Elements of time-frequency analysis

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observing
“chirps”

- **Waves and vibrations** — Bird songs, bats, music ("glissando"), speech, "whistling atmospherics", tidal waves, gravitational waves, wide-band impulses propagating in a dispersive medium, pendulum, diapason (string, pipe) with time-varying length, vibroseismics, radar, sonar, Doppler effect, ....

- **Biology and medicine** — EEG (epilepsy), uterine EMG (contractions), ...

- **Disorder and critical phenomena** — Coherent structures in turbulence, accumulation of earthquake precursors, "speculative bubbless" prior a financial crash, ...

- **Mathematical special functions** — Weierstrass, Riemann, ...
describing
Definition

A “chirp” is any complex-valued signal reading
\[ x(t) = a(t) \exp\{i \varphi(t)\}, \]
where \( a(t) \geq 0 \) is a low-pass amplitude whose evolution is slow as compared to the phase oscillations \( \varphi(t) \).

**Slow evolution?** — Usual heuristic conditions assume that:

1. \( |\dot{a}(t)/a(t)| \ll |\dot{\varphi}(t)| \): the amplitude is **almost constant** at the scale of a pseudo-period \( T(t) = 2\pi/|\dot{\varphi}(t)| \).
2. \( |\ddot{\varphi}(t)|/\dot{\varphi}^2(t) \ll 1 \): the pseudo-period \( T(t) \) is itself **slowly varying** from one oscillation to the next.
modulations

- **Monochromatic wave** — In the case of a harmonic model $x(t) = a \cos(2\pi f_0 t + \varphi_0)$, observing $x(t)$ leads in an **unambiguous** way to the amplitude $a$ and to the frequency $f_0$.

- **Amplitude and frequency modulations** — Moving to an **evolutive** model amounts (intuitively) to achieve the transformation $a \cos(2\pi f_0 t + \varphi_0) \rightarrow a(t) \cos \varphi(t)$ with $a(t)$ variable and $\varphi(t)$ nonlinear. In an **observation** context, the unicity of the representation is however lost since

$$a(t) \cos \varphi(t) = \left[ \frac{a(t)}{b(t)} \right] [b(t) \cos \varphi(t)] =: \tilde{a}(t) \cos \tilde{\varphi}(t)$$

for any function $0 < b(t) < 1$. 

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Monochromatic wave — The real-valued harmonic model can be written as

\[ x(t) = a \cos(2\pi f_0 t + \varphi_0) = \text{Re} \{ a \exp i(2\pi f_0 t + \varphi_0) \} , \]

with

\[ a \exp i(2\pi f_0 t + \varphi_0) = x(t) + i (Hx)(t) \]

and where \( H \) is the Hilbert transform (quadrature).

Interpretation

A monochromatic wave (prototype of a “stationary” deterministic signal) is described, in the complex plane, by a rotating vector whose modulus and rotation speed are constant along time.
instantaneous amplitude and frequency

**Generalisation** — A wave modulated in amplitude and in frequency (prototype of a “nonstationary” deterministic signal) is described, in the complex plane, by a rotating vector whose modulus and rotation speed are varying along time, complexification mimicking the “stationary” case:

\[ x(t) \rightarrow z_x(t) := x(t) + i (Hx)(t). \]

**Definition (Ville, ’48)**

The instantaneous amplitude and frequency follow from this complex-valued representation, called analytic signal, as:

\[ a_x(t) := |z_x(t)| \quad ; \quad f_x(t) := (d/dt) \arg z_x(t)/2\pi. \]
limitations

- **Multiple components** — By construction, the instantaneous frequency can only attach **one** frequency value at a given time $\Rightarrow$ weighted average in the case of multicomponent signals.

    
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    [freqinst2.m]

- **Trends** — Same problem with a monocomponent signal with a DC component or a very low frequency trend.

    
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    [freqinsttrend.m]

    Possible improvement with an “osculating” Fresnel representation (Aboutajdine et al., ’80).

    [freqinstosc.m]

- **Noise** — **Differential** definition very sensitive to additive noise, even faint.

    
    Noise — **Differential** definition very sensitive to additive noise, even faint.

    [freqinst1b.m]
A process \( \{x(t), t \in \mathbb{R}\} \) is said to be (second order) **stationary** if its statistical properties (of orders 1 and 2) are independent of some absolute time.

- **Mean value** — The expectation \( \mathbb{E}\{x(t)\} \) is constant (\( \rightarrow 0 \))
- **Covariance** — The covariance function \( r_x(t, t') := \mathbb{E}\{x(t)x(t')\} \) is such that

\[
r_x(t, t') =: \gamma_x(t - t').
\]
Result (Cramér)

\[ x(t) = \int e^{i2\pi ft} dX(f) \]

with \( E\{dX(f) dX(f')\} = \delta(f - f') d\Gamma_x(f) df' \)

- **Simplification** — \( d\Gamma_x(f) \) abs. cont. wrt Lebesgue
  \( \Rightarrow d\Gamma_x(f) =: S_x(f) df \) with \( S_x(f) \) **power spectral density**.

- **Duality (Bochner, Wiener, Khintchine)** — One thus gets

\[ r_x(\tau) = \int e^{i2\pi f\tau} d\Gamma_x(f) \left( = \int e^{i2\pi f\tau} S_x(f) df \right). \]
nonstationarities

- **Spectral representation** — Always valid, but **without** the orthogonality of spectral increments ⇒ the spectral distribution is no more diagonal but a function of **two** frequencies.

- **Covariance** – Depends explicitly of **two** times (e.g., one **absolute** time and one **relative** time).

**Interpretation**

*The “power spectrum density” becomes **time-dependent** ⇒ time-frequency.*
**Chirp Spectrum**

**Stationary Phase** — In the case where the phase derivative $\dot{\varphi}(t)$ is monotonous, one can approach a chirp spectrum

$$X(f) = \int a(t) e^{i(\varphi(t) - 2\pi ft)} \, dt$$

by its **stationary phase approximation** $\tilde{X}(f)$, leading to

$$|\tilde{X}(f)| \propto a(t_s) |\ddot{\varphi}(t_s)|^{-1/2},$$

with $t_s$ such that $\dot{\varphi}(t_s) = 2\pi f$.

**Interpretation**

*The “instantaneous frequency” curve $\dot{\varphi}(t)$ puts in a one-to-one correspondence one time and one frequency. The spectrum follows by weighting the **visited frequencies** by the corresponding residence durations.*
time-frequency interpretation

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Elements of time-frequency analysis
representing
intuition

Idea

Give a mathematical sense to musical notation

Aim

Write the “musical score” of a signal with multiple, evolutive components with that additional constraint of getting, in the case of an isolated chirp \( x(t) = a(t) \exp\{i\varphi(t)\} \), a localized representation

\[
\rho(t, f) \sim a^2(t) \delta(f - \dot{\varphi}(t)/2\pi).
\]
The example of the short-time FT — One defines the \textbf{local} quantity

$$F_x^{(h)}(t, f) = \int x(s) h(s - t) e^{-i2\pi fs} \, ds,$$

where $h(t)$ is some short-time observation window.

\textbf{Measurement} — The representation results from an interaction between the signal and a \textbf{measurement device} (the window $h(t)$).

\textbf{Trade-off} — A short window favors the “resolution” in time at the expense of the “resolution” in frequency, and vice-versa.

[spectrodemo.m]
adaption

- **Chirps** — Adaptation to **pulses** if \( h(t) \to \delta(t) \) and to **tones** if \( h(t) \to 1 \) \( \Rightarrow \) adapting the analysis to arbitrary chirps suggests to make \( h(t) \) (**locally**) depending on the signal.

- **Linear chirp** — In the linear case \( f_x(t) = f_0 + \alpha t \), the equivalent frequency width \( \delta f_S \) of the **spectrogram** \( S_x^{(h)}(t, f) := |F_x^{(h)}(t, f)|^2 \) behaves as:

\[
\delta f_S \approx \sqrt{\frac{1}{\delta t_h^2} + \alpha^2 \delta t_h^2}
\]

for a window \( h(t) \) with an equivalent time width \( \delta t_h \) \( \Rightarrow \) minimum for \( \delta t_h \approx 1/\sqrt{\alpha} \) (but \( \alpha \) **unknown**...).
self-adaptation and Wigner-Ville distribution

- **Matched filtering** — If one takes for the window $h(t)$ the time-reversed signal $x_-(t) := x(-t)$, one readily gets that $F_{x_-(t,f)} = W_x(t/2, f/2)/2$, where

$$W_x(t, f) := \int x(t + \tau/2) \overline{x(t - \tau/2)} e^{-i2\pi f \tau} d\tau$$

is the **Wigner-Ville Distribution** (Wigner, '32; Ville, '48).

- **Linear chirps** — The WVD **perfectly** localizes on **straight lines** of the plane:

$$x(t) = \exp\{i2\pi (f_0 t + \alpha t^2 / 2)\} \Rightarrow W_x(t, f) = \delta (f - (f_0 + \alpha t)) .$$

- **Remark** — Localization via self-adaptation leads to a **quadratic** transformation (energy distribution).
Mirror symmetry — Indexing the analyzed signal with respect to a local frame as $x_t(s) := x(s + t)$, one gets:

$$W_x(t, f) := \int \left[ x_t(\tau/2) x_t(-\tau/2) \right] e^{-i2\pi f \tau} d\tau,$$

Phase signal — If $x_t(s) = \exp\{i\varphi_t(s)\}$, $W_x(t, f)$ is, as a function of $t$, the FT of a phase signal

$\Phi_t(\tau) := \varphi_t(\tau/2) - \varphi_t(-\tau/2)$, with “instantaneous frequency”

$$\tilde{f}_{x_t}(\tau) = \frac{1}{2\pi} \frac{\partial}{\partial \tau} \Phi_t(\tau) = \frac{1}{2} [f_{x_t}(\tau/2) + f_{x_t}(-\tau/2)]$$

Localization — It follows that $\tilde{f}_{x_t}(\tau) = f_0$ if $f_{x_t}(\tau) = f_0 + \alpha \tau$, for any modulation rate $\alpha$. 

[spectrovsWV.m]
further properties

- **Energy**
  \[ \iint W_x(t, f) \, dt \, df = \| x \|^2 \]

- **Marginals**
  \[ \int W_x(t, f) \, dt = |X(f)|^2; \quad \int W_x(t, f) \, df = |x(t)|^2 \]

- **Unitarity ("Moyal’s formula")**
  \[ \iint W_x(t, f) W_y(t, f) \, dt \, df = |\langle x, y \rangle|^2 \]

- **Conservation of supports, covariance wrt scaling, linear filtering and modulation, etc.**
further properties

- **Local moments**
  \[
  \int f \ W_x(t, f) \, df / |x(t)|^2 = f_x(t); \quad \int t \ W_x(t, f) \, dt / |X(f)|^2 = t_x(f)
  \]

**Interpretation**

\( W_x(t, f) \) **quasi-probability (joint) density** of energy in time and frequency:

\[
W_x(t, f) = W_x(t|f) \int W_x(t, f) \, dt = W_x(f|t) \int W_x(t, f) \, df
\]

\[
f_x(t) = \mathbb{E}\{f|t\}; \quad t_x(f) = \mathbb{E}\{t|f\}
\]

- **Limitation** — \( W_x(t, f) \in \mathbb{R} \) but \( \notin \mathbb{R}_+ \).
interferences

- **Quadratic superposition** — For any pair of signals \( \{x(t), y(t)\} \) and coefficients \((a, b)\), one gets

\[
W_{ax+by}(t, f) = |a|^2 W_x(t, f) + |b|^2 W_y(t, f) + 2 \text{Re} \left\{ a \overline{b} W_{x,y}(t, f) \right\},
\]

with

\[
W_{x,y}(t, f) := \int x(t + \tau/2) \overline{y(t - \tau/2)} e^{-i2\pi f \tau} d\tau
\]

- **Drawback** — Interferences between **disjoint** component reduce readability.

- **Advantage** — Inner interferences between **coherent** components guarantee localization.
interferences

- **Janssen’s formula (Janssen, ’81)** — It follows from the **unitarity** of $W_x(t, f)$ that:

$$|W_x(t, f)|^2 = \int \int W_x \left( t + \frac{\tau}{2}, f + \frac{\xi}{2} \right) W_x \left( t - \frac{\tau}{2}, f - \frac{\xi}{2} \right) d\tau \, d\xi$$

- **Geometry (Hlawatsch & F., ’85)** — Contributions located in any two points of the plane plan interfere to create a third contribution

  1. midway of the segment joining the two components
  2. oscillating (positive and negative values) in a direction perpendicular to this segment
  3. with a “frequency” proportional to their “time-frequency distance”.

[WV2trans.m, WVinterf.m]
interferences and readability

somme des WV (N = 16)

WV de la somme (N = 16)
interferences and localization

sum(WV) (N = 16)

WV(sum) (N = 16)
classes of quadratic distributions

Observation

Many quadratic distributions have been proposed in the literature since more than half a century (e.g., spectrogram and DWV): none fully extends the notion of spectrum density to the nonstationary case.

Principle of conditional unicity — Classes of quadratic distributions of the form $\rho_x(t, f) = \langle x, K_{t,f}x \rangle$ can be constructed based on covariance requirements:

\[
\begin{align*}
x(t) \quad \rightarrow \quad & \rho_x(t, f) \\
(Tx)(t) \quad \rightarrow \quad & \rho_{Tx}(t, f) = (\tilde{T}\rho_x)(t, f)
\end{align*}
\]
classes of quadratic distributions

- **Cohen’s class** — Covariance wrt shifts
  \[(T_{t_0,f_0}x)(t) = x(t - t_0) \exp \{i2\pi f_0 t \} \text{ leads to Cohen’s class (Cohen, ’66):}
  \]
  \[
  C_x(t, f) := \int \int W_x(s, \xi) \Pi(s - t, \xi - f) \, ds \, d\xi,
  \]
  with \( \Pi(t, f) \) “arbitrary” (and to be specified via additional constraints).

- **Variations** — Other choices possibles, e.g.,
  \[(T_{t_0,f_0}x)(t) = (f/f_0)^{1/2} x(f(t - t_0)/f_0) \rightarrow \text{affine class (Riou & F, ’92), etc.}\]
an alternative interpretation of Cohen’s class

- **Duality between distribution and correlation** — In the “stationary” case, the **frequency** energy distribution can be estimated as the Fourier image of the **time** correlation \( \langle x, T_\tau x \rangle \), possibly weighted.

- **Extension** — In the “nonstationary” case, one must consider a **time-frequency correlation** \( A_x(\xi, \tau) \propto \langle x, T_{\tau,\xi} x \rangle \) (ambiguity function) which, after weighting and Fourier transformation, leads again to Cohen’s class:

\[
C_x(t, f) = \int \int \varphi(\xi, \tau) A_x(\xi, \tau) e^{-i2\pi(\xi t + \tau f)} \, d\xi \, d\tau.
\]
why “Cohen-type” classes?

- **Unification** — Specifying a kernel (i.e., $\Pi(t, f)$) defines a distribution: unifying framework or most propositions of the literature (Wigner-Ville, spectrogram, Page, Levin, Rihaczek, etc.).

- **Parameterization** — Properties of a distribution are directly connected with admissibility conditions of the associated kernel $\Rightarrow$ simplified possibility of evaluation and design.
an example of definition

**Spectrogram** — If we consider the case of the **spectrogram** with window \( h(t) \), one can write:

\[
S_x^{(h)}(t, f) = \left| \int x(s) h(s - t) e^{-i 2\pi f s} \, ds \right|^2
\]

\[
= \left| \langle x, T_{t,f} h \rangle \right|^2
\]

\[
= \iint W_x(s, \xi) W_{T_{t,f} h}(s, \xi) \, ds \, d\xi
\]

\[
= \iint W_x(s, \xi) W_h(s - t, \xi - f) \, ds \, d\xi
\]

⇒ a spectrogram is a member of Cohen’s class, with kernel

\[
\Pi(t, f) = W_h(t, f)
\]
Marginal in time — If one wants to have $\int C_x(t, f) \, df = |x(t)|^2$, one can write:

\[
\int C_x(t, f) \, df = \int \left( \int \varphi(\xi, \tau) A_x(\xi, \tau) e^{-i2\pi(\xi t + \tau f)} \, d\xi \, d\tau \right) \, df \\
= \int \varphi(\xi, 0) A_x(\xi, 0) e^{-i2\pi \xi t} \, d\xi \\
= \int \varphi(\xi, 0) \left( \int |x(\theta)|^2 e^{i2\pi \xi \theta} \, d\theta \right) e^{-i2\pi \xi t} \, d\xi \\
= \int |x(\theta)|^2 \left( \int \varphi(\xi, 0) e^{i2\pi \xi (\theta - t)} \, d\xi \right) \, d\theta
\]

⇒ the associated kernel must necessarily satisfy

\[\varphi(\xi, 0) = 1, \forall \xi\]

(true for Wigner-Ville but not for spectrograms)
Cohen’s class and smoothing

- **Spectrogram** — Given a low-pass window \( h(t) \), one gets the smoothing relation:

\[
S^{(h)}_{\chi}(t, f) := |F^{(h)}_{\chi}(t, f)|^2 = \int \int W_{\chi}(s, \xi) W_h(s-t, \xi-f) \, ds \, d\xi
\]

- **From Wigner-Ville to spectrograms** — A generalization amounts to choose a smoothing function \( \Pi(t, f) \) allowing for a continuous and separable transition between Wigner-Ville and a spectrogram (smoothed pseudo-Wigner-Ville distributions):

\[
\text{Wigner} \rightarrow \text{Ville} \rightarrow \text{PWVL} \rightarrow \text{spectrogram}
\]

\[
\delta(t) \delta(f) \quad g(t) H(f) \quad W_h(t, f)
\]
Definition (Martin, '82)

One of the most "natural" extensions of the power spectrum density is given by the Wigner-Ville Spectrum:

\[ W_x(t, f) := \int r_x \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) e^{-i2\pi f \tau} \, d\tau \]

- **Interpretation** — FT of a local correlation.
- **Properties** — PSD if \( x(t) \) stationary, marginals, etc.
- **Relation with the WVD** — Under simple conditions, one has \( W_x(t, f) = \mathbb{E}\{ W_x(t, f) \} \).
estimation of the Wigner-Ville spectrum

**Aim**

*Approach* $\mathbb{E}\{W_x(t, f)\}$ on the basis of only one realization.

- **Assumption** — **Local** stationnarity (in time and in frequency).
- **Estimators** — Smoothing of the DWV:
  
  $$\hat{W}_x(t, f) = (\Pi \ast \ast W_x)(t, f)$$

  i.e., Cohen’s classe.

- **Properties** — **Statistical** (bias-variance) and **geometrical** (localization) trade-offs, both controlled by $\Pi(t, f)$. 
  
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  Elements of time-frequency analysis
global vs. local

- **Global approach** — The Wigner-Ville Distribution localizes perfectly on **straight lines** of the plane (linear chirps). One can construct other distributions localizing on more general **curves** (ex.: **Bertrand**’s distributions adapted to hyperbolic chirps).

- **Local approach** — A different possibility consists in revisiting the smoothing relation defining the spectrogram and in considering localization wrt the instantaneous frequency as it can be measured **locally**, at the scale of the short-time window ⇒ **reassignment**.
**Principle** — The key idea is (1) to replace the geometrical center of the smoothing time-frequency domain by the center of mass of the WVD over this domain, and (2) to **reassign** the value of the smoothed distribution to this local centroïd:

\[
S_x^{(h)}(t, f) \mapsto \int \int S_x^{(h)}(s, \xi) \delta \left( t - \hat{t}_x(s, \xi), f - \hat{f}_x(s, \xi) \right) \, ds \, d\xi.
\]

**Remark** — Reassignment has been first introduced for the only spectrogram (Kodera et al., '76), but its principle has been further generalized to **any** distribution resulting from the smoothing of a localizable mother-distribution (Auger & F., '95).
reassignment

Wigner-Ville

spectrogram

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reassignment

Wigner-Ville

reassigned spectrogram
reassignment in action

- **Spectrogram — Implicit** computation of the local centroids (Auger & F., '95):

\[
\hat{t}_x(t, f) = t + \text{Re} \left\{ \frac{F_x^{(Th)}}{F_x^{(h)}} \right\}(t, f)
\]

\[
\hat{f}_x(t, f) = f - \text{Im} \left\{ \frac{F_x^{(Dh)}}{F_x^{(h)}} \right\}(t, f),
\]

with \((Th)(t) = t \: h(t)\) and \((Dh)(t) = (dh/dt)(t)/2\pi\).

- **Beyond spectrograms** — Possible generalizations to other smoothings (smoothed pseudo-Wigner-Ville, scalogram, etc.).
independence wrt window size

spectro
window = 21
63
127 points

signal model
128 points

Wigner-Ville

reass. spectro
window = 21
63
127 points
an example of comparison

- Signal model
- Wigner–Ville (log scale)
- Spectrogram (log scale)
- Reassigned spectrogram

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comparison with noise
comparison with noise

signal (RSB = 20 dB)

pseudo–Wigner–Ville

pseudo–Wigner–Ville liss

spectrogramme

spectrogramme rallou
comparison with noise

signal (RSB = 13 dB)

pseudo-Wigner-Ville

pseudo-Wigner-Ville liss

spectrogramme

spectrogramme rallou
comparison with noise

- Signal (RSB = 7 dB)
- Pseudo-Wigner-Ville
- Pseudo-Wigner-Ville liss
- Spectrogramme
- Spectrogramme rallou
comparison with noise

.signal (RSB = 0 dB)

pseudo–Wigner–Ville

pseudo–Wigner–Ville liss

spectrogramme

spectrogramme rallou

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reassignment and estimation

- **Advantage** — Very good properties of localization for chirps (> spectrogram).
- **Limitation** — High sensitivity to noise (< spectrogram).

**Aim**

*Reduce fluctuations while preserving localization.*

**Idea (Xiao & F., ’06)**

*Adopt a multiple windows approach.*
Observing, describing, representing time-frequency, from Fourier to Wigner beyond Wigner the stochastic case localization time-frequency decisions

Back to spectrum estimation

Stationary processes — The power spectrum density can be viewed as:

$$S_x(f) = \lim_{T \to \infty} \mathbb{E} \left\{ \frac{1}{T} \left| \int_{-T/2}^{+T/2} x(t)e^{-i2\pi ft} \, dt \right|^2 \right\}$$

In practice — Only one, finite duration, realization ⇒ crude periodogram (squared FT) = non consistent estimator with large variance.
classical way out (Welch, ’67)

- **Principle** — Method of *averaged periodograms*

\[ \hat{S}_{x,K}^{(W)}(f) = \frac{1}{K} \sum_{k=1}^{K} S_{x}^{(h)}(t_k, f) \]

with \( t_{k+1} - t_k \) of the order of the width of the window \( h(t) \).

- **Bias-variance trade-off** — Given \( T \) (finite), increasing \( K \) ⇒ reduces variance, but increases bias
multitaper solution (Thomson, ’82)

- **Principle** — Computing

\[
\hat{S}_{x,K}(f) = \frac{1}{K} \sum_{k=1}^{K} S_x^{(h_k)}(0, f)
\]

with \( \{h_k(t), k \in \mathbb{N}\} \) a family of orthonormal windows extending over the whole support of the observation \( \Rightarrow \) reduced variance, without sacrificing bias

- **Nonstationary extension** — Multitaper spectrogram

\[
\hat{S}_{x,K}(f) \rightarrow S_{x,K}(t, f) := \frac{1}{K} \sum_{k=1}^{K} S_x^{(h_k)}(t, f)
\]

- **Limitation** — Localization controlled by most spread spectrogram.
Multitaper reassignment

Idea

Combining the advantages of reassignment \((\text{wrt localization})\) with those of multitapering \((\text{wrt fluctuations})\):

\[
S_{x,K}(t, f) \rightarrow RS_{x,K}(t, f) := \frac{1}{K} \sum_{k=1}^{K} RS_{x}^{(h_k)}(t, f)
\]

1. **coherent averaging of chirps** (localization independent of the window)
2. **incoherent averaging of noise** (different TF distributions for different windows)
in practice

- **Choice of windows — Hermite functions**
  
  \[ h_k(t) = (-1)^k \frac{e^{-t^2/2}}{\sqrt{\pi^{1/2} 2^k k!}} (D^k \gamma)(t); \gamma(t) = e^{t^2} \]

  rather than **Prolate Spheroidal Wave functions**

- **Two main reasons**
  1. WVD with **elliptic symmetry** and **maximum concentration** in the plane.
  2. **recursive** computation of \( h_k(t), (\mathcal{T} h_k)(t) \) and \( (D h_k)(t) \) ⇒ better implementation in **discrete-time**. In particular:

  \[ (D h_k)(t) = (\mathcal{T} h_k)(t) - \sqrt{2(k+1)} h_{k+1}(t) \]
example 1

1 taper

sample spectro.
sample reass. spectro.
sample Wigner

10 tapers

sample mean spectro.
sample mean reass. spectro.

average mean spectro.
average mean reass. spectro.
average Wigner

10 samples

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Elements of time-frequency analysis
example 2

spectro. (M = 1)

spectro. (M = 2)

spectro. (M = 3)

spectro. (M = 4)

spectro. (M = 5)

spectro. (M = 6)

reass. spectro. (M = 1)

reass. spectro. (M = 2)

reass. spectro. (M = 3)

reass. spectro. (M = 4)

reass. spectro. (M = 5)

reass. spectro. (M = 6)
detection/estimation of chirps

- **Optimality** — Matched filtering, maximum likelihood, contrast, ... basic ingredient = **correlation** “received signal — copy of emitted signal”.

- **Time-frequency interpretation** — **Unitarity** of a time-frequency distribution $\rho_x(t, f)$ guarantees the equivalence:

$$|\langle x, y \rangle|^2 = \langle \langle \rho_x, \rho_y \rangle \rangle.$$ 

- **Chirps** — Unitarity + localization $\Rightarrow$ detection/estimation via **path integration** in the plane (e.g., Wigner-Ville and linear chirps).

[detectTF.m]
**VIRGO example**

chirp de binaire coalescente + référence pour le filtre adapté

observation bruitée, SNR = −10 dB

enveloppe de la sortie du filtre adapté
VIRGO example (Chassande-Mottin & F., ’98)
time-frequency detection?

- **Language** — The time-frequency viewpoint offers a natural language for addressing detection/estimation problems beyond nominal situations.

- **Robustness** — Incorporation of uncertainties in the chirp model by replacing the integration curve by a domain (example of post-newtonian approximations in the case of gravitational waves).
Localization of a moving target — When estimating a delay by matched filtering with some unknown Doppler effect, estimations of delay and Doppler are coupled ⇒ bias and contrast loss at the detector output.

Addressed problem — Suppress bias on delay and minimize contrast loss.

Signal design — Specification of performance via a geometric interpretation of the time-frequency structure of a chirp.

[dopptol.m, faTFdopp.m]
monographs

collective books

preprints & Matlab codes

- [http://tftb.nongnu.org/](http://tftb.nongnu.org/)
- [http://perso.ens-lyon.fr/patrick.flandrin/](http://perso.ens-lyon.fr/patrick.flandrin/)
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pendulum
\[ \ddot{\theta}(t) + \left(\frac{g}{L}\right) \dot{\theta}(t) = 0 \]

- **Constant length** — \( L = L_0 \Rightarrow \) small oscillations are sinusoidal, with **constant** period \( T_0 = 2\pi \sqrt{L_0/g} \).

- **“Slowly” varying length** — \( L = L(t) \Rightarrow \) small oscillations are quasi-sinusoidal, with **varying** pseudo-period \( T(t) \sim 2\pi \sqrt{L(t)/g} \).
gravitational waves

Corotating Neutron Stars

Radiate Gravity Waves and Merge

To form a Black Hole
gravitational waves

gravitational wave

time
bat echolocation
bat echolocation

bat echolocation call + echo

bat echolocation call (heterodyned)
bat echolocation

- **System** — *(Active)* navigation system, natural sonar
- **Signals** — Ultrasound acoustic waves, *transient* (a few ms) and “*wideband*” (some tens of kHz between 40 and 100kHz)
- **Performance** — Close to optimality, with *adaption* of the waveforms to multiple tasks (detection, estimation, recognition, interference rejection,...)
Doppler effect
Doppler effect

- Moving monochromatic source — Differential perception of the emitted frequency.

\[ f + \Delta f \quad f - \Delta f \quad "chirp" \]
Riemann function
Riemann function

\[ \sigma(t) := \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t \]