
TD 1: Topology issues in product spaces and Banach spaces

Exercise 1 (General topology).

1. Let $f : E \to F$ be a continuous map between topological spaces. Show that f is sequently continuous. Namely, show that if the sequence $(x_n)_n$ converges to x in E then the sequence $(f(x_n))_n$ converges to $f(x)$ in F. Can we claim that if f is sequently continuous then f is continuous ?
2. Let $f : E \to F$ be a map between topological spaces. The function f is said to be continuous at $x \in E$ if for all open set $\mathcal{V}$ containing $f(x)$, there exists an open set $\mathcal{U}$ containing x and such that $f(\mathcal{U}) \subset \mathcal{V}$. Check that, in this definition, "open set" can be replaced by "neighbourhood".
3. Let X be a set, $(F_i)_{i\in I}$ be a family of topological spaces and $f_i : X \to F_i$ be some functions.
   (a) Prove that the "coarsest topology that makes the functions f_i continuous" exists.
   (b) Let $g : E \to X$ be a function defined on a topological space E. Check that g is continuous if and only if for all $i \in I$, $f_i \circ g$ is continuous.
   (c) Let $(x_n)_n$ be a sequence in X. Prove that $(x_n)_n$ converges to x if and only if for all $i \in I$, $(f_i(x_n))_n$ converges to $f_i(x)$.
4. Let $(F_i)_{i\in I}$ be a family of topological spaces. We define the product topology on $\prod_{i\in I} F_i$ as the "coarsest topology" making the projections continuous. Show that this topology is generated by the cylinder sets, *i.e.* the sets of the form $C_J = \prod_{i\in I} U_i$, where each U_i is open in F_i and $U_i = F_i$, except for a finite number of indexes $i \in J$.

Exercise 2 (A theorem of Hörmander). Let $1 \leq p, q < \infty$ and

$$T : (L^p(\mathbb{R}^n), \|\cdot\|_p) \to (L^q(\mathbb{R}^n), \|\cdot\|_q),$$

be a continuous linear operator which commutes with the translations, that is, which satisfies $\tau_h T = T\tau_h$ for all $h \in \mathbb{R}^n$, where $\tau_h f = f(\cdot - h)$. The purpose of this exercice is to prove the following property: if $q < p < \infty$, then the operator T is trivial.

1. Let u be a function in $L^p(\mathbb{R}^n)$. Prove that $\|u + \tau_h u\|_p \to 2^{1/p}\|u\|_p$ as $\|h\| \to \infty$.
   Hint: you may decompose u as the sum of a compactly supported function and of a function with arbitrarily small L^p norm.
2. Check that if C stands for the norm of operator T, then we have that for all $u \in L^p(\mathbb{R}^n)$,
   $$\|Tu\|_q \leq 2^{1/p-1/q} C\|u\|_p,$$
   and conclude.
3. Can you give the example of a non-trivial such operator T when $p \leq q$?

Exercise 3 (Fourier coefficients of L^1 functions). For any function f in $L^1(\mathbb{T})$, we define the function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t, \quad n \in \mathbb{Z}.$$

We denote by c_0 the space of complex valued functions on $\mathbb{Z}$ tending to 0 at $\pm\infty$.

1. Check that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
2. Prove that, for all $f \in L^1(\mathbb{T})$, $\hat{f} \in c_0$.
   Hint: Recall that the trigonometric polynomials $\sum_{k=-n}^{n} a_k e^{ikt}$ *are dense in* $L^1(\mathbb{T})$.

Now we study the converse question: is every element of c_0 the sequence of Fourier coefficients of a function in $L^1(\mathbb{T})$?

3. Prove that $\Lambda : f \to \hat{f}$ defines a bounded linear map from $L^1(\mathbb{T})$ to c_0.
4. Prove that the function Λ is injective.
5. Show that the function Λ is not onto.
   Hint: You may use the Dirichlet kernel $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$, *whose* $L^1(\mathbb{T})$ *norm goes to* $+\infty$ *as* $n \to +\infty$.

Exercise 4 (Equivalence of norms).

1. Let E be a vector space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that both $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are Banach spaces. Assume the existence of a finite constant $C > 0$ such that
$$\forall x \in E, \quad \|x\|_1 \leqslant C\|x\|_2.$$
   Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. Let K be a compact subset of $\mathbb{R}^n$. We consider a norm N on the space $\mathcal{C}^0(K, \mathbb{R})$ such that $(\mathcal{C}^0(K, \mathbb{R}), N)$ is a Banach space, and satisfying that any sequence of functions $(f_n)_n$ in $\mathcal{C}^0(K, \mathbb{R})$ that converges for the norm N also converges pointwise to the same limit. Prove that the norm N is then equivalent to the norm $\|\cdot\|_\infty$.

Exercise 5 (A Rellich-like theorem). Let us consider E the following subspace of $L^2(\mathbb{R})$
$$E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E < +\infty\big\}, \quad \text{where} \quad \|u\|_E = \|(\sqrt{1+x^2})u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})}.$$
The aim of this exercice is to prove that the unit ball B_E of E is relatively compact in $L^2(\mathbb{R})$, with
$$B_E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E \leq 1\big\}.$$
In the following, we denote by ϕ a non-negative $\mathcal{C}^\infty$ function such that $\phi^{-1}(\{0\}) = \mathbb{R} \setminus [-2, 2]$ and $\phi^{-1}(\{1\}) = [-1, 1]$.

1. Considering the cut-off $\phi_R(x) = \phi(x/R)$, show that $\sup_{u \in B_E} \|(1 - \phi_R)u\|_{L^2(\mathbb{R})}$ converges to 0 as $R \to +\infty$.
2. We define $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ and τ_h the translation operator (see Exercice 2). Show that for all $R \geq 1$ and $\varepsilon > 0$, there exists $C_{\varepsilon,R} > 0$ such that for all $h \in \mathbb{R}$ and $u \in E$,
$$\|\tau_h((\phi_R u) * \psi_\varepsilon) - (\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C_{\varepsilon,R}|h|\|u\|_E \quad \text{and} \quad \|(\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C_{\varepsilon,R}\|u\|_E.$$
3. Show that for any sequence $(u_n)_n$ in B_E, there exists a subsequence $(u_{n'})_{n'}$ such that for any $R, \varepsilon^{-1} \in \mathbb{N}^*$, the sequence $((\phi_R u_{n'}) * \psi_\varepsilon)_{n'}$ converges in $L^2(\mathbb{R})$ as $n' \to \infty$.
   Hint: Use Cantor's diagonal argument.
4. Conclude.
5. Let us now consider the set $B_{H^1} \subset L^2(\mathbb{R})$ defined by
$$B_{H^1} = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})} \leq 1\big\}.$$
   Is B_{H^1} relatively compact in $L^2(\mathbb{R})$?