

TD 5: WEAK TOPOLOGIES (II)

EXERCISE 1. Let E and F be two Banach spaces, and $T : E \rightarrow F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

EXERCISE 2. Let E be a separable real normed vector space. Let $(u_n)_n$ be a dense sequence in $B_E(0, 1)$. By considering the following metric d on the unit ball of E^* ,

$$d(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} |(f - g)(u_n)|, \quad f, g \in B_{E^*}(0, 1),$$

prove that the weak-* topology on $B_{E^*}(0, 1)$ is metrizable.

EXERCISE 3 (Goldstine lemma). Let X be a Banach space. For any $x \in X$, let us define the evaluation $\text{ev}_x : \varphi \in X^* \mapsto \varphi(x) \in \mathbb{R}$. We can therefore consider the following application

$$J : \begin{cases} X & \rightarrow X^{**} \\ x & \mapsto \text{ev}_x \end{cases}$$

For any normed vector space E , we denote by B_E its closed unit ball.

1. Check that J is an isometry and that $J(X)$ is strongly closed in X^{**} .
2. Let E be a normed vector space. Determine all the linear forms on E^* which are continuous for the weak-* topology $\sigma(E^*, E)$.
3. By using the Hahn-Banach theorem, prove that $J(B_X)$ is dense in $B_{X^{**}}$ for the weak-* topology $\sigma(X^{**}, X^*)$.

EXERCISE 4.

1. In $\ell^\infty(\mathbb{N})$ we consider

$$C = \{x \in \ell^\infty(\mathbb{N}) : \liminf_n x_n \geq 0\}.$$

Show that C is strongly closed but not weakly-* closed.

2. Let E be a normed vector space. Show that an hyperplane $H \subset E^*$ which is closed for the weak-* topology $\sigma(E^*, E)$ is the kernel of $\text{ev}_x : \varphi \mapsto \varphi(x)$ for some $x \in E$.

EXERCISE 5. Let $(E, \|\cdot\|)$ be a reflexive space and B_E be its unit ball. Show that for all $\varphi \in E^*$, there exists $x_\varphi \in B_E$, such that $\|\varphi\|_{E^*} = |\varphi(x_\varphi)|$, *i.e.* the supremum in the definition of the norm operator is in fact a maximum.

EXERCISE 6. The aim of this exercise is to prove by two different methods that the space $(C^0([0, 1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0, 1]$ is not reflexive.

1. Method by compactness.

(a) Define $\varphi \in C^0([0, 1])^*$ by

$$\varphi(f) = \int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt, \quad f \in C^0([0, 1]),$$

and show that $\|\varphi\| = 1$.

(b) Prove that $|\varphi(f)| < 1$ for all $f \in C^0([0, 1])$ such that $\|f\|_\infty \leq 1$.

(c) Conclude that the space $C^0([0, 1])$ is not reflexive.

2. Method by separability.

(a) Prove that if E is a Banach space and its dual E^* is separable, then E is separable.

(b) Show that $C^0([0, 1])$ is separable.

(c) Prove that $C^0([0, 1])^*$ is not separable.

Hint: Consider the functions $\delta_t : C^0([0, 1]) \rightarrow \mathbb{R}$ defined by $\delta_t(f) = f(t)$ for any $t \in [0, 1]$.

(d) Conclude that $C^0([0, 1])$ is not isomorphic to $C^0([0, 1])^{**}$ as Banach spaces.

Remark: This is stronger than not being reflexive.

EXERCISE 7.

1. Let E be a reflexive, separable Banach space. Let $(u_n)_n$ be a bounded sequence in E . Show that one can extract a subsequence $(u_{n'})_{n'}$ which converges weakly in E .

2. Does this result hold when E is not reflexive ?

EXERCISE 8. Let E be a reflexive Banach space and $I : E \rightarrow \mathbb{R}$ be a continuous, convex and coercive functional, in the sense that there exist $\alpha > 0$ and $M \geq 0$ such that for all $x \in E$,

$$I(x) \geq \alpha\|x\|_E - M.$$

We also consider $A \subset E$ a non-empty, closed and convex set. Prove that the functional I admits a minimum on A .

EXERCISE 9. Let B denote the closed unit ball of $L^1([0, 1])$. Recall that a function $f \in B$ is called an extreme point if, whenever $f = \theta f_1 + (1 - \theta)f_2$ with $\theta \in (0, 1)$ and $f_1, f_2 \in B$, one has $f_1 = f_2$. Prove that B does not admit extremal points. Deduce that there is no isometry between $L^1([0, 1])$ and the topological dual of a normed vector space.

Hint: We admit Krein-Milman's theorem, stating that any non-empty convex compact subset of any l.c.t.v.s coincides with the closed convex envelop of its extremal points.