TD 11: Sobolev spaces

Exercise 1.

1. Let $K : \mathbb{R}^{2n} \to \mathbb{C}$ be a continuous function. Assume that there exists $A > 0$ such that
$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)| \,\mathrm{d}y \leq A \quad \text{and} \quad \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)| \,\mathrm{d}x \leq A.$$
For all $u \in C_0^\infty(\mathbb{R}^n)$, we set
$$(Pu)(x) = \int_{\mathbb{R}^n} K(x,y)u(y) \,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$
Schur's lemma: prove that P can be extended as a bounded operator $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for any $p \in [1, +\infty]$, with an operator norm smaller than A.
2. *Application:* Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $N \geq 1$ be a positive integer. We define the Fourier multiplier $\varphi(N^{-1}D_x)$ by
$$\varphi(N^{-1}D_x)u := \mathscr{F}^{-1}(\varphi(N^{-1}\cdot)) * u, \quad u \in C_0^\infty(\mathbb{R}^n).$$
Prove that the operator $\varphi(N^{-1}D_x)$ can be extended as a bounded operator $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for any $p \in [1, +\infty]$, and that there exists a positive constant $A > 0$ not depending on the integer N such that for all $u \in L^p(\mathbb{R}^n)$,
$$\|\varphi(N^{-1}D_x)u\|_{L^p(\mathbb{R}^n)} \leq A\|u\|_{L^p(\mathbb{R}^n)}.$$

Exercise 2.

1. Show that $H^{s_1}(\mathbb{R}^n)$ embeds continuously into $H^{s_2}(\mathbb{R}^n)$ for $s_1 \geq s_2$.
2. Check that $\delta_0 \in H^s(\mathbb{R}^n)$ for $s < -n/2$.
3. When $s \in \mathbb{N}^*$ is a nonnegative integer, the Sobolev space is also given by
$$H^s(\mathbb{R}^n) = \big\{u \in L^2(\mathbb{R}^n) : \forall|\alpha| \leq s,\, \partial^\alpha u \in L^2(\mathbb{R}^n)\big\}.$$

Exercise 3.

1. Prove that if $s > n/2$, the space $H^s(\mathbb{R}^n)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^n)$, the space of continuous functions u on $\mathbb{R}^n$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
2. State an analogous result in the case where $s > n/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.
3. Let us now consider $s \in (n/2, n/2 + 1)$.
   (a) Show that for all $\alpha \in [0,1]$ and all $x, y, \xi \in \mathbb{R}^n$:
$$\big|e^{ix\cdot\xi} - e^{iy\cdot\xi}\big| \leq 2^{1-\alpha}|x-y|^\alpha|\xi|^\alpha.$$

(b) Deduce that for all $\alpha \in (0, s - n/2)$, there exists a constant $C(\alpha) > 0$ such that for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$,

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(\alpha)\|u\|_{H^s(\mathbb{R}^n)}.$$

(c) Conclude that $H^s(\mathbb{R}^n)$ embeds continuously to $C^\alpha(\mathbb{R}^n)$, the space of α-Hölder functions.

Exercise 4. Assuming that s belongs to $[0, n/2)$, the purpose of this exercice is to prove that $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, where $p = 2n/(n - 2s)$. To that end, let us recall that for all $u \in L^p(\mathbb{R}^n)$,

$$\|u\|^p_{L^p(\mathbb{R}^n)} = \int_0^\infty p\lambda^{p-1}\big|\{|u| > \lambda\}\big|\,\mathrm{d}\lambda.$$

Considering $u \in \mathscr{S}(\mathbb{R}^n)$ and $A_\lambda > 0$, we set $u_{1,\lambda} = \mathscr{F}^{-1}(\mathbb{1}_{|\xi|<A_\lambda}\widehat{u})$ and $u_{2,\lambda} = \mathscr{F}^{-1}(\mathbb{1}_{|\xi|\geq A_\lambda}\widehat{u})$.

1. Prove that

$$\forall x \in \mathbb{R}^n, \quad |u_{1,\lambda}(x)| \leq CA_\lambda^{(n-2s)/2}\|u\|_{H^s(\mathbb{R}^n)}.$$

   Deduce that there exists some A_λ such that $|\{|u_{1,\lambda}| > \lambda/2\}| = 0$.

2. Show that for this choice of A_λ,

$$\|u\|^p_{L^p(\mathbb{R}^n)} \leq 4p\int_0^\infty \lambda^{p-3}\|u_{2,\lambda}\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}\lambda.$$

3. Conclude.

Exercise 5. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathscr{S}(\mathbb{R}^3)$,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\,\|u\|^{1/2}_{H^1(\mathbb{R}^3)}\|u\|^{1/2}_{H^2(\mathbb{R}^3)}.$$

Hint: Considering $R > 0$, use the following decomposition

$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi|\leq R}\langle\xi\rangle|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle} + \int_{|\xi|>R}\langle\xi\rangle^2|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle^2}.$$

Exercise 6 (Trace on an hyperplane). Let us consider the function

$$\gamma_0 : \varphi(x', x_n) \in C_0^\infty(\mathbb{R}^n) \mapsto \varphi(x', x_n = 0) \in C_0^\infty(\mathbb{R}^{n-1}).$$

Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^n)$ to $H^{s-1/2}(\mathbb{R}^{n-1})$.
Hint: For all $\varphi \in C_0^\infty(\mathbb{R}^n)$, begin by computing the Fourier transform of the function $\gamma_0\phi$.

Exercise 7 (An estimate). Let $0 < \alpha < 1$ and $p > 1$ be positive real numbers. Show that there exists a positive constant $C_{\alpha,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\bigg(\iint_{\mathbb{R}^n\times\mathbb{R}^n}\bigg(\frac{|u(x) - u(y)|}{|x - y|^\alpha}\bigg)^p\frac{\mathrm{d}x\mathrm{d}y}{|x - y|^d}\bigg)^{1/p} \leq C_{\alpha,p}\|u\|^{1-\alpha}_{L^p(\mathbb{R}^n)}\|\nabla u\|^\alpha_{L^p(\mathbb{R}^n)}.$$

Hint: Consider the two regions $\{|x - y| > R\}$ and $\{|x - y| \leq R\}$, where $R > 0$ is to be chosen.