An alternative model for Figures 2 and 3 in “Multitaper Time-Frequency Reassignment for Nonstationary Spectrum Estimation and Chirp Enhancement,”

by J. Xiao & P. Flandrin

The “box” model used in Figures 2 and 3 is unattainable because of uncertainty relations. As a substitute, and as suggested by one of the reviewers, we can instead consider the WVS of a suitably modulated and filtered white Gaussian noise (wGn).

Let us then define the process $x(t)$ to be analyzed as

$$x(t) = m(t) \int_{-\infty}^{+\infty} h(t-s)\, e(s)\, ds,$$

where $m(t)$ is some amplitude modulation, $h(t)$ the impulse response of some linear filter and $e(t)$ wGn such that $\mathbb{E}\{e(t)\} = 0$ and $\mathbb{E}\{e(t)e(s)\} = \sigma^2\, \delta(t-s)$.

Writing $x(t) = m(t)\, y(t)$ with $y(t) := (h * e)(t)$ and using basic covariance properties of the WVD, we have

$$W_x(t,f) = \int_{-\infty}^{+\infty} W_m(t, f-f')\, W_y(t,f')\, df'$$

and

$$W_y(t,f) = \int_{-\infty}^{+\infty} W_h(t-t', f)\, W_e(t',f)\, dt'.$$

Since $e(t)$ is wGn, we have $\mathbf{W}_e(t,f) := \mathbb{E}\{W_e(t,f)\} = \sigma^2$. This leads to $\mathbf{W}_y(t,f) = \sigma^2\, |H(f)|^2$ (with $H(f)$ the Fourier transform of $h(t)$) and it readily follows that the WVS of the output process $x(t)$ reads

$$\mathbf{W}_x(t,f) = \sigma^2 \int_{-\infty}^{+\infty} W_m(t, f-f')\, |H(f')|^2\, df'.$$

If we now specify $m(t)$ and $H(f)$ by choosing a Gaussian model for both according to $m(t) = \exp\{-\alpha\, t^2\}$ and $H(f) = \exp\{-\beta(f-f_0)^2\}$ (where f_0 stands for some central frequency), we get

$$\mathbf{W}_x(t,f) = \sqrt{\frac{2\pi}{\alpha}}\, \sigma^2\, e^{-2\alpha t^2} \int_{-\infty}^{+\infty} e^{-(2\pi^2/\alpha)(f-f')^2}\, e^{-2\beta(f'-f_0)^2}\, df',$$

and evaluating the above integral leads to the final result:

$$\mathbf{W}_x(t,f) = \frac{\sigma^2}{\sqrt{1+\alpha\beta/\pi^2}}\, e^{-2\alpha t^2}\, e^{-\frac{2\beta}{1+\alpha\beta/\pi^2}(f-f_0)^2}.$$

It has to be remarked that this (valid) model corresponds to a non-negative WVS and that its equivalent time-frequency support (ellipse area) is controlled by the bandwidth-duration product $1/\alpha\beta$.

Using this model, we can get as a substitute to Figures 2 and 3 of the paper the following Figures 1 and 2, respectively. The corresponding Matlab codes are `MTFR_tvnoiseG.m`, `tvnoiseG.m` and `wGnmodelG.m`.

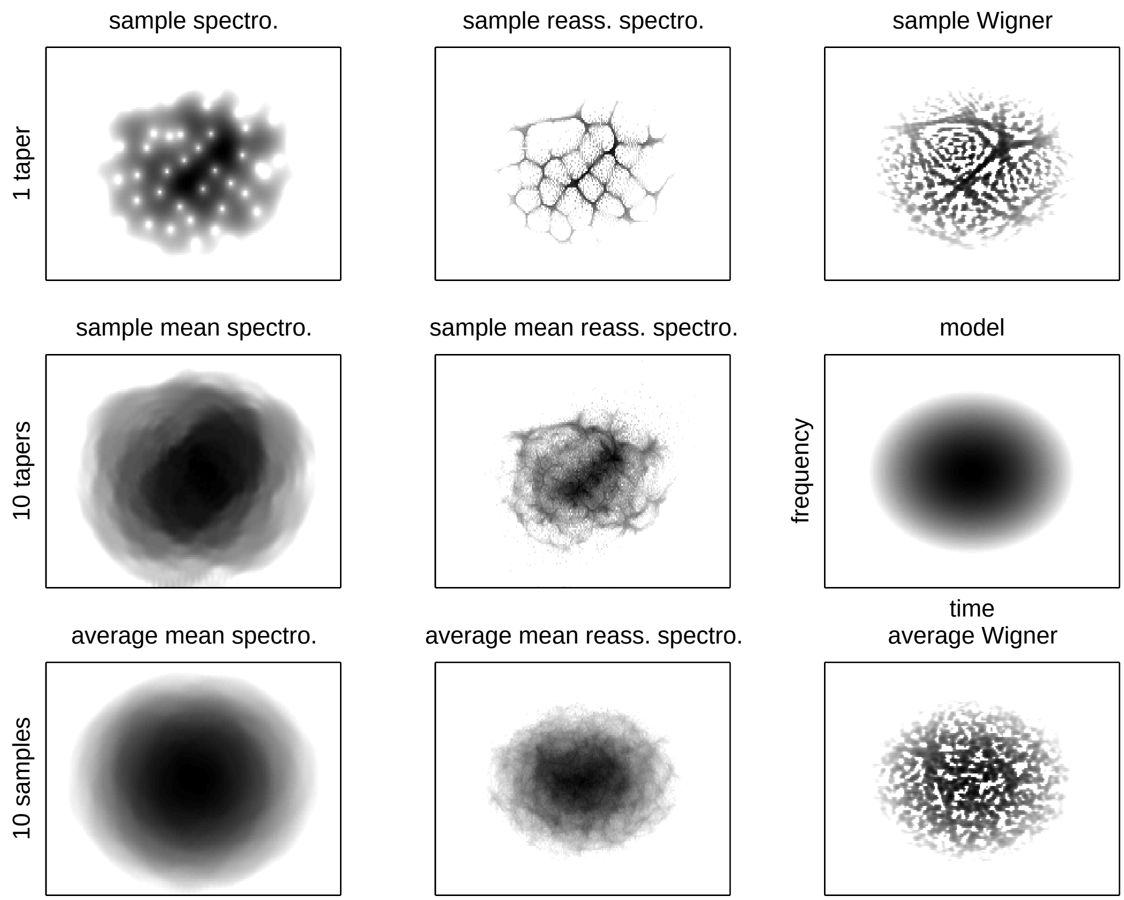

Figure 1: Comparison of noise WVS estimates. Each diagram represents a WVS estimate in the case of a Gaussian filtered wGn modulated in time by a Gaussian window (the corresponding WVS model is given in the middle row of the right column). The first row consists of a spectrogram, its reassigned version and the WVD, based on one realization. The corresponding multitaper estimates (10 Hermite functions) are given in the middle row, whereas the bottom row displays ensemble averages of such estimates (10 independent realizations), together with the empirical WVS estimate on the same data set. In each diagram, time is horizontal, frequency vertical and the energy is coded logarithmically with a dynamic range of 30 dB.

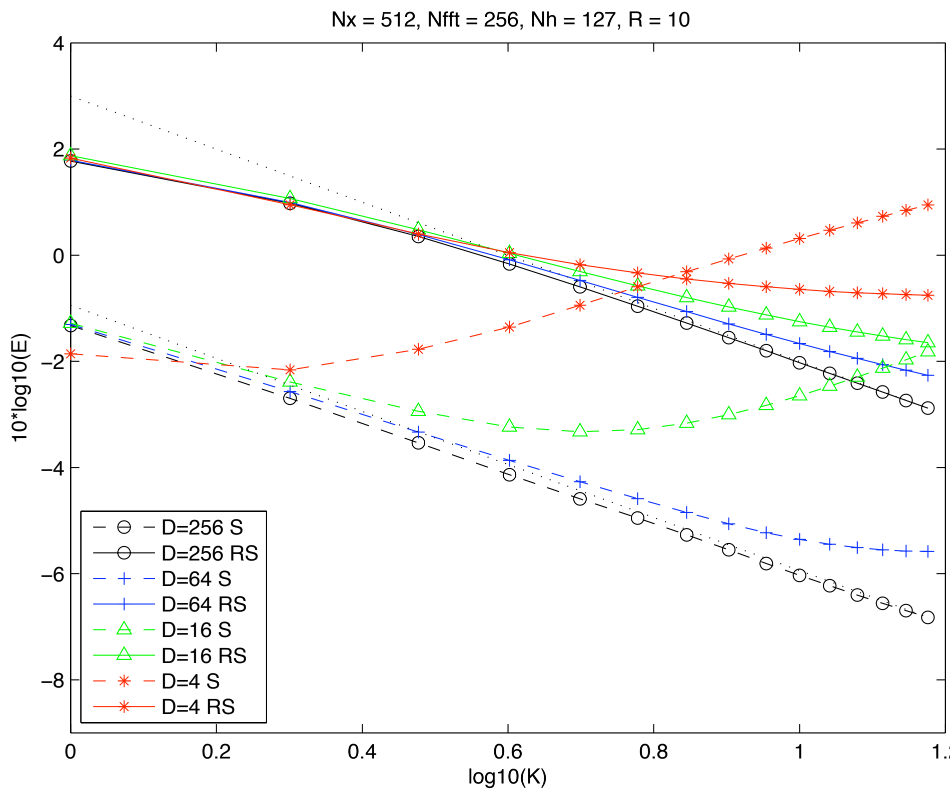

Figure 2: Error measures in WVS multitaper estimates. The figure plots, as a function of the number of tapers K, the error measure attached to multitaper (reassigned) spectrograms when the model is the WVS of a Gaussian filtered wGn modulated in time by a Gaussian window, extending over an equivalent elliptic domain of area D. The simulations have been conducted (with up to $K = 15$ Hermite tapers, each of length 127) on the basis of 10 independent realizations of 512 data points each, with 256 frequency bins over the whole frequency range $[0, 1/2)$. In the pure wGn situation ($D = 256$), asymptotic decays in $K^{-1/2}$ (see text) have been superimposed as dotted lines.