0.1 TIME-FREQUENCY REASSIGNMENT[1]

Time-frequency and time-scale representations [1] aim to extract relevant information from a signal by representing it over a two-dimensional plane. These tools have been extensively studied in the past twenty years, resulting today in many useful analysis methods. Among them, the spectrogram and the smoothed versions of the Wigner-Ville distribution [1] are probably the most widely used, but their applicability is limited by localization trade-offs, which may be troublesome in some applications. For the spectrogram, a shorter analysis window yields a better time resolution and henceforth a poorer frequency resolution, as a consequence of the Gabor-Heisenberg inequality [1]. For the smoothed versions of the Wigner-Ville distribution, a larger smoothing kernel yields reduced cross-terms, but also a poorer localization of the signal components [2].

These shortcomings must be overcome in order to obtain time-frequency "pictures" that can be both easily read by non-experts and easily included in a signal processing application. This is exactly what the *reassignment principle* has been devised for. Initially introduced in 1976 by Kodera, Gendrin and de Villedary [3], this idea first remained little known and rarely used. But recently, advances obtained during the eighties in time-frequency analysis have made its rebirth possible [4], which considerably extended its applicability, both conceptually and computationally.

0.1.1 Basic principle

For a sake of simplicity, we will first present the basics of reassignment in the case of the spectrogram, which was the only case considered by Kodera *et al* [3]. Its application to other representations will be discussed afterwards. The spectrogram, which is the squared modulus of the short–time Fourier transform

$$S_x^h(t,f) = \left|F_x^h(t,f)\right|^2, \tag{0.1.1}$$

$$F_x^h(t,f) = \int x(u)\,h^*(t-u)\,e^{-i2\pi fu}\,du \tag{0.1.2}$$

can also be expressed as a two-dimensional smoothing of the Wigner–Ville distribution [1]

$$S_x^h(t,f) = \iint W_x(u,\nu)\,W_h(t-u,f-\nu)\,du\,d\nu. \tag{0.1.3}$$

In these expressions, t and f are respectively the time and frequency running variables, $x(t)$ is the analysed signal, and $h(t)$ is the analysing window. All integrals have integration bounds running from $-\infty$ to $+\infty$. The latter expression shows explicitly that the value of the spectrogram at a given point (t,f) is a weighted sum of all the Wigner–Ville distribution values at the neighboring points $(t-u,f-\nu)$. The number $S_x^h(t,f)$ is therefore the sum

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of a whole energy distribution located around its geometrical center (t, f). Reasoning with a mechanical analogy, the situation is as if the total mass of an object was assigned to its geometrical center, an arbitrary point which except in the very specific case of an homogeneous distribution, has no reason to suit the actual distribution. A much more meaningful choice is to assign the total mass of an object — as well as the spectrogram value $S_x^h(t,f)$ — to the *center of gravity* of their respective distribution. This is exactly what the reassignment performs : at each time–frequency point (t, f) where a spectrogram value is computed, we also compute the coordinates $(\widehat{t}, \widehat{f})$ of the local centroid of the Wigner–Ville distribution W_x, as seen through the time–frequency window W_h centered at (t, f):

$$\widehat{t}_x(t,f) \;=\; \frac{1}{S_x^h(t,f)} \iint u\, W_x(u,\nu)\, W_h(t-u, f-\nu)\, du\, d\nu \tag{0.1.4}$$

$$\widehat{f}_x(t,f) \;=\; \frac{1}{S_x^h(t,f)} \iint \nu\, W_x(u,\nu)\, W_h(t-u, f-\nu)\, du\, d\nu. \tag{0.1.5}$$

Then, the spectrogram value $S_x^h(t,f)$ is moved from (t, f) to $(\widehat{t}, \widehat{f})$. This leads us to define the reassigned spectrogram as

$$\check{S}_x^h(t,f) = \iint S_x^h(u,\nu)\, \delta\left(t - \widehat{t}_x(u,\nu)\right)\, \delta\left(f - \widehat{f}_x(u,\nu)\right)\, du\, d\nu. \tag{0.1.6}$$

Originally, the reassignment operators $\widehat{t}$ and $\widehat{f}$ have been equivalently related to the phase of the STFT, an information which is generally discarded when computing the spectrogram:

$$\widehat{t}_x(t,f) \;=\; -\frac{1}{2\pi}\frac{\partial \varphi}{\partial f}(t,f) \tag{0.1.7}$$

$$\widehat{f}_x(t,f) \;=\; f + \frac{1}{2\pi}\frac{\partial \varphi}{\partial t}(t,f). \tag{0.1.8}$$

with $\varphi(t,f) = \arg F_x^h(t,f)$. These expressions may be interpreted respectively as the local group delay and the local instantaneous frequency of the signal observed inside the time-frequency domain imposed by the analysis window h. But it has been shown in [4] that a much more efficient implementation is possible thanks to a third expression involving two additional STFTs with particular analysis windows :

$$\widehat{t}_x(t,f) \;=\; t - \Re\left\{\frac{F_x^{th}(t,f)}{F_x^h(t,f)}\right\}, \tag{0.1.9}$$

$$\widehat{f}_x(t,f) \;=\; f + \Im\left\{\frac{F_x^{dh/dt}(t,f)}{2\pi F_x^h(t,f)}\right\} \tag{0.1.10}$$

As presented here, the reassignment principle can be used with a large number of distributions, beyond the spectrogram case. For example, if the WVD of the short–time window $h(t)$ in eq. (0.1.3) is replaced by an arbitrary (low–pass) kernel $\Pi(u,\nu)$, one recognizes the

general form of the quadratic time–frequency energy distributions that are covariant under time and frequency shifts, referred to as the Cohen's class [1] :

$$\rho_x^{\Pi}(t,f) = \iint W_x(u,\nu)\,\Pi(t-u,f-\nu)\,du\,d\nu. \tag{0.1.11}$$

The local centroids are then given by

$$\widehat{t}_x(t,f) \;=\; \frac{1}{\rho_x^{\Pi}(t,f)} \iint u\,W_x(u,\nu)\,\Pi(t-u,f-\nu)\,du\,d\nu \tag{0.1.12}$$

$$\widehat{f}_x(t,f) \;=\; \frac{1}{\rho_x^{\Pi}(t,f)} \iint \nu\,W_x(u,\nu)\,\Pi(t-u,f-\nu)\,du\,d\nu. \tag{0.1.13}$$

and the corresponding reassigned distribution becomes

$$\check{\rho}_x^{\Pi}(t,f) = \iint \rho_x^{\Pi}(u,\nu)\,\delta\left(t-\widehat{t}_x(u,\nu)\right)\,\delta\left(f-\widehat{f}_x(u,\nu)\right)\,du\,d\nu. \tag{0.1.14}$$

From a theoretical point of view, this reassigned representation is no longer bilinear, but it still remains an energy distribution covariant under time and frequency shifts. One of the most important properties of the reassignment principle is that the application of the reassignment process defined by eqs (0.1.12), (0.1.13) and (0.1.14) to any distribution of the Cohen's class yields perfectly localized distributions for chirp signals, frequency tones and impulses, since the WVD does so, and since the centroid of a linear distribution necessary lies on the line. When applied to multicomponent signals, reassignment improves readability by overcoming — to a certain extent — the usual trade-off between cross-term level and localization: the underlying *smoothing* of the standard distribution guarantees some cross-term reduction, whereas reassignment acts as a *squeezing* that re-focuses the signal terms that had been spread out by smoothing (see Figure 1).

Among the examples of Cohen's class members studied in [4], the case of the smoothed pseudo Wigner–Ville distribution yields a very versatile signal analysis tool, with independently adjustable time and frequency smoothings:

$$SPWV_x^{g,h}(t,f) = \iint g(t-u)H(f-\nu)\,W_x(u,\nu)\,du\,d\nu \tag{0.1.15}$$

Its reassigned version can be computed easily with two additional SPWDs:

$$\widehat{t}_x(t,f) \;=\; t - \frac{SPWV_x^{tg,h}(t,f)}{SPWV_x^{g,h}(t,f)}, \tag{0.1.16}$$

$$\widehat{f}_x(t,f) \;=\; f + i\,\frac{SPWV_x^{g,dh/dt}(t,f)}{2\pi\,SPWV_x^{g,h}(t,f)} \tag{0.1.17}$$

A different kind of generalization can be obtained when switching to time-scale energy distributions of the affine class [1], i.e., the quadratic distributions covariant under time shifts and dilations:

$$\Omega_x^{\Pi}(t,a) = \iint W_x(u,\nu)\,\Pi\left(\frac{t-u}{a},a\nu\right)\,du\,d\nu \tag{0.1.18}$$

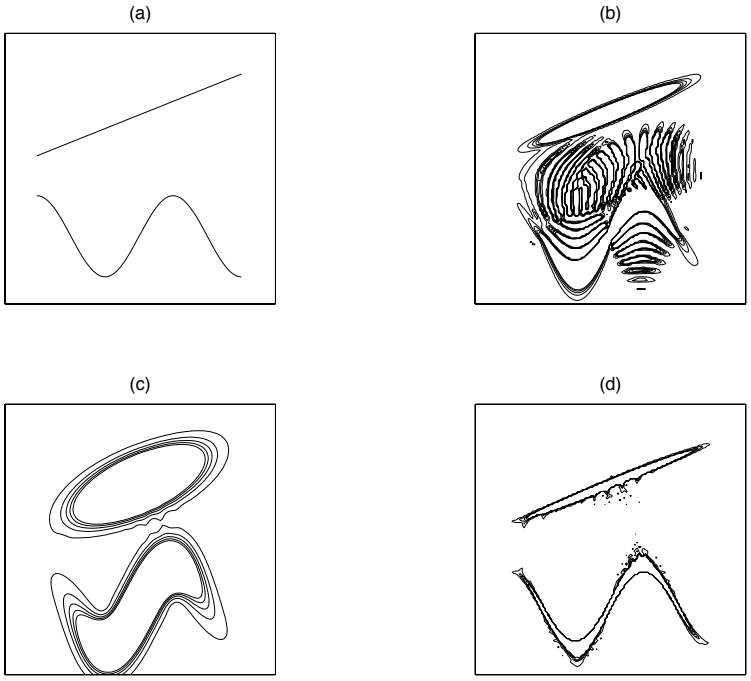

Figure 1: (a) Instantaneous frequency laws of a two-component signal; (b) Wigner-Ville distribution; (c) spectrogram; (d) reassigned spectrogram.

Within this framework, the reassignment operator in time is given directly by

$$\widehat{t}_x(t,a) = \frac{1}{\Omega_x^{\Pi}(t,a)} \iint u\, W_x(u,\nu)\, \Pi\left(\frac{t-u}{a}, a\nu\right) du\, d\nu, \tag{0.1.19}$$

whereas the reassignment operator in scale requires an intermediate step in the frequency domain

$$\begin{aligned} \widehat{a}_x(t,a) &= \frac{f_0}{\widehat{f}_x(t,a)}, \quad \text{with} \quad f_0 = \iint f\, \Pi(t,f)\, dt\, df \\ \text{and} \quad \widehat{f}_x(t,a) &= \frac{1}{\Omega_x^{\Pi}(t,a)} \iint \nu\, W_x(u,\nu)\, \Pi\left(\frac{t-u}{a}, a\nu\right) du\, d\nu \end{aligned} \tag{0.1.20}$$

The most important case among this class is the scalogram (the squared modulus of the wavelet transform) [1], obtained by choosing for Π the WVD of the chosen wavelet. Simple and efficient expressions of the reassignment operators also exist in this case [4] [5].

0.1.2 Variations and related approaches

0.1.2.1 Two variations

The original purpose of the reassignment principle was the design of time-frequency distributions with increased readability. But some useful information on the signal structure can also be directly extracted from the reassignment operators, as shown by the following two extensions:

Signal/noise discrimination and supervised reassignment. When the analysed signal includes broadband noise, the reassignment process yields peaked areas in noise-only regions, whereas rather smooth energy distributions are expected there. For such situations, an improved reassignment algorithm referred to as *supervised reassignment* [6] has been designed. This approach first attempts to discriminate between "signal+noise" and "noise only" regions in the time-frequency plane by means of a detector applied to the reassignment operators. Reassignment is then only performed for the points considered to belong to "signal+noise" regions.

Signal component extraction and differential reassignment. Many signal processing problems such as denoising and signal classification can be solved by a relevant tiling of the time-frequency plane, so as to isolate each signal "component" (although this concept is not clearly defined). For such applications, a new reassignment process called *differential reassignment* [7] has been considered. Whereas the original reassignment principle moves each value by one finite jump, differential reassignment considers each time-frequency point as the starting point of an elementary particle whose velocity field is deduced from the reassignment operators. The final points called *asymptotic reassignment points* are gathered and lead to a time-frequency map in which each cell indicates a signal component.

0.1.2.2 Related approaches

Although original in many respects, the concept of reassignment is obviously connected with several approaches that have been proposed independently. We lack space to discuss these interactions precisely, but we cite:

- The instantaneous frequency density [8], which yields at each time sample an histogram of the frequency reassignment operator of the spectrogram.
- The extraction of ridges and skeletons out of the phase structure of the wavelet transform [9] [10]. These ridges are made of the fixed points of the reassignment operators, either horizontally ($\widehat{a}_x(t,a) = a$) or vertically ($\widehat{t}_x(t,a) = t$).
- The synchrosqueezed plane [11], which also moves the scalogram values, but by a scale displacement only.

0.1.3 Summary and Conclusions

Reassignment can be viewed as the second step of a process whose goal is to build a readable time–frequency representation. It consists of:

1. a *smoothing*, whose main purpose is to rub out oscillatory interferences, but whose drawback is to smear localized components;

2. a *squeezing*, whose effect is to refocus the contributions which survived the smoothing.

As a result, this approach yields — without a drastic increase in computational complexity — enhanced contrast (when compared to smoothed distributions such as spectrograms) with a much reduced level of interferences (when compared to the Wigner-Ville distribution). This is especially true when the signal–noise ratio is not too low, and when the signal components are not "too close" to each other. Finally, MATLAB implementations of the algorithms discussed here are included in a freeware available at
`http://iut-saint-nazaire.univ-nantes.fr/~auger/tftb.html`

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