École Normale Supérieure de Lyon **Année 2023 - 2024**

Département de Mathématiques **M1 - Analyse avancée**

TD 10: Fourier transform and tempered distributions

Exercise 1. Let $A \in S_n^{++}(\mathbb{R})$ be a definite positive real matrix. Prove that the function u defined on $\mathbb{R}^n$ by $u(x) = e^{-\langle Ax,x\rangle}$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and that its Fourier transform is given by

$$\forall \xi \in \mathbb{R}^n, \quad \widehat{u}(\xi) = \sqrt{\frac{\pi^n}{\det A}}\, e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle}.$$

Hint: Begin by considering the case $n = 1$, and diagonalize the matrix A to treat the general case.

Exercise 2.

1. Let $A \subset \mathbb{R}^n$ be a measurable subset with finite measure. Prove that $\widehat{\mathbb{1}_A}$ belongs to $L^2(\mathbb{R}^n)$ but not to $L^1(\mathbb{R}^n)$.
2. Are there two functions $f, g \in \mathscr{S}(\mathbb{R}^n)$ not being identically equal to zero and satisfying the relation $f * g = 0$? Same question for some functions f et g with compact supports.
3. Prove that the equation $f * f = f$ has no non trivial solution in $L^1(\mathbb{R}^n)$, but has an infinite number of solutions in $L^2(\mathbb{R}^n)$.

Exercise 3. By computing the Fourier transform of the functions $f = \mathbb{1}_{[-1/2,1/2]}$ and $f * f$, show that

$$\int_{\mathbb{R}} \left(\frac{\sin t}{t}\right)^2 \mathrm{d}t = \pi.$$

Exercise 4 (Heisenberg's uncertainty principle). Prove that for all $f \in \mathscr{S}(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$,

$$\inf_{a\in\mathbb{R}} \big\|(x_j - a)f\big\|^2_{L^2(\mathbb{R}^n)} \inf_{b\in\mathbb{R}} \big\|(\xi_j - b)\widehat{f}\big\|^2_{L^2(\mathbb{R}^n)} \geq \frac{(2\pi)^n}{4}\|f\|^4_{L^2(\mathbb{R}^n)},$$

When is this inequality an equality ?

Exercise 5. Let us consider the interval $I = [-1, 1]$ and the following subspace of $L^2(I)$

$$\mathrm{BL}^2(I) = \big\{u \in L^2(\mathbb{R}) : \widehat{u} = 0 \text{ almost everywhere on } \mathbb{R} \setminus I\big\}.$$

1. Prove that $\mathrm{BL}^2(I)$ is a Hilbert space.
2. Check that $\mathrm{BL}^2(I) \subset C^0_{\to 0}(\mathbb{R})$ and that the corresponding embedding is continuous.
3. Let us consider the continuous extension of $x \mapsto \sin x/x$, denoted sinc.
   (a) Prove that the family $(\pi^{-1/2}\tau_{2\pi k}\,\mathrm{sinc})_{k\in\mathbb{Z}}$ is a Hilbert basis of $\mathrm{BL}^2(I)$.
   (b) Prove (sampling theorem) that any element $u \in \mathrm{BL}^2(I)$ can be decomposed as follows

$$u(x) = \sum_{k\in\mathbb{Z}} u(2\pi k)\,\mathrm{sinc}(x - 2\pi k),$$

   the convergence being uniform in $\mathbb{R}$, and also holds in $L^2(\mathbb{R})$.

Exercise 6. Give the example of a function $f \in C^\infty(\mathbb{R})$ such that

(i) There is no polynomial P such that $|f| \leq |P|$.

(ii) The linear form
$$\varphi \in \mathscr{S}(\mathbb{R}) \mapsto \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x,$$
defines a tempered distribution.

Exercise 7. Prove that the following distributions are tempered and compute their Fourier transform:

1. δ_0
2. 1,
3. H (Heaviside),
4. $\mathrm{p.\,v.}(1/x)$,
5. $|x|$ in $\mathbb{R}$.

Indication : $\mathrm{p.\,v.}(1/x)$ *is an odd distribution, so its Fourier transform is also odd.*

Exercise 8. The aim of this exercice is to compute the Fourier transform of the following tempered distribution on $\mathbb{R}^2$
$$\langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}} \varphi(x,x)\,\mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}^2).$$

1. Let $\psi \in \mathscr{S}(\mathbb{R}^2)$. Prove that
$$\langle \widehat{T}, \psi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\varepsilon\to 0^+} I_\varepsilon \quad \text{où} \quad I_\varepsilon = \int_{\mathbb{R}} e^{-\varepsilon x^2}\widehat{\psi}(x,x)\,\mathrm{d}x.$$
2. By using the expression of $\widehat{\psi}(x,x)$, show that
$$I_\varepsilon = 2\sqrt{\pi}\int_{\mathbb{R}^2} e^{-\zeta^2}\psi(\xi, 2\sqrt{\varepsilon}\zeta - \xi)\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$
3. Deduce the expression of $\widehat{T}$.

Exercise 9. We consider the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, & (t,x) \in \mathbb{R}^* \times \mathbb{R}^n \\ u_{t=0} = u_0. \end{cases} \tag{1}$$

1. For $u_0 \in \mathcal{S}(\mathbb{R}^n)$, solve the equation (1) in $C^0(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^*, \mathscr{S}(\mathbb{R}^n))$.
2. Justify that the Fourier transform of the function $e^{it|\xi|^2}$ is well defined.
3. Show that for $\alpha \in \mathbb{C}$ with negative real part,
$$\mathscr{F}^{-1}(e^{\alpha|\xi|^2}) = \frac{1}{(-4\alpha\pi)^{d/2}} e^{\frac{|x|^2}{4\alpha}}.$$
4. Check that this formula also holds in $\mathcal{S}'(\mathbb{R}^n)$ when $\alpha \in i\mathbb{R}$.
5. Deduce that there exists a constant $C > 0$ such that for all $t > 0$,
$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{t^{d/2}}\|u_0\|_{L^1(\mathbb{R}^n)}.$$