Elements of signal decompositions

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decomposing signals

chirps and atomic decompositions

- **Fourier** — The usual Fourier transform (FT) can be formally written as $(\mathcal{F}x)(f) := \langle x, e_f \rangle$, with $e_f(t) := \exp\{i2\pi ft\}$, so that:
$$x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df.$$
- **Extensions** — Replace complex exponentials by chirps, considered as **warped** versions of monochromatic waves, or by "chirplets" (short duration chirps) $\Rightarrow$ *modified (short-time) FT, or WT)*.

from Fourier to Mellin

Definition

A **Mellin Transform** *(MT) of a signal* $x(t) \in L^2(\mathbb{R}^+, t^{-2\alpha+1}dt)$ *can be defined as the projection:*

$$(\mathcal{M}x)(s) := \int_0^{+\infty} x(t)\, t^{-i2\pi s-\alpha}\, dt =: \langle x, c \rangle.$$

- Analysis over **hyperbolic** chirps $c(t) := t^{-\alpha} \exp\{i2\pi s \log t\}$.
- $\dot{\varphi}_c(t)/2\pi = s/t \Rightarrow$ the Mellin parameter s can be interpreted as a **hyperbolic modulation rate**.
- The MT can also be viewed as a **warped** FT, since $\tilde{x}(t) := e^{(1-\alpha)t}\, x(e^t) \Rightarrow (\mathcal{M}x)(s) = (\mathcal{F}\tilde{x})(s)$.

from "gaborets" and "wavelets" to "chirplets"

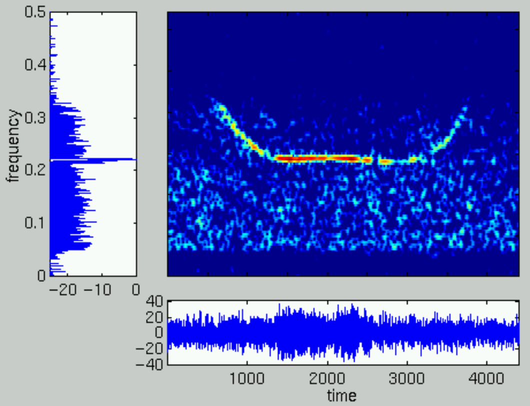
Definition

Localization + **modulation** *lead to 4-parameters representations, e.g., of the form* $\langle x, x_{t,f,\alpha,\gamma} \rangle$ *with*

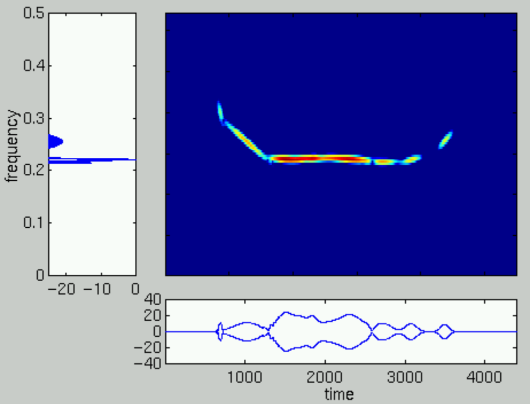
$$x_{t,f,\alpha,\gamma}(s) \propto \exp\{-\pi(\gamma + i\alpha)(s-t)^2 + i2\pi f(s-t)\}.$$

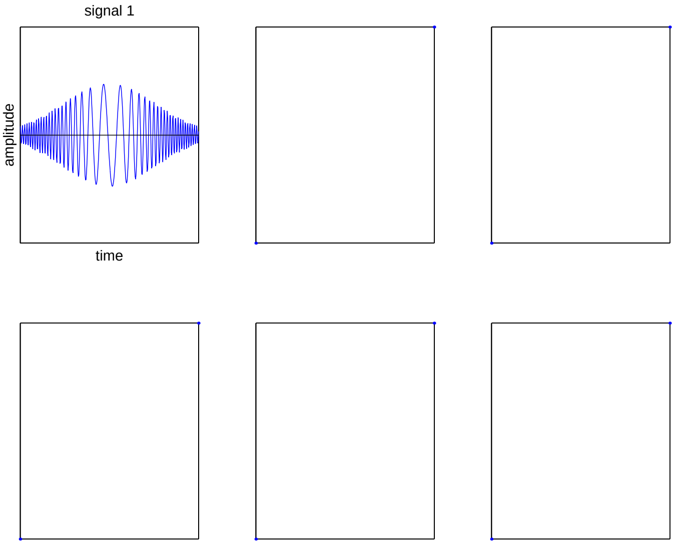
- **Decomposition as an estimation problem** — Chirplets constitutive of a signal can be sequentially **identified** by techniques such as "matching (ou basis) pursuit" (Mallat & Zhang, '93; Chen & Donoho, '99). They can also be **estimated** in a maximum likelihood sense (O'Neill & F., '98–'00).
- **"Parametric-type" limitations** — Trade-off between **dictionary richness** and **computational burden**.

observation



approximation with 8 chirplets

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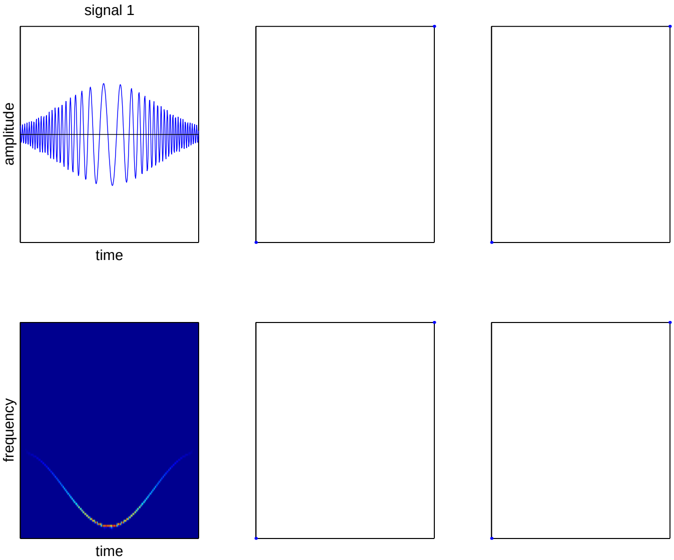
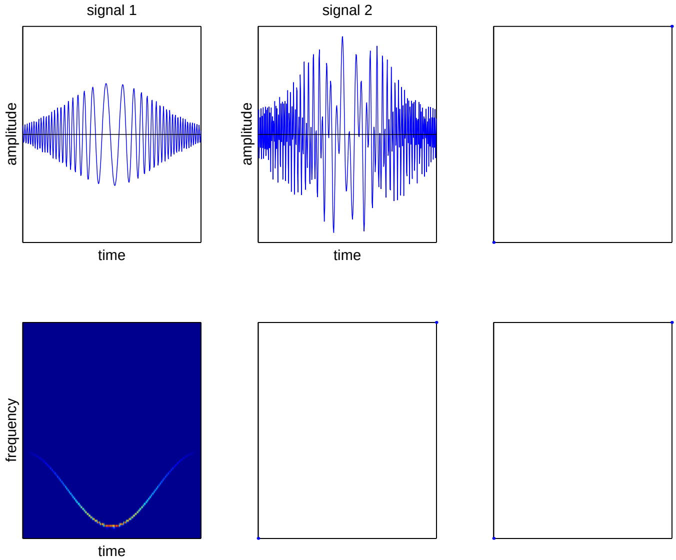
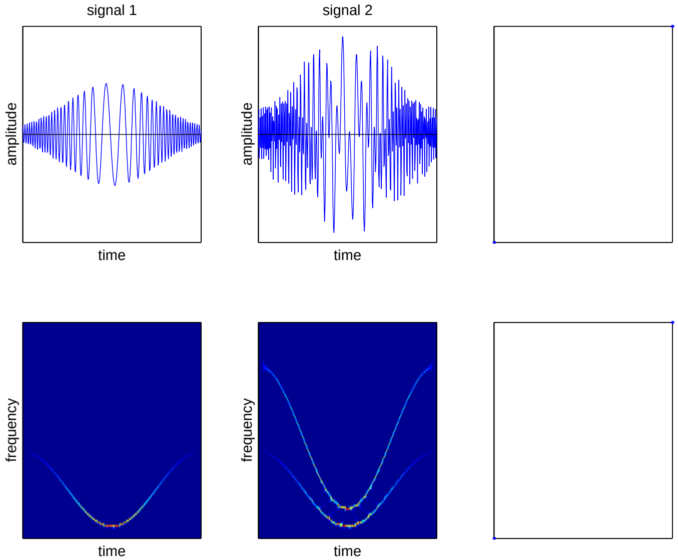
amplitude
time


signal 1
amplitude
time
frequency
time


signal 1
signal 2
amplitude
amplitude
time
time
frequency
time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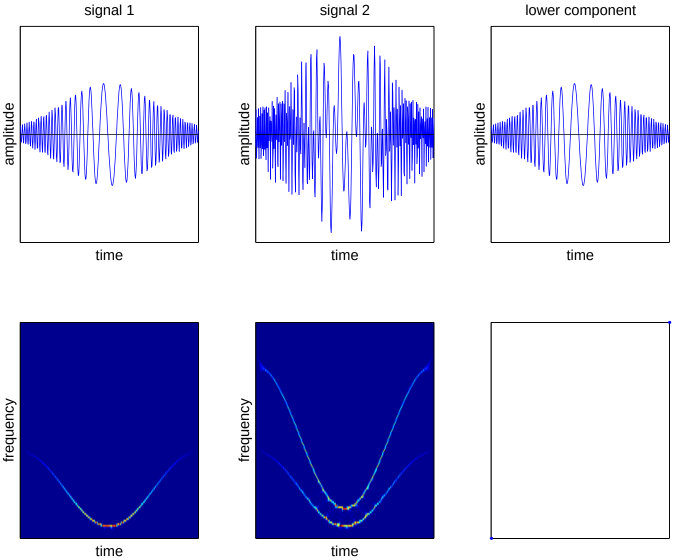
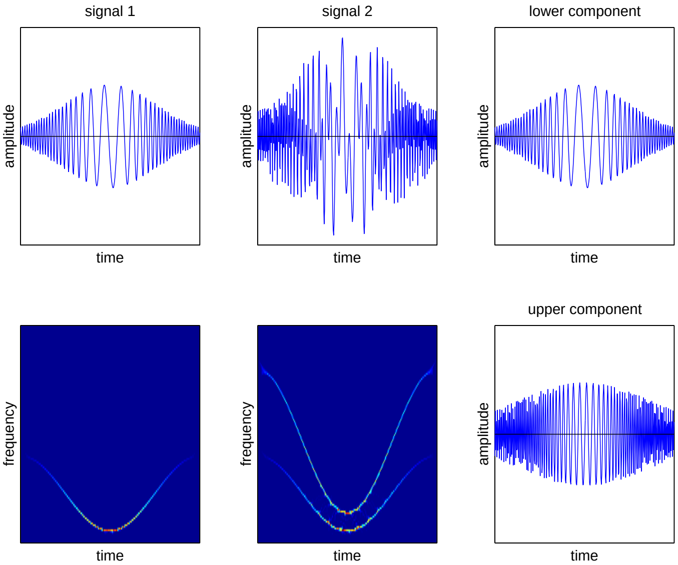
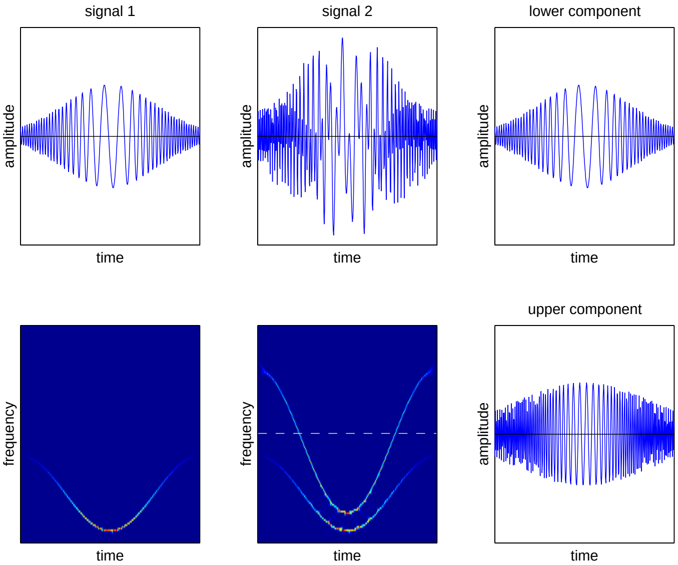
signal 2
amplitude
time
amplitude
time
frequency
time
frequency
time


signal 1
signal 2
lower component
amplitude
time
amplitude
time
amplitude
time
frequency
time
frequency
time


signal 1
amplitude
time
signal 2
amplitude
time
lower component
amplitude
time
frequency
time
frequency
time
upper component
amplitude
time


signal 1
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time
signal 2
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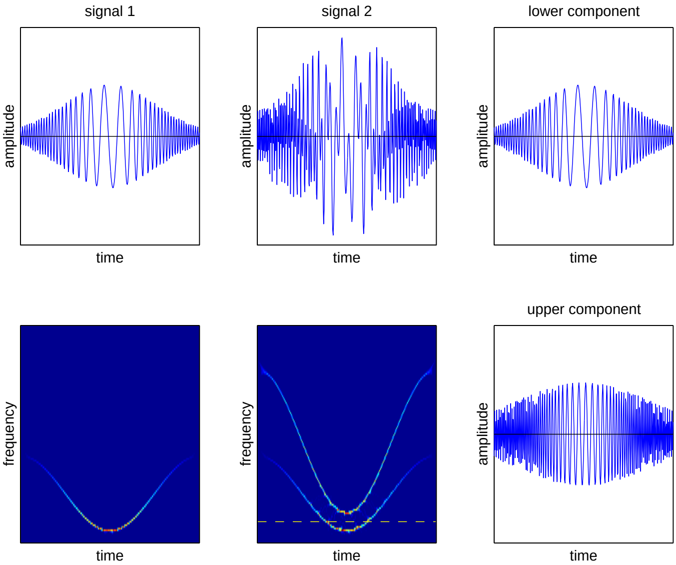
time
lower component
amplitude
time
frequency
time
frequency
time
upper component
amplitude
time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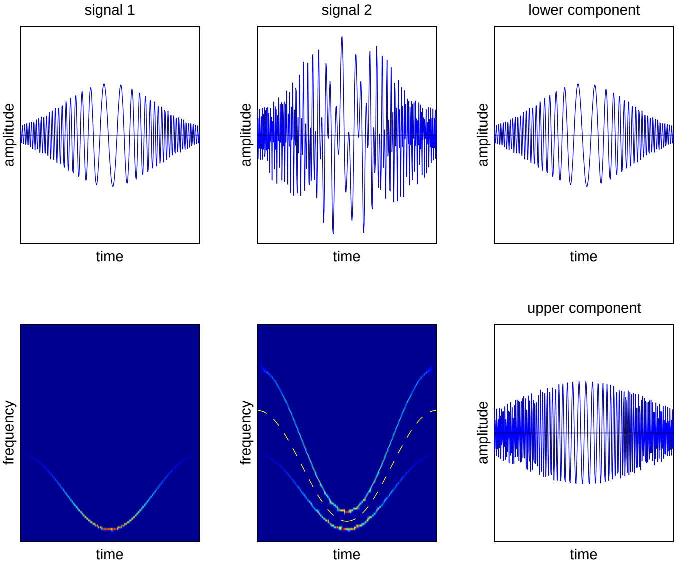
time
signal 2
amplitude
time
lower component
amplitude
time
frequency
time
frequency
time
upper component
amplitude
time


signal 1
amplitude
time
signal 2
amplitude
time
lower component
amplitude
time
frequency
time
frequency
time
upper component
amplitude
time

"EMD" (*Empirical Mode Decomposition*)

Aim

Given an observation $x(t)$, get a representation of the form:

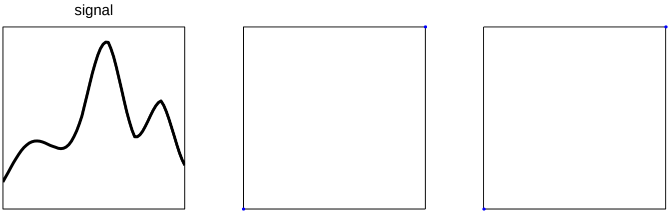
$$x(t) = \sum_{k=1}^{K} a_k(t)\, \psi_k(t),$$

where the $a_k(t)$'s measure "amplitude modulations" and the $\psi_k(t)$'s "oscillations".

principle

Idea

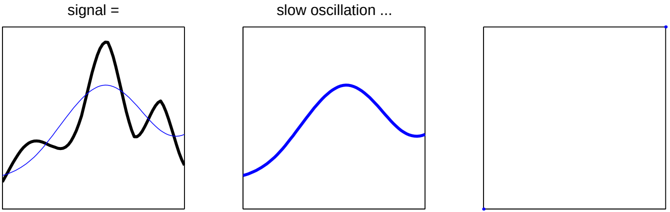
"signal = fast oscillations superimposed to slow oscillations".



principe

Idea

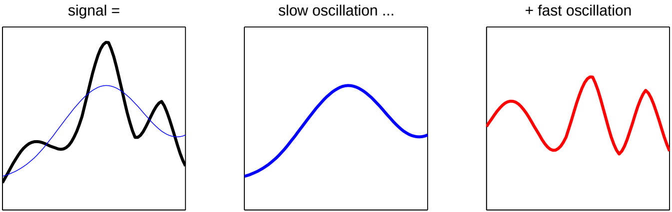
"signal = fast oscillations superimposed to slow oscillations".



principe

Idea

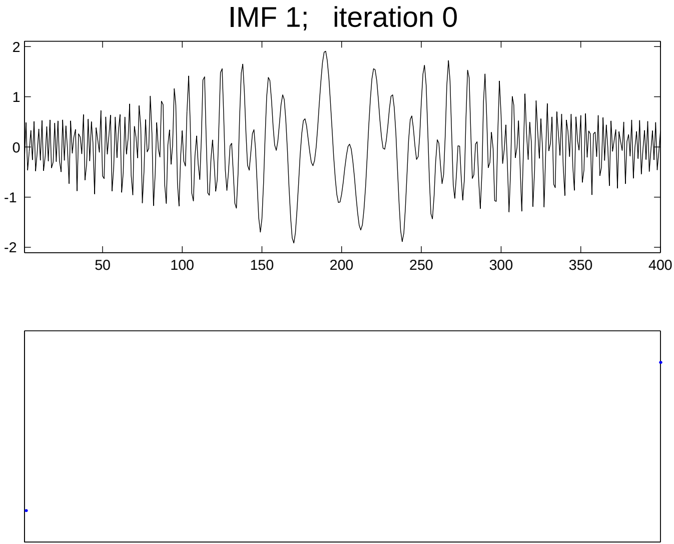
"signal = fast oscillations superimposed to slow oscillations".



Huang's algorithm

Implementation (Huang *et al.*, '98) — Identify (locally) the fastest oscillation, subtract it to the initial signal and iterate on the residual:

1. identify local maxima and minima in the signal
2. deduce an upper and a lower envelope by interpolation (cubic splines)
    1. subtract the mean envelope from the signal
    2. iterate until $\#\{\text{extrema}\} = \#\{\text{zeroes}\} \pm 1$
3. subtract the so-obtained mode from the signal
4. iterate on the residual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1
0
-1
-2
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100
150
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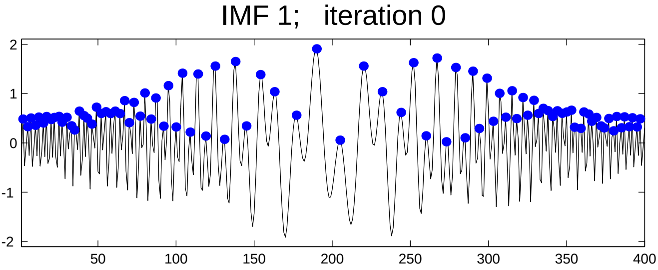
350
400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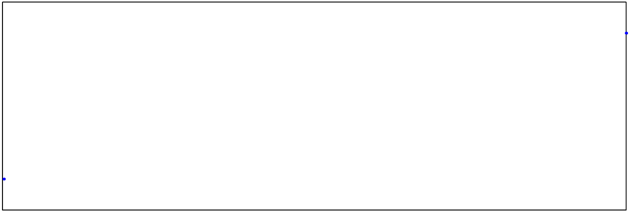

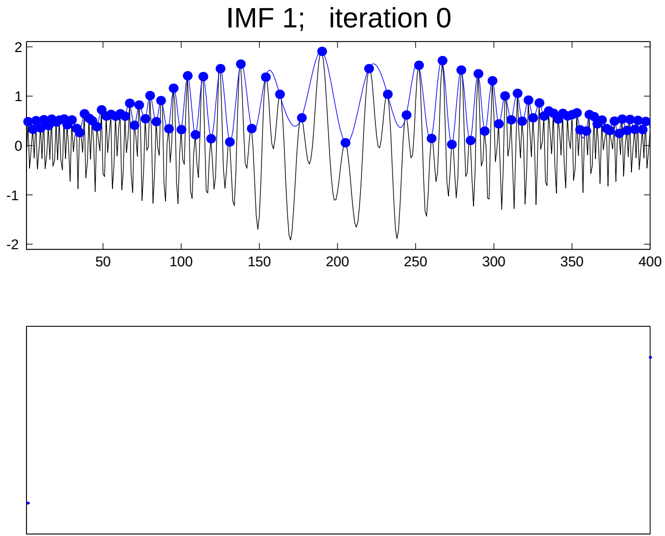
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2
1
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-1
-2
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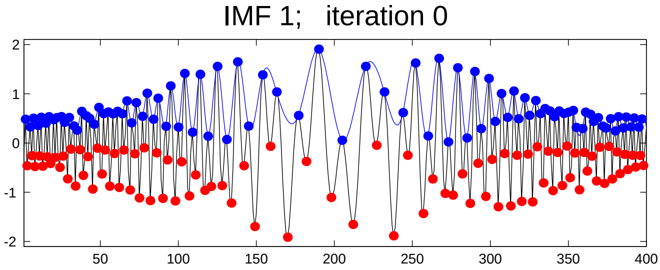
350
400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2
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-1
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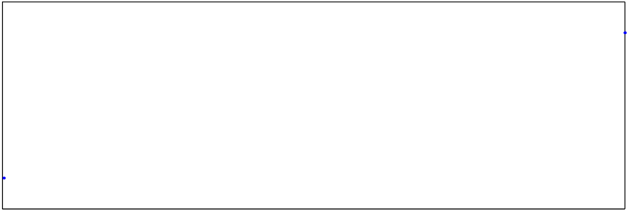

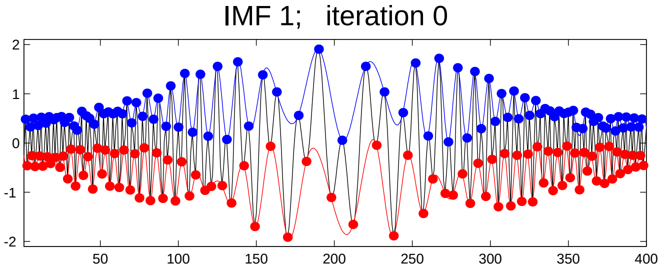
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1
0
-1
-2
50
100
150
200
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350
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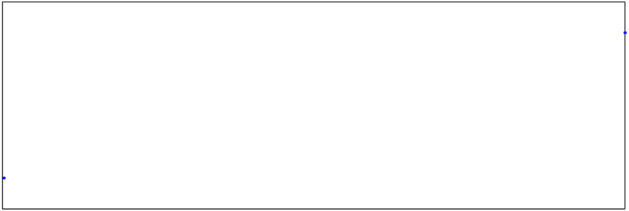

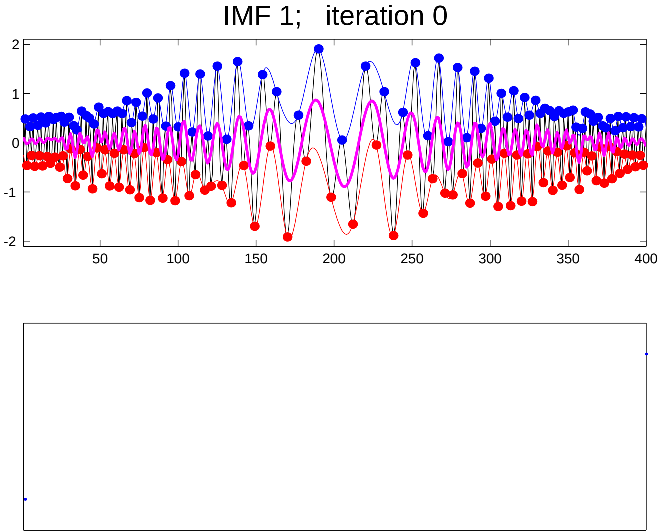
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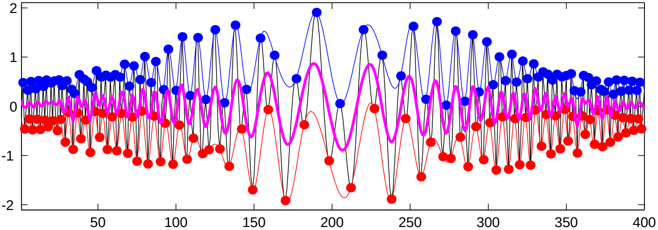
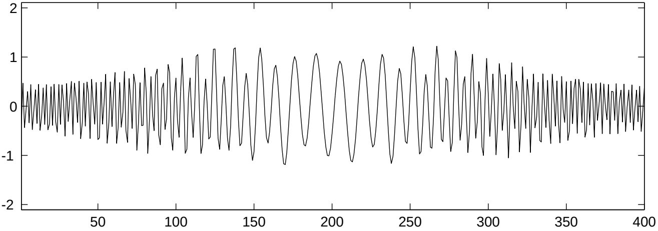
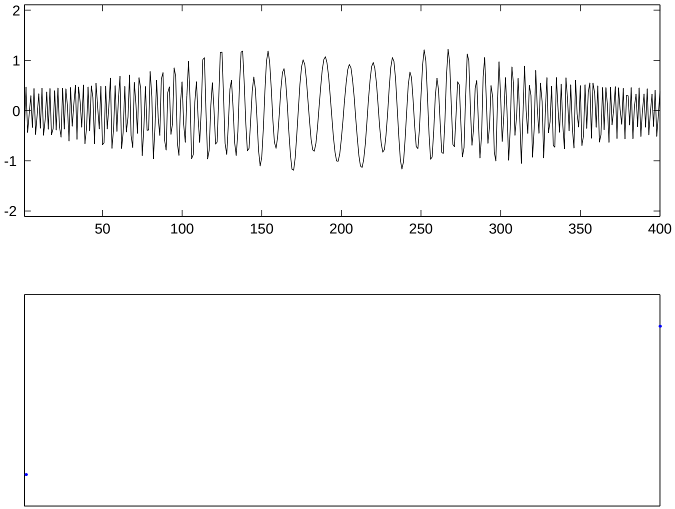
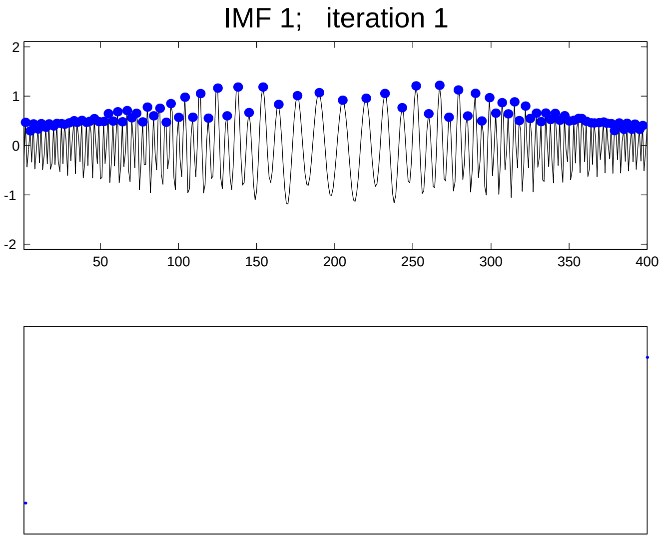


residue



IMF 1; iteration 1

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2
1
0
-1
-2
50
100
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200
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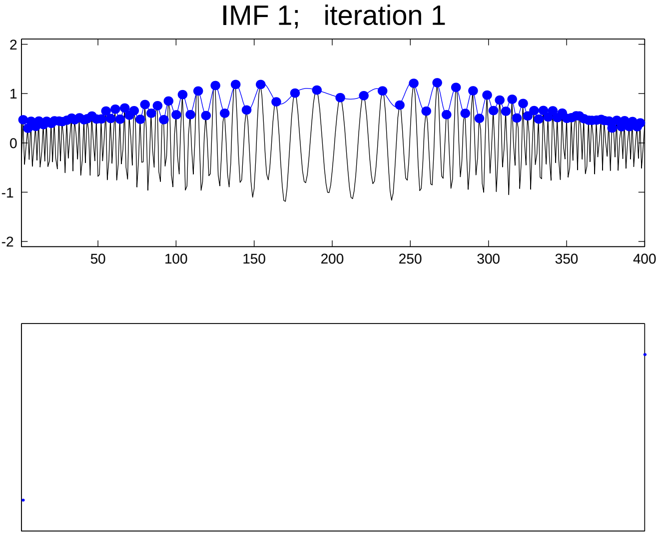
300
350
400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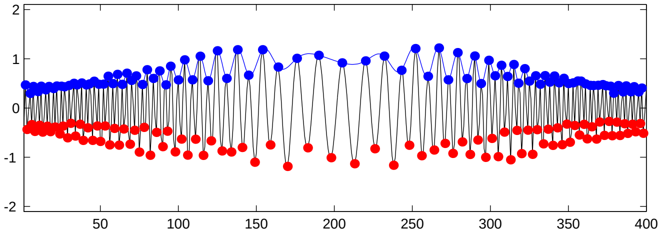
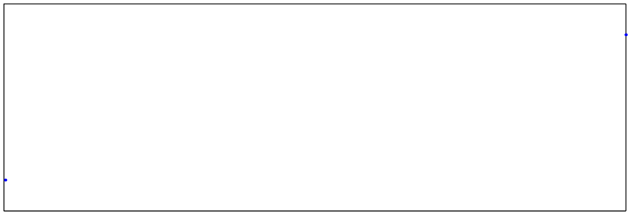
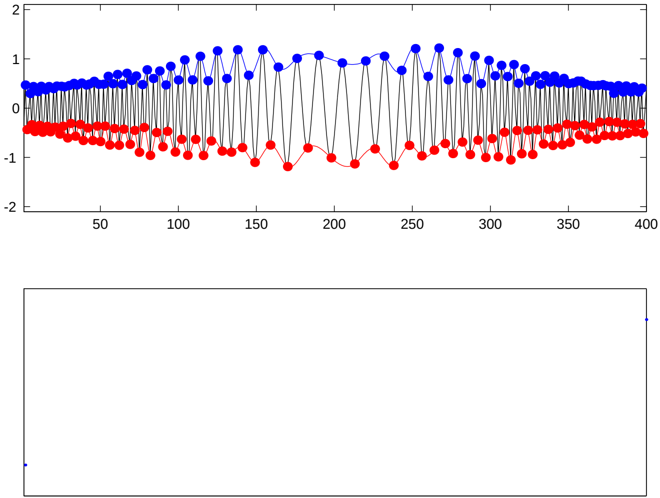
1
0
-1
-2
50
100
150
200
250
300
350
400

IMF 1;   iteration 1

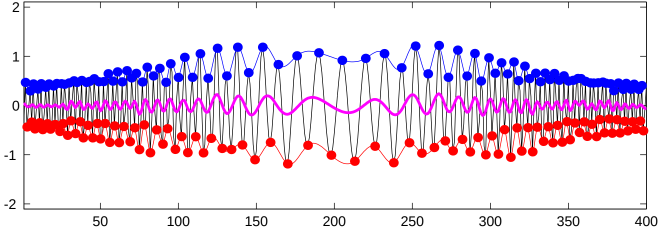


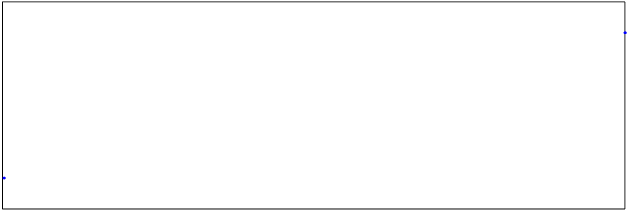


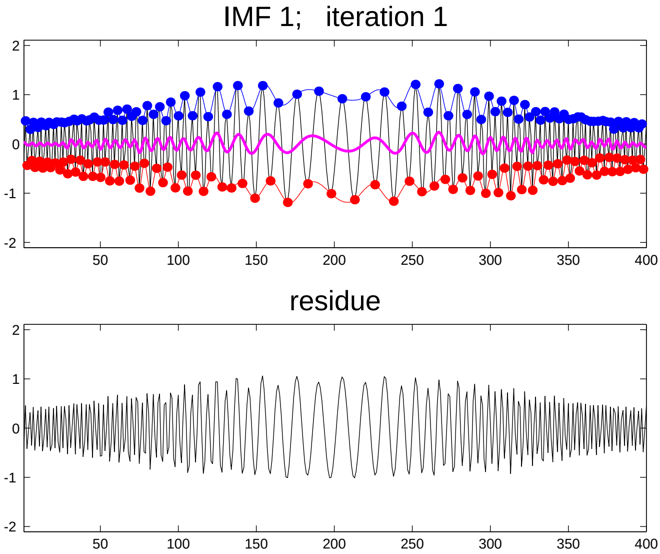
IMF 1; iteration 1



IMF 1; iteration 1






IMF 1; iteration 1
2
1
0
-1
-2
50
100
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200
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400
residue
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0
-1
-2
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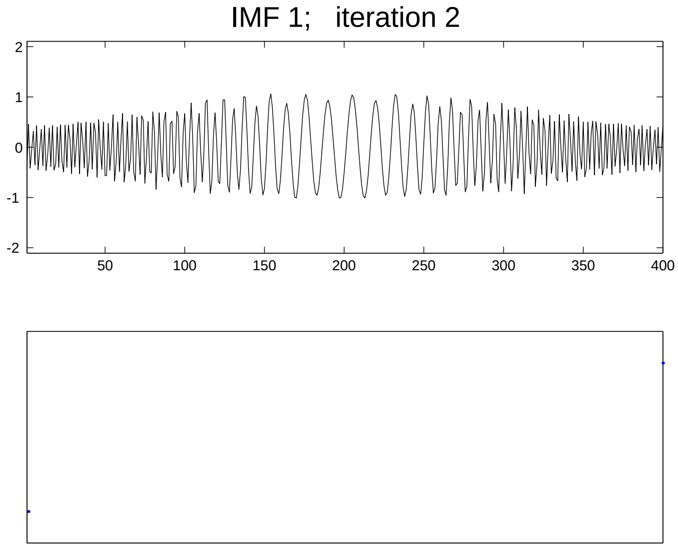
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400

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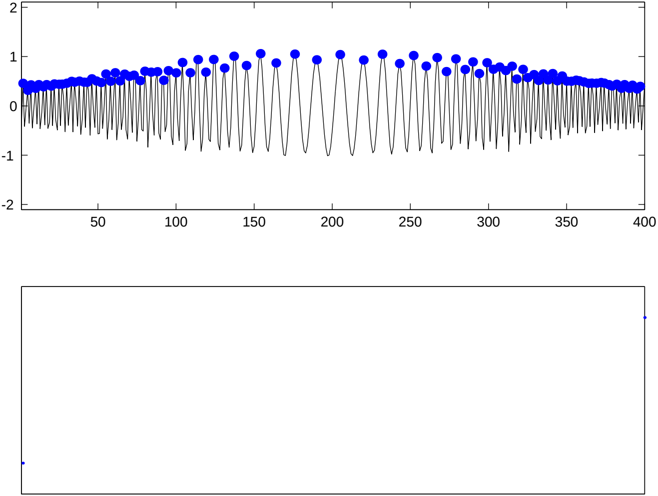

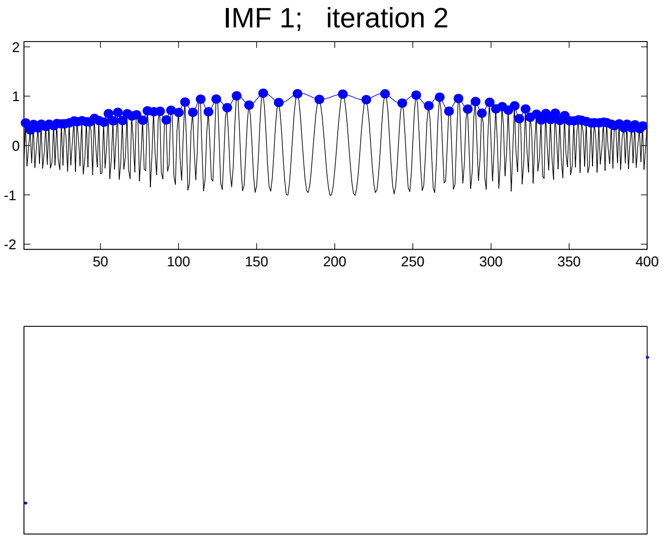
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400

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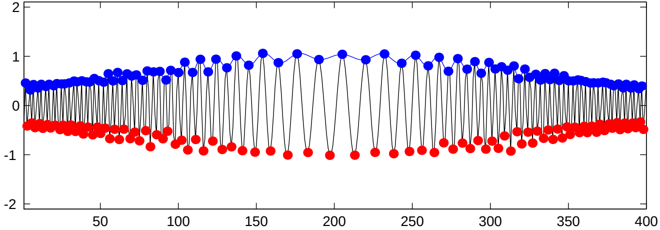

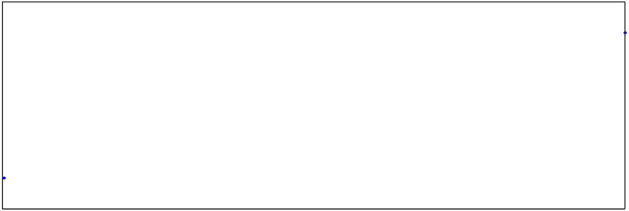

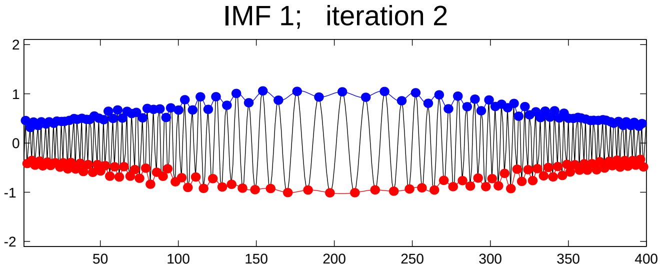
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2
1
0
-1
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50
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300
350
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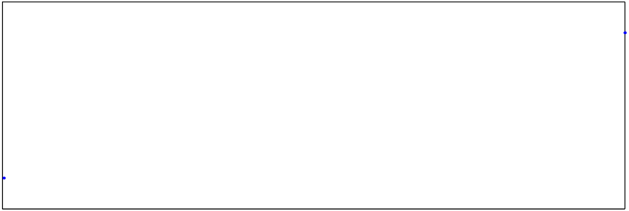

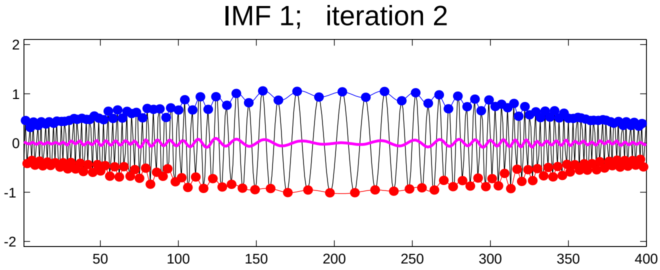
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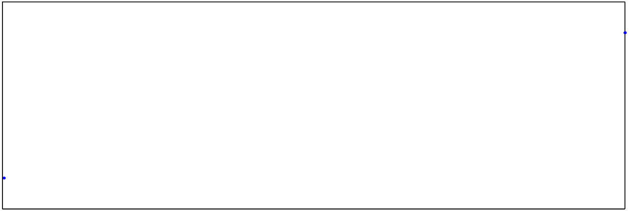
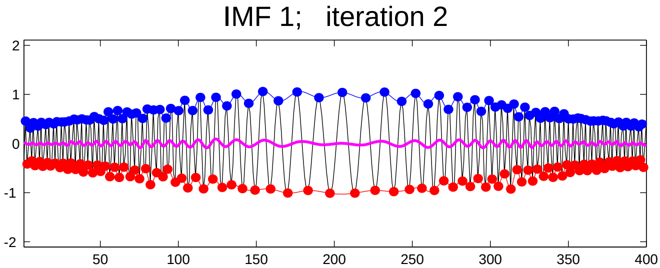
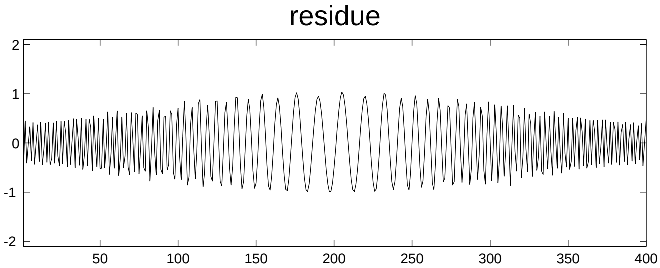
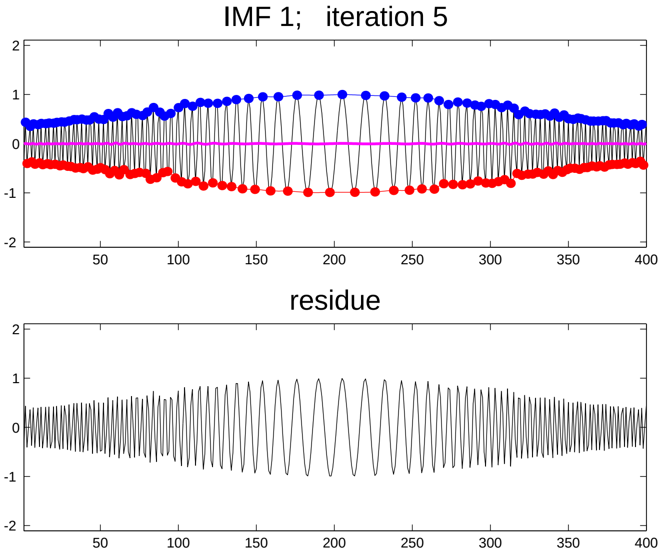
2
1
0
-1
-2
50
100
150
200
250
300
350
400




IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400


residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400


IMF 1; iteration 5
2
1
0
-1
-2
50
100
150
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300
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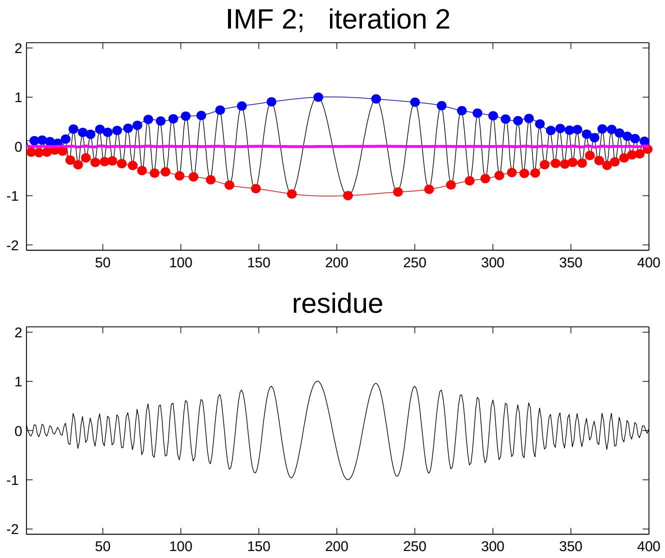
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400


IMF 2; iteration 2
2
1
0
-1
-2
50
100
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200
250
300
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400
residue
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0
-1
-2
50
100
150
200
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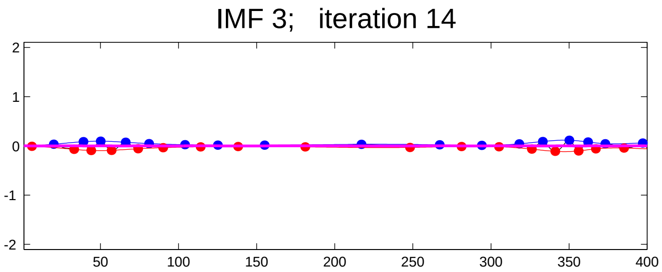
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0
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-2
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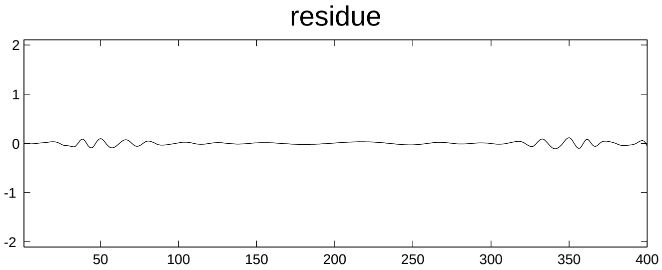
350
400


residue
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-1
-2
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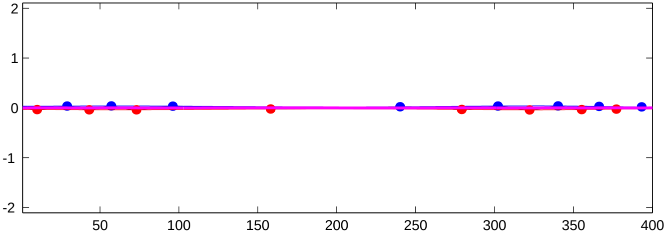
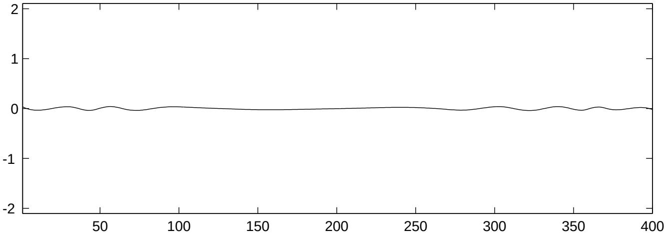
150
200
250
300
350
400

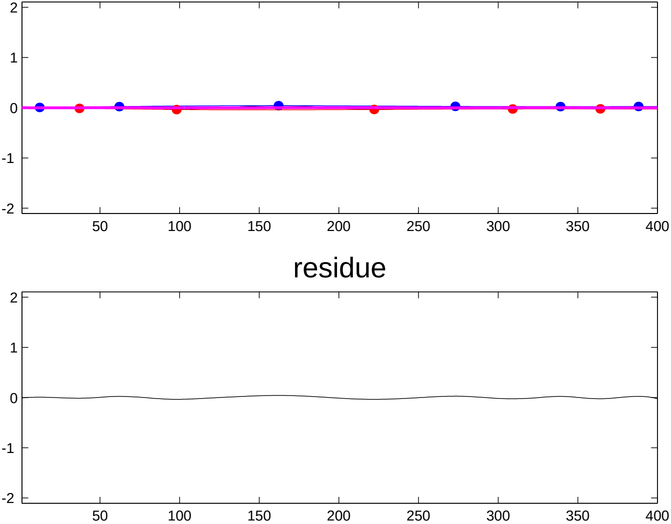
IMF 4; iteration 42

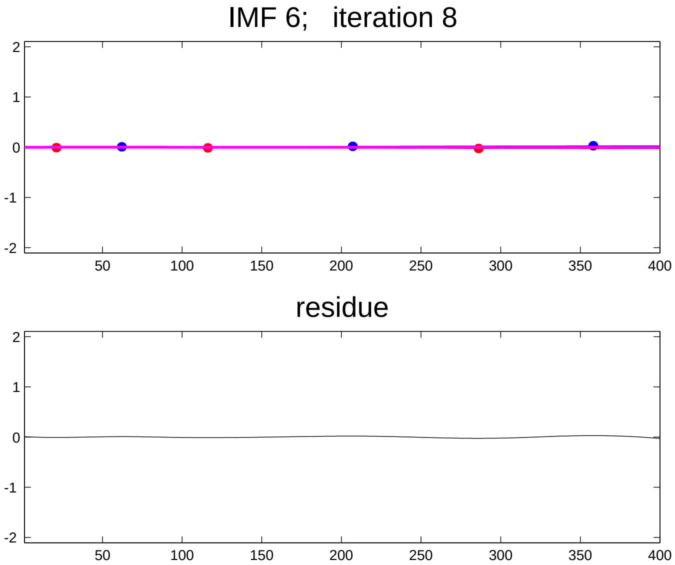


residue



IMF 5; iteration 13

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2
1
0
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100
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residue
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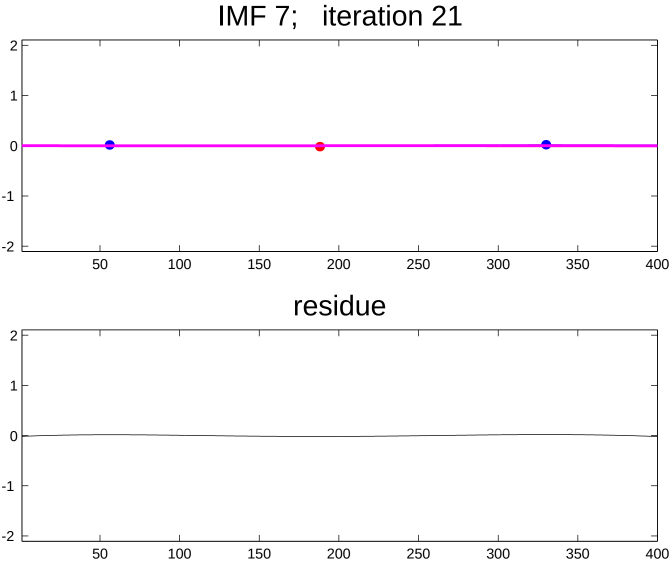
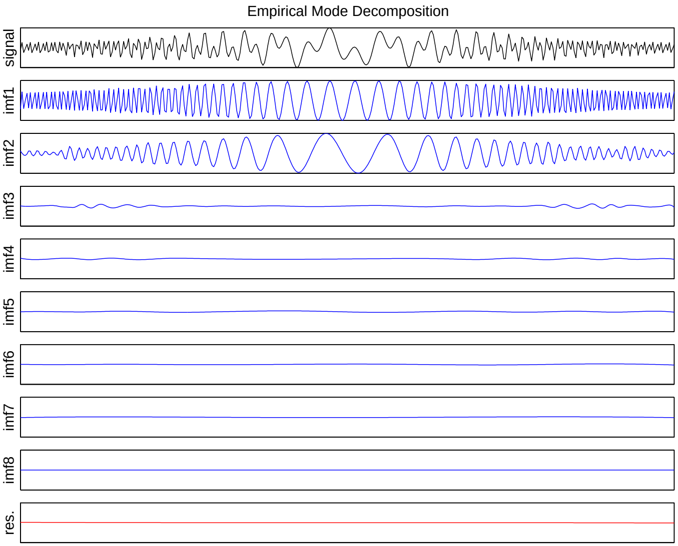
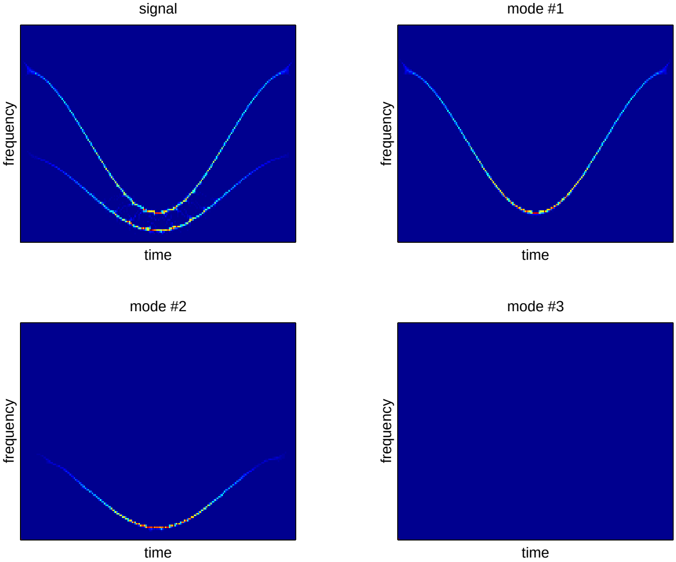

IMF 7; iteration 21
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

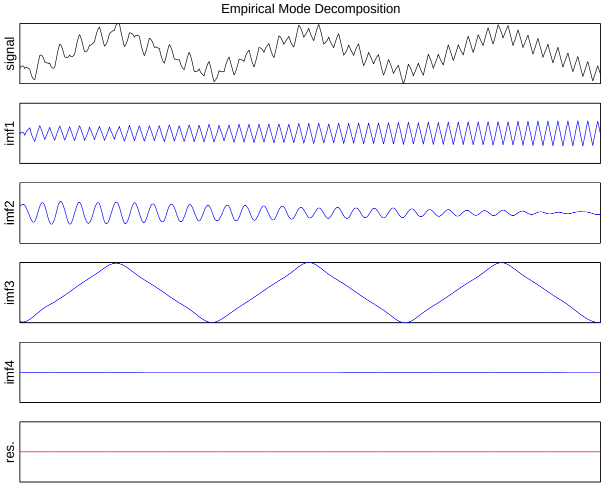

Empirical Mode Decomposition
signal
imf1
imf2
imf3
imf4
imf5
imf6
imf7
imf8
res.


signal
mode #1
mode #2
mode #3
frequency
time

some features

- **Locality** — The method operates at the scale of **one** oscillation.
- **Adaptativity** — The decomposition is fully **data-driven**.
- **Arbitrary oscillation** — No assumption on the harmonic structure of oscillations $\Rightarrow$ 1 **non linear** oscillation $=$ 1 mode.
- **Multiresolution** — The iterative process explores **sequentially** the "natural" constitutive scales of a signal.
- **Performance evaluation** — The decomposition is defined as the output of the algorithm (**no analytic definition**) $\Rightarrow$ need of **numerical simulations** in well-controlled situations.

non linear oscillations



multiresolution

- **Stochastic frequency approach** — Decomposition and spectrum analysis, mode by mode, of a wideband noise.
- **Model** — Fractional Gaussian noise (fGn), with spectrum density $\mathcal{S}(f) \sim |f|^{1-2H}$, with $0 < H < 1$ (Hurst exponent).

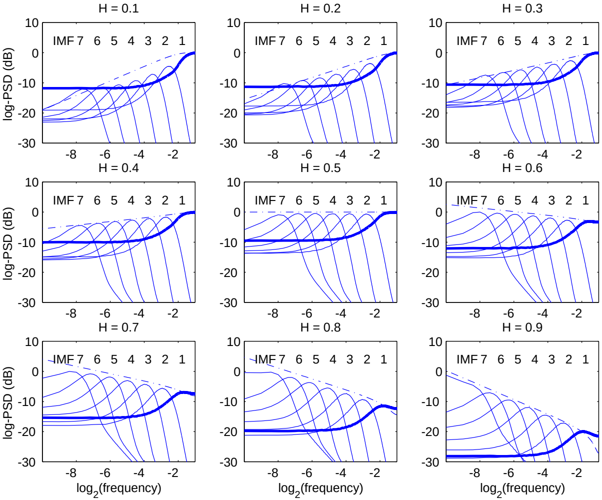
Result

"Spontaneous" emergence of a quasi-dyadic, self-similar, filterbank structure (F., Gonçalvès et Rilling, '03) :

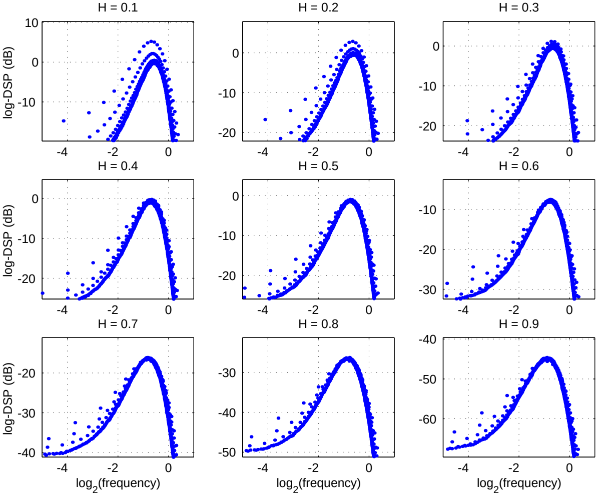
$$\mathcal{S}_{k',H}(f) = \rho_H^{\alpha(k'-k)} \mathcal{S}_{k,H}(\rho_H^{k'-k} f)$$

for any $k' > k \geq 2$*, with* $\alpha - 2H - 1$ *and* $\rho_H \approx 2$*.*

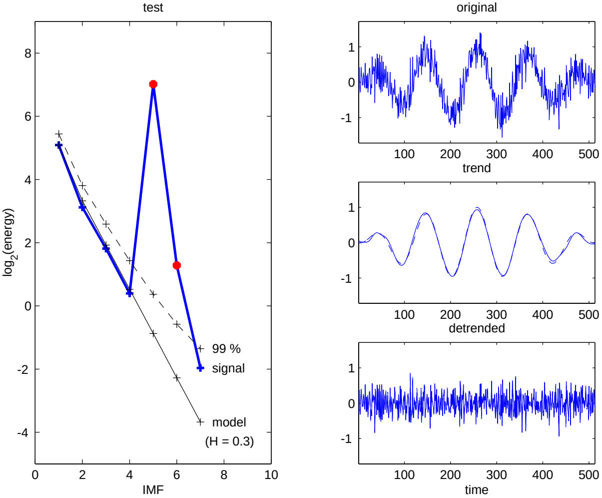
multiresolution



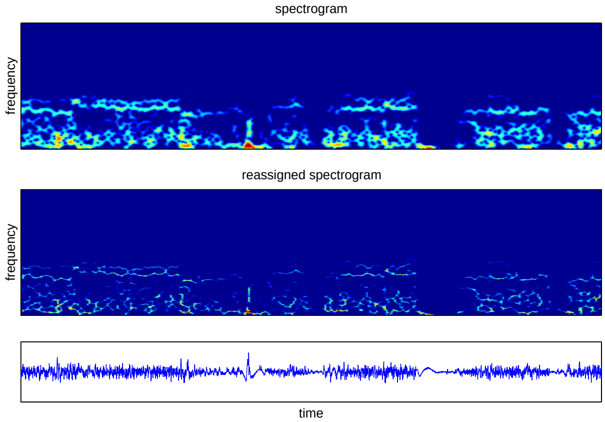
multiresolution



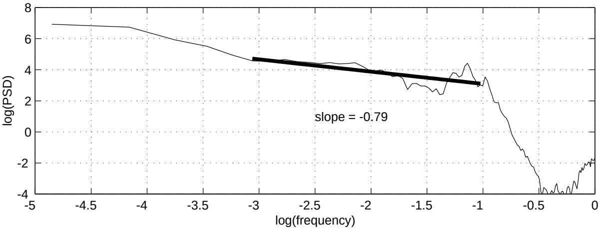
a toy example



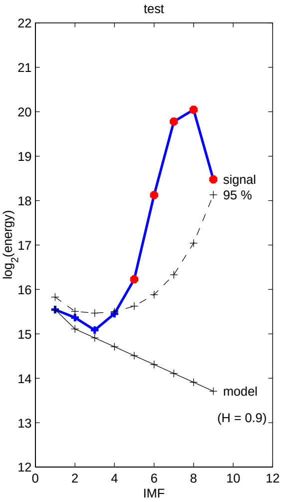
a real data example (heart rate variability)

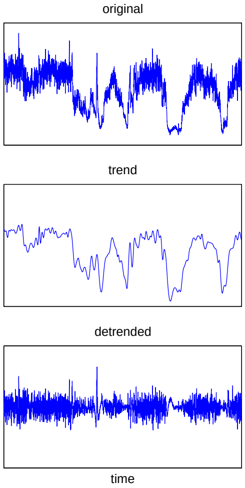


a real data example (heart rate variability)

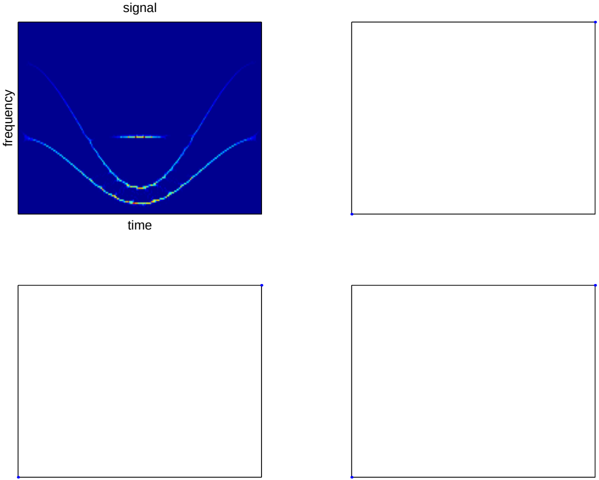


a real data example (heart rate variability)

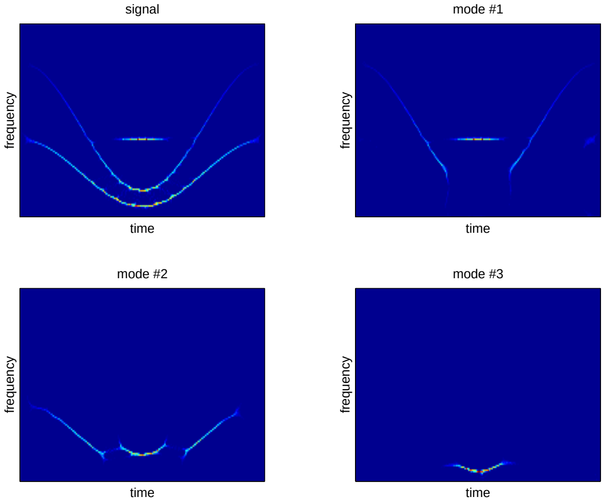




an example of limitation



an example of limitation



collective book

- N.E. Huang & S.S.P. Shen (*eds.*), *Hilbert-Huang Transform: Introduction and Applications*, World Scientific, 2005.

preprints & Matlab codes

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