the chirplet in time and frequency, respectively, whereas α is its chirp rate and $\gamma > 0$ its (inverse squared) duration.

Chirplet decompositions — Since a direct evaluation of all the inner products $\langle x, x_{t_0,f_0,\alpha,\gamma}\rangle$ would be computationally much too expensive,[12] more efficient strategies have been developed. "Matching"[43,33] (or "basis"[19]) "pursuit" is one such strategy in which, at each step of the algorithm, the largest inner product is identified and the corresponding chirplet contribution removed, so that the process can be iterated on the residual. Another point of view consists in estimating chirplet parameters in the maximum likelihood sense.[48,50] In this case, the advantage can be shown to be in terms of statistical efficiency in the one chirplet case, while approximate (and computationally efficient) solutions can be obtained when dealing with multiple chirplets.[50]

4. TIME-FREQUENCY

4.1. Time-frequency as a paradigm

Beyond the specific technicalities of the aforementioned modifications to Fourier-type signal decompositions, the common denominator of all approaches is that the time-frequency plane appears as a natural representation space for chirps (and especially multiple chirp signals), with expected energy localizations along curves of the plane that can be interpreted as the "instantaneous frequencies" of the different components.

Rather than focusing a priori on (segments of) pre-determined curves of the plane—a point of view which amounts to addressing the signal description problem in an essentially 1D way—, we will hereafter reconsider it from a truely 2D perspective, offering signals various ways of structuring their complexity in the time-frequency plane. The key issue is therefore shifted to how properly choosing a time-frequency representation with potential localization properties for given chirps.

4.2. From Fourier to Wigner-Ville, via short-time analyses

In order to concile both time and frequency aspects, the easiest (and oldest) way of introducing a time-dependence in a spectral representation is to make it local by substituting to the ordinary FT the quantity $F_x^{(h)}(t,f) := \langle x, h_{tf}\rangle$, where $h(t)$ is some "window" and $h_{tf}(\tau) := h(\tau - t)\exp\{i2\pi f\tau\}$. The main drawback of any such "short-time Fourier transform" (STFT) is that it necessarily introduces some extraneous ingredient (the window $h(t)$), which may be poorly adapted to the analyzed signal. An intuitive improvement amounts therefore to make the window depend on the signal, with the simple choice $h(t) \equiv x_-(t) := x(-t)$, as suggested by the "matched filtering" principle. Doing so, it is straightforward to check that we are in fact led to $F_x^{(x_-)}(t,f) \equiv W_x(t/2, f/2)/2$, where

$$W_x(t,f) := \int_{-\infty}^{+\infty} x(t+\tau/2)\,\overline{x(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau \tag{9}$$

is nothing but the usual Wigner-Ville distribution (WVD).[21,28,65,67]

4.3. Classes of distributions from covariance principles

By construction, a WVD is quadratic in the signal and is an energy distribution. More generally, a systematic way of constructing classes of solutions consists in imposing some (very general) a priori structure to the desired distribution, and in deducing more and more restrictive parameterizations from the progressive imposition of further requirements considered as "natural".[28] Albeit not strictly necessary, the usual framework for energy distributions $\rho_x(t,f)$ such that

$$\iint_{-\infty}^{+\infty} \rho_x(t,f)\,dt\,df = \|x\|_2^2 \tag{10}$$

is quadratic:

$$\rho_x(t,f) = \int\int_{-\infty}^{+\infty} K(s,s';t,f)\, x(s)\, \overline{x(s')}\, ds\, ds', \tag{11}$$

and it then suffices to impose additional covariance constraints to appropriately reduce the space of admissible solutions. In brief, this approach amounts to imposing the commutative relation $\rho_{\mathbf{T}x}(t,f) = (\tilde{\mathbf{T}}\rho)_x(t,f)$, in which $\mathbf{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ stands for some transformation operator acting on signals (and $\tilde{\mathbf{T}} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ for the corresponding operator acting on time-frequency distributions). In other words, the desired distribution is asked to "follow" a signal in the transformations that it undergoes.

The simplest example is that of shifts, in both time and frequency, for which the covariance principle leads the most general kernel to take on the simpler form $K(s,s';t,f) = K_0(s-t, s'-t)\exp\{-i2\pi f(s-s')\}$, where $K_0(s,s')$ is some arbitrary two-dimensional function. A remarkable result of this approach is that it then ends up exactly with Cohen's class[21,28]:

$$C_x(t,f) := \int\int_{-\infty}^{+\infty} \varphi(\xi,\tau)\, x(s+\tau/2)\, \overline{x(s-\tau/2)}\, e^{i2\pi[(s-t)\xi - f\tau]}\, ds\, d\xi\, d\tau, \tag{12}$$

provided that

$$\varphi(\xi,\tau) := \int_{-\infty}^{+\infty} K_0\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-i2\pi\xi t}\, dt. \tag{13}$$

The central role played, in classical time-frequency analysis, by Cohen's class (whose the WVD (9), as well as the spectrogram, i.e., the squared modulus of a STFT, is a member) is therefore reinforced by the constructive argument according to which it is the class of all quadratic time-frequency distributions that are shift-covariant.

Generalizing the approach, a deductive construction of other classes of distributions can be obtained on the basis of covariance requirements different from shifts. In particular, imposing covariance with respect to shifts in time and dilations leads—in the space of analytic signals—to the so-called "affine" class.[8,55] Other choices may be considered, at will: covariance requirements with respect to frequency-dependent shifts (nonlinear group delays) lead, e.g., to "hyperbolic" and "power" classes.[52]

5. TIME-FREQUENCY LOCALIZATION OF CHIRPS

The non-unicity of a time-frequency distribution leaves room for specific choices on the basis of additional requirements within a given class. In this respect, localization properties play a special role and may point towards relevant uses of specific distributions when analyzing chirps.

5.1. Wigner and linear chirps

A by-product of the interpretation of the WVD as a self-adapted STFT is that the standard localization property of the Fourier representation on pure tones, namely that $x(t) = \exp\{i2\pi f_0 t\} \Rightarrow X(f) = \delta(f - f_0)$, is now extended to any linear FM in the WVD case, since we then have $x(t) = \exp\{i2\pi(f_0 t + \alpha t^2/2)\} \Rightarrow W_x(t,f) = \delta(f - (f_0 + \alpha t))$.

The localization property of the WVD on straight lines of the plane is a direct consequence of the quadratic nature of the representation, and its admits a simple geometrical interpretation[28,30] which allows for a number of generalizations. Let $x(t) = a(t)\exp\{i\varphi(t)\}$ be a chirp, its WVD can be written as the FT of a modified chirp, whose phase is $\Phi_t(\tau) := \varphi(t+\tau/2) - \varphi(t-\tau/2)$. The corresponding "instantaneous frequency" identifies therefore, at each time instant t, to the quantity $(\partial\Phi_t/\partial\tau)(\tau)/2\pi = [f_x(t+\tau/2) + f_x(t-\tau/2)]/2$, a quantity which exactly coincides with $f_x(t)$ if and only if $\varphi(t)$ is a polynomial of degree at most two, i.e., if the chirp is linear.

A different insight on the same issue of localization for linear chirps can be gained from Janssen's interference formula[37]:

$$W_x^2(t,f) = \int\int_{-\infty}^{+\infty} W_x(t+\tau/2, f+\xi/2)\, W_x(t-\tau/2, f-\xi/2)\, d\tau\, d\xi,$$

according to which a non-zero value of the WVD at a given time-frequency point results from the superposition of other non-zero WVD contributions which are symmetrically located with respect to the considered point. From this perspective, localization on straight lines is clear, since a straight line is the only curve of the plane defined as the locus of all of its mid-points. More generally, we see that localization is a concept which directly follows from the

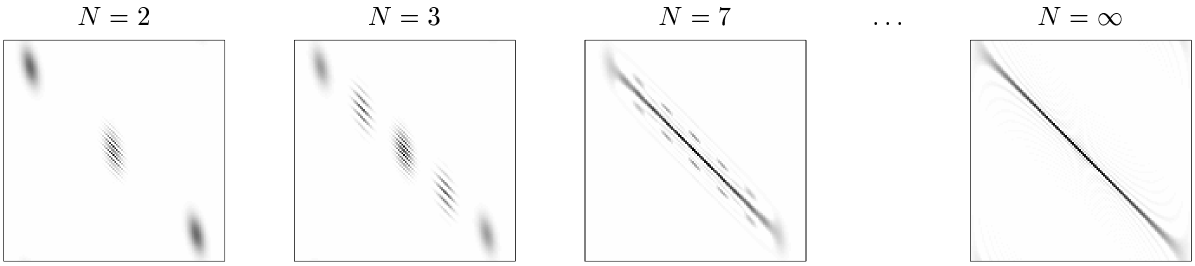

Figure 2. *The localization property of the Wigner-Ville distribution on straight lines of the time-frequency plane can be seen as the result of a contructive interference process. Cross-terms of the WVD being located midway between any two interacting components, the Figure illustrates how an increasing number N of aligned wave packets creates an increasing number of cross-terms that are aligned too. In the limit where $N \to \infty$, this leads to a perfect localization of the distribution along the line, which is the locus of all of its mid-points. (In each image, time is horizontal, frequency is vertical, and amplitude is coded with gray levels (only positive values are displayed).*

quadratic nature of the transform and, hence, that it is just another facet of interference phenomena[35] attached to quadratic distributions: in a nutshell, and as illustrated in Figure 2, *localization is nothing but the emergence of a constructive interference process.*

5.2. Quadratic generalizations

While retaining the overall philosophy put forward by the specific case of the WVD, variations can therefore be proposed, with modified "mid-points geometries" leading to modified WVD's localizing on nonlinear curves of the plane. More precisely, in the case of analytic signals, it is known[8] that localization on power-law group delays of the type $t_X(f) = t_0 + c\, f^{k-1}$ $(k \leq 0)$ can be achieved with adapted "Bertrand distributions" of the form

$$B_X^{(k)}(t,f) := \int_{-\infty}^{+\infty} \underbrace{X\left(f\lambda_k(u)\right)}_{\text{dilation}} \underbrace{\overline{X\left(f\lambda_k(-u)\right)}}_{\text{compression}} \underbrace{f\, \mu_k(u)\, e^{i2\pi f t \zeta_k(u)}\, du}_{\text{modified Fourier}},$$

where $\lambda_k(u) := [k(e^{-u}-1)/(e^{-ku}-1)]^{1/(k-1)}$ (with $k \neq 0$ or 1, and continuous extensions when $k = 0$ or 1), $\zeta_k(u) := \lambda_k(u) - \lambda_k(-u)$ and $\mu_k(u) := \dot{\zeta}_k(u)\, \sqrt{\lambda_k(u)\lambda_k(-u)}$. In fact, such distributions differ only slightly from the usual WVD (9) which can be equivalently expressed as :

$$W_X(t,f) = \int_{-\infty}^{+\infty} \underbrace{X\left(f + \xi/2\right)}_{\text{shift forward}} \underbrace{\overline{X\left(f - \xi/2\right)}}_{\text{shift backward}} \underbrace{e^{i2\pi\xi t}\, d\xi}_{\text{Fourier}}.$$

In the WVD case, interference terms appear midway between interacting components,[35] and are controlled by the arithmetic mean rule $(a,b) \mapsto A(a,b) = (a+b)/2$. In the Bertrand case, the construction rule turns out[31] to be controlled by the "generalized logarithmic mean"[61] $(a,b) \mapsto L_k(a,b) = [(a^k - b^k)/k(a-b)]^{1/(k-1)}$, with $k \neq 0, 1$ and continuous extensions when $k = 0$ or 1. In accordance with the formal equivalence $B_X^{(2)} \equiv W_X$, we have $L_2 \equiv A$, whereas varying k allows for interpreting localization on power-law curves as the result of a modified geometry, based on a notion of mean different from the usual arithmetic one (L_{-1}, for instance, is the geometric mean), see Figure 3.

5.3. Warped quadratic distributions

Among the various classes of distributions that can be obtained from a covariance requirement with respect to frequency-dependent shifts, the Altes distribution[28,52]

$$\check{Q}_X(s,f) := f \int_{-\infty}^{+\infty} X(f\, e^{u/2})\, \overline{X(f\, e^{-u/2})}\, e^{i2\pi s u}\, du \tag{14}$$

plays, within the hyperbolic class, a role as central as the WVD within Cohen's class. In fact, one can check that both distributions are intimately related since we have $\check{Q}_X(s,f) = W_{\widetilde{X}}(s, \log f)$, with $\widetilde{X}(f) := X(e^f) e^{f/2}$. It thus follows

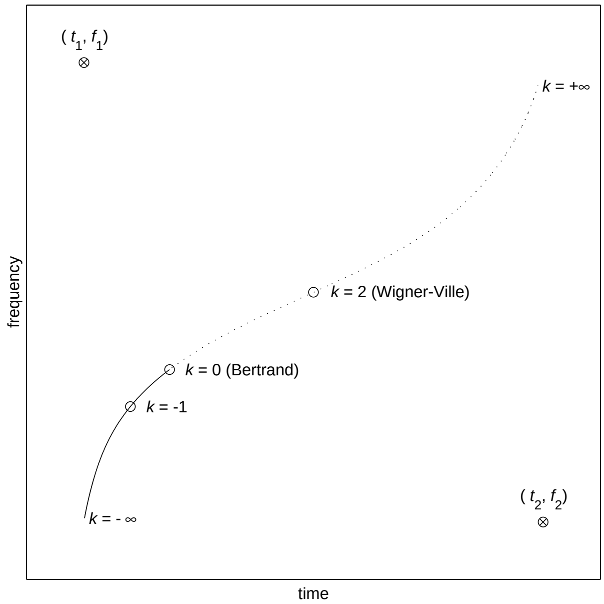

Figure 3. *In the case of Bertrand distributions of index $k \in \mathbb{R}$, two components located in (t_1, f_1) and (t_2, f_2) interfere to create a cross-term whose location is controlled by a generalized logarithmic mean rule. The Figure gives the time-frequency trajectory of this "mid-point" when k varies from $-\infty$ to $+\infty$. In a first approximation (for details, see[31]), one can infer from this diagram that, for a given k, chirp localization can therefore be achieved along the matched power-law functions passing through the three considered points.*

that Altes distributions may be perfectly localized, in their Mellin variable s, for specific chirps with hyperbolic group delays. From a time-frequency perspective, the formal identification $s \equiv tf$ guarantees that the associated time-frequency version $Q_X(t,f) := \check{Q}_X(tf,f)$ of the Altes distribution can be perfectly localized along hyperbolae of the plane.

This idea of getting localized distributions from warping can be pushed further and adapted to more general cases, on the basis of very general unitary equivalence arguments.[5]

5.4. Quartic and higher-order generalizations

As we have seen, the WVD combines two ingredients: it is a quadratic distribution of the signal, but arguments of the signal entering the cross-product are linear in the variable onto which the Fourier transform applies. Localization on straight lines of the plane is then the result of this combination, justifying that at least one of those two ingredients has to be relaxed for ensuring localization on nonlinear curves. The Bertrand distribution was an instance of such a modification, with a quadratic transform involving nonlinearly warped spectra. Another way can however be explored: it consists in generalizing the idea of self-adaptation thanks to which a signal-based STFT gave birth to the WVD. Transposing the approach to quadratic distributions, the idea is to start from some generalized form of the WVD (as offered, e.g., by the kernel-based framework of Cohen's class[21,28]), with some explicit signal dependence in the parameterization.[20] This point of view paves the road (together with a fresh interpretation) for polynomial distributions,[10] amongst which the simplest ones are quartic in the signal:

$$Q_x(t,f) = \int_{-\infty}^{+\infty} x\,(t + b_1\tau)\ \overline{x\,(t + b_2\tau)}\, x\,(t + b_3\tau)\ \overline{x\,(t + b_4\tau)}\, e^{-i2\pi f\tau}\, d\tau,$$

where the b_i's are real-valued free parameters. As expected, convenient choices of these parameters can be made so as to guarantee a perfect localization in the case of unimodular chirps with a cubic phase (i.e., quadratic FM signals), with preferred solutions in terms of computational simplicity and minimum spreading for quartic phases.[49] Although it can be extended conceptually to higher-order chirps with higher-order distributions, this approach, however, becomes quickly totally unefficient in terms of analysis, computational complexity and readability.

5.5. Locally adapted distributions

A common remark that can be addressed to all of the above-mentioned ways of making a quadratic transform signal-dependent is that they all involve the analyzed signal as a whole, being therefore much too global to be universally effective (unless in very specific signal classes).

Reassignment — Among the many possibilities of locally adapting a distribution to a signal, one is of special interest and of very general applicability: it is referred to as "reassignment".[2,15,18,39,40] In order to explain what reassignment consists in, it is better to start with a re-interpretation of conventional spectrograms. Classically, a spectrogram is defined as the squared modulus of a STFT: $S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2$, but it is well-known[21,28] that it can be expressed as well as a smoothed WVD, according to:

$$S_x^{(h)}(t,f) = \iint_{-\infty}^{+\infty} W_x(\tau,\xi)\, W_h(\tau - t, \xi - f)\, d\tau\, d\xi.$$

This relation makes explicit the fact that a spectrogram value cannot be considered as pointwise. In fact, this value rather results from the summation of all WVD contributions within some time-frequency domain defined as the essential time-frequency support of W_h, properly centered at the location of the considered point of interest. A whole distribution of values is therefore summarized by a single number, and this number is assigned to the geometrical center of the domain over which the distribution is considered. Reasoning with a mechanical analogy, the situation is as if the total mass of an object were assigned to its geometrical center, an arbitrary point which—except in the very specific case of an homogeneous distribution over the domain—has no reason to suit the actual distribution. A much more meaningful choice is to assign the total mass to the center of gravity of the distribution within the domain, and this is precisely what reassignment does: at each point where a spectrogram value is computed, we also compute the local centroïd of the WVD, as seen through the time-frequency window defined by the local kernel, and the distribution value is moved from the point where it has been computed to this centroïd.

In the case of linear FM signals, reassigned spectrograms inherit therefore of the perfect localization property of the WVD, since the centroïd of any segment of a line distribution necessarily belongs to the line. This property still remains effective in the case of multicomponent linear chirps as long as no more than one chirp is "seen" through the same time-frequency smoothing window. Similarly, almost perfect localization is achieved for nonlinear chirps which are locally linear within the window. Finally, one must add that, although it had been historically introduced for spectrograms only,[39,40] reassignment is by no way restricted to this sole family of distributions: its principle can be applied as well to very general settings (Cohen's class, affine class...), in fact to any distribution which can be expressed as a smoothed version of some mother-distribution with localization properties.[2,18]

An example of the effectiveness of reassignment is given in Figure 4[†]. The analyzed signal is in this case Riemann's function:

$$\sigma(t) := \sum_{n=1}^{\infty} \frac{\sin \pi n^2 t}{n^2}, \tag{15}$$

a 2-periodic function which has been shown[36] to admit a local approximation, in the vicinity of $t = 0$, in terms of power-law chirps with $\alpha = -3/2$ and $\beta = -1$.

Ridges and skeletons — A technique related to reassignment, referred to as "ridges and skeletons", has been developed for both Gabor and wavelet transforms.[13,23] Behind the idea of a "ridge" is the intuition that, in the case of chirp signals, the largest contributions should lie in the plane along specific trajectories. What has been shown is that such trajectories can be identified from the phase of the transform, and that the corresponding coefficients convey most of the information present in a signal, in the sense that they allow for its almost perfect reconstruction.[13,23] By structure, reassignment has much to share with "ridges and skeletons" and can be viewed as a form of generalization: indeed, for a fixed time location, the frequency location of a ridge is nothing but the fixed point of a (frequency only) reassignment operator.[15]

Connecting power-law chirps and oscillating singularities, the concept of "ridges and skeletons" justifies that the largest wavelet coefficients live outside from the influence cone centered at the time of occurrence of the singularity,

[†]It has to be remarked that reassigned distributions can be equipped with efficient algorithms.[2] MATLAB codes for reassigned distributions (as the one used for producing the Figure) are available as part of a freeware time-frequency toolbox.[3]

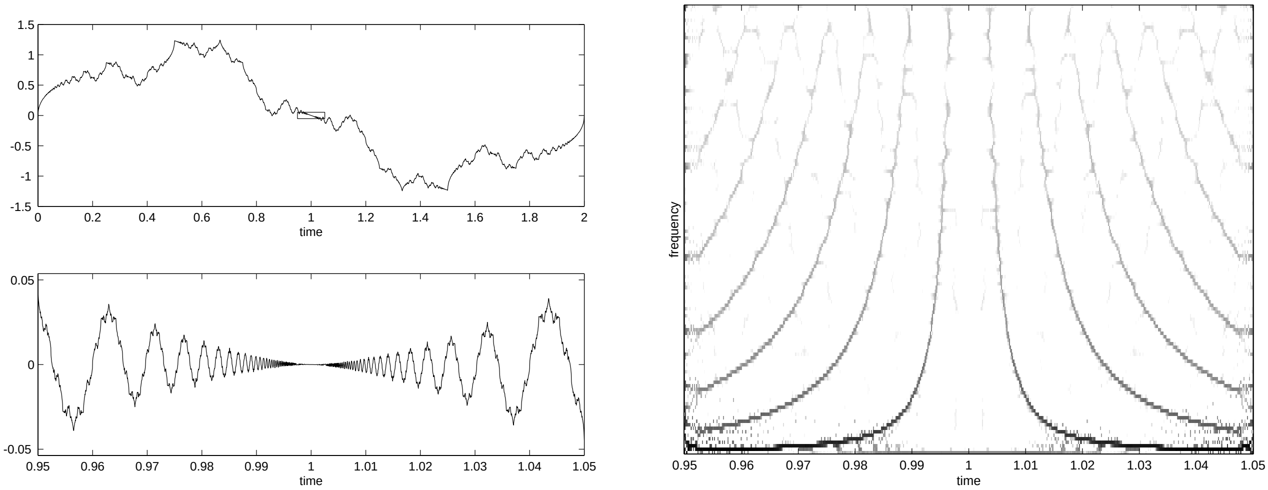

Figure 4. *The Riemann function (15) is an example of a mathematical object that can be locally expanded in terms of power-law chirps at certain points. The top left diagram displays the complete (*2*-periodic) function over the fundamental interval* $[0, 2]$. *A (detrended) magnification of the restriction of the function within the box centered in* $(1, 0)$ *is given in the bottom left diagram. This clearly evidences a local chirping behavior, whose rich multi-component structure is revealed by the reassigned spectrogram plotted in the right diagram.*

thus forbidding the use of "classical" wavelet-based estimations aimed at Hölder singularities.[42] Recognizing this fact has been the starting point of rigorous mathematical developments[36] which basically amount to considering coefficients located along suitable trajectories of the plane rather than within the influence cone.

Revealing instantaneous frequencies — Both reassigned distributions and "ridges and skeletons" are powerful techniques for evidencing time-varying structures in signals. In the spectrogram case, the local centroïds involved in the reassignment process have for coordinates the group delay and the instantaneous frequency of the signal, as seen through the local time-frequency window W_h. This offers a reversed perspective (which could now be referred to as "think global, act local") to the concept of instantaneous frequency: as opposed to the standard definition based on the analytic signal, reassignment examplifies the idea that this notion can rather be viewed as a form of emergence of energy concentration along trajectories on the plane. As such, the time-frequency paradigm is not used for estimating pre-defined quantities, but rather for revealing relevant structures composing a signal.[29]

6. CONCLUSION

This paper has surveyed a number of issues related to chirps, and has advocated the explicit use of time-frequency tools for their analysis. In particular, it has been shown how a matched and/or adapted distribution may be localized in the plane along a curve which is an image of the frequency history of a chirp. In many circumstances, localizing a chirp is not the ultimate goal, but rather a pre-requisite for simplifying a further processing (one can mention, e.g., the problem of detecting a chirp via a coherent path integration in the plane,[6,17] or of synthesizing music with additive techniques[26]). Whatever the objective, it has been argued that the time-frequency plane is a convenient representation space for chirps, whose internal structure can be *revealed* via the emergence of localized contributions and which, as notes on a musical score, can be used as a natural *language* for numerous time-varying signals.

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