Bisymmetric difference means

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February 1, 1999

Abstract — We consider the difference means $\{D_{a,b}(x,y)\ ; a,b \in \mathbb{R}, x,y \in \mathbb{R}_+\}$ introduced by Stolarsky and we characterize all pairs (a,b) guaranteeing that the property of bisymmetry : $D_{a,b}\left(D_{a,b}(x_1,x_2), D_{a,b}(x_3,x_4)\right) = D_{a,b}\left(D_{a,b}(x_1,x_3), D_{a,b}(x_2,x_4)\right)$ is satisfied. We show that the power means with exponent $k \in \mathbb{R}$ are the only bisymmetric difference means.

Difference means have been introduced by Stolarsky [7] as a way of generalizing the logarithmic mean. For any pair of distinct positive numbers x and y, the basic form of their difference mean is given by

$$D_{a,b}(x,y) = \left(\frac{b}{a}\,\frac{x^a - y^a}{x^b - y^b}\right)^{\frac{1}{a-b}} \tag{1}$$

if $ab(a-b)(x-y) \neq 0$, with a continuous extension to the whole domain defined by $\{(a,b,x,y)\ |a,b \in \mathbb{R}\ ; x,y \in \mathbb{R}_+\}$ (see [6] for detailed expressions). Apart from the logarithmic mean $D_{0,1}$, numerous standard means happen to be special cases of difference means : for instance, $D_{2,1}$, $D_{0,0}$ and $D_{-2,-1}$ correspond to the arithmetic, geometric and harmonic means, respectively, whereas power means with exponent $k \neq 0$ can be obtained as $D_{2k,k}$. Difference means offer therefore a nice and versatile framework for most of the useful means, as well as a unified setting for evaluating their properties.

By construction, difference means are *symmetric* in the sense that, for any pair of numbers $x, y \in \mathbb{R}_+$, we have $D_{a,b}(x,y) = D_{a,b}(y,x)$. However, it is not guaranteed that, for any $x_1, x_2, x_3, x_4 \in \mathbb{R}_+$, they all satisfy the further property of *bisymmetry* :

$$D_{a,b}\left(D_{a,b}(x_1, x_2), D_{a,b}(x_3, x_4)\right) = D_{a,b}\left(D_{a,b}(x_1, x_3), D_{a,b}(x_2, x_4)\right). \tag{2}$$

From a point of view of interpretation, such a property may nevertheless be desirable since it corresponds to an independence of the mean with respect to ordering. The purpose of this note is to make explicit the conditions on a and b which guarantee the bisymmetry of a difference mean. These conditions are given by the following Theorem and Corollary (which generalize a result given in [4] for the specific case $a = 1$) :

Theorem *Difference means $D_{a,b}(x, y)$ are bisymmetric if and only if $a + b = 0$, $a = 2b$ or $b = 2a$.*

Corollary *Power means with exponent $k \in \mathbb{R}$ are the only bisymmetric difference means.*

Proof of the Theorem. A result due to Aczél [1] states that, for a single-valued function of two variables $M(x, y)$, the conditions

1. Strict monotony : $x_1 < x_2 \Rightarrow M(x_1, y) < M(x_2, y)$ and $y_1 < y_2 \Rightarrow M(x, y_1) < M(x, y_2)$;
2. Continuity ;
3. Reflexivity : $M(x, x) = x$;
4. Symmetry : $M(x, y) = M(y, x)$;
5. Bissimmetry : $M(M(x_1, x_2), M(x_3, x_4)) = M(M(x_1, x_3), M(x_2, x_4))$

are necessary and sufficient for the existence of an increasing and continuous function $\varphi(x)$ such that

$$M(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \tag{3}$$

i.e., for identifying M with a Kolmogorov-Nagumo "quasi-arithmetic" mean [5].

If we consider the difference mean $D_{a,b}(x,y)$, we know [7] that conditions 1 to 4 are satisfied, with the consequence that requiring the extra condition of bisymmetry guarantees for (1) a quasi-arithmetic representation of the form (3). We can therefore apply to it the characterization due to de Finetti [3], according to which a function $M(x,y)$ is a quasi-arithmetic mean (i.e., admits a representation of the form (3)) if and only if

$$\frac{\partial M(x,y)/\partial x}{\partial M(x,y)/\partial y} = \frac{g(x)}{g(y)}, \tag{4}$$

with

$$g(x) = d\varphi(x)/dx. \tag{5}$$

In the case of the difference mean (1), we have

$$\frac{\partial D_{a,b}(x,y)/\partial x}{\partial D_{a,b}(x,y)/\partial y} = \frac{h(x,y)}{h(y,x)},$$

with

$$h(x,y) = (a-b)x^{a+b-1} - ax^{a-1}y^b + bx^{b-1}y^a. \tag{6}$$

Setting $x = \alpha y$, we get

$$\frac{\partial D_{a,b}(x,y)/\partial x}{\partial D_{a,b}(x,y)/\partial y} = \psi(\alpha),$$

with

$$\psi(\alpha) = \frac{(a-b)\alpha^{a+b-1} - a\alpha^{a-1} + b\alpha^{b-1}}{(a-b) - a\alpha^b + b\alpha^a}. \tag{7}$$

This implies that the ratio involved in (4) must satisfy

$$g(\alpha y) = \psi(\alpha)g(y). \tag{8}$$

We are thus led to

$$\begin{aligned} g(\alpha\beta y) &= \psi(\alpha)g(\beta y) \\ &= \psi(\alpha)\psi(\beta)g(y) \end{aligned}$$

on one hand, and

$$g(\alpha\beta y) = \psi(\alpha\beta)g(y)$$

on the other hand, so that the function $\psi(x)$ must verify the equation

$$\psi(\alpha\beta) = \psi(\alpha)\psi(\beta),$$

the (continuous) solution of which is of the form [2]

$$\psi(\alpha) = \alpha^c,\ c \in \mathbb{R}. \tag{9}$$

Plugging this result in (7), the problem reduces to finding the triples (a, b, c) which are solution of the equation

$$(a-b)\alpha^{a+b-1} - a\alpha^{a-1} + b\alpha^{b-1} - (a-b)\alpha^c + a\alpha^{b+c} - b\alpha^{a+c} = 0. \tag{10}$$

Since this equation is trivially satisfied for $\alpha = 0$, and since it must be verified for any $\alpha \neq 0$, we can factor out any of the powers of α, e.g., α^{a+c} in the above equation (10), so that it simplifies to

$$(a-b)\alpha^{b-c-1} - a\alpha^{-(c+1)} + b\alpha^{b-c-a-1} - (a-b)\alpha^{-a} + a\alpha^{b-a} = b. \tag{11}$$

The left-hand side of (11) must therefore be independent of α, leading by differentiation to the new condition :

$$\begin{aligned}(a-b)(b-c-1)\alpha^{b-c-2} + a(c+1)\alpha^{-(c+2)} + b(b-c-a-1)\alpha^{b-c-a-2} &\quad + \\ a(a-b)\alpha^{-(a+1)} + a(b-a)\alpha^{b-a-1} &\quad = \quad 0.\end{aligned} \tag{12}$$

As compared to the initial equation (10), its simplified form (12) is of the very same nature, except that it involves only five terms instead of six. Iterating four times the procedure used for obtaining (12) from (10), we clearly see that we will finally end up with a one term equation, involving—as a pre-factor of some power of α—the product of the different powers which appear in (11). It is thus enough to start from (11) and to study, case by case, the different possibilities of making either $b-c-1$, $c+1$ or $b-c-a-1$ vanish (the cases $a = 0$, $b = 0$ or $b = a$ are also solutions of (10), but they have not to be considered here since, according to (6), they would all lead to $h(x, y) = 0$ and would therefore correspond to a function $\varphi(x)$ which would be constant and, hence, not invertible).

1. Case $c = b - 1$.

   In this case, the procedure outlined above leads to the equation

   $$ab\alpha^{a-2b} - ab\alpha^{-b} + a(a-b)\alpha^{b} = a(a-b),$$

   and therefore (since we have $a \neq 0$ and $b \neq 0$) to the only solution $b = a/2$.

2. Case $c = -1$.

   The equation to be considered in this situation reads

   $$(a-b)\alpha^{a+b} - a\alpha^{a} + b\alpha^{b} + a\alpha^{-2} - b\alpha^{a-1} = a - b,$$

   with the general solution $b = -a$ (since the specific solution attached to $a = 1$ naturally leads to $b = -1$, and happens to be nothing but a special case of $a + b = 0$).

3. Case $c = b - a - 1$.

   Proceeding as for Case 1, we find that we must have $b = 2a$.

Proof of the Corollary. We can remark that, thanks to the symmetry property $D_{a,b}(x,y) = D_{b,a}(x,y)$, the solutions of Cases 1 and 3 define in fact the same difference mean. In the case where $b = a/2$ or $b = 2a$, the functions $\varphi_{a,b}(x)$ which are such that we can write

$$D_{a,b}(x,y) = \varphi_{a,b}^{-1}\left(\frac{\varphi_{a,b}(x) + \varphi_{a,b}(y)}{2}\right), \tag{13}$$

can be evaluated according to (4), (5) and (6). The result is that (up to an affine transformation)

$$\varphi_{a,a/2}(x) = x^{a/2} \tag{14}$$

and

$$\varphi_{a,2a}(x) = x^{a}, \tag{15}$$

leading to difference means which both belong to the class of power means with exponent $k \neq 0$:

$$M_k(x,y) = \left(\frac{x^k + y^k}{2}\right)^{1/k}. \tag{16}$$

(We have indeed $D_{a,a/2} = M_{a/2}$ and $D_{a,2a} = M_a$, and we assumed that $a \neq 0$.)

If we now turn back to the case $b = -a$, the evaluation of $h(x, y)$ in (6) leads to consider $g(x) = x^{-1}$ in (5), with the consequence that (up to an affine transformation)

$$\varphi_{a,-a}(x) = \log x. \tag{17}$$

In this case, the associated difference mean reduces, for any $a \in \mathbb{R}$, to :

$$D_{a,-a}(x, y) = \sqrt{xy}, \tag{18}$$

i.e., to the geometric mean, which is itself the power mean of exponent 0. Power means with exponent $k \in \mathbb{R}$ are therefore the only bisymmetric difference means.

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