TD 6: Introduction to defect measures

Exercise 1. Let $K = [-\pi, \pi]$.

1. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a continuous function supported in the interval $[-1, 1]$ such that
$$\int_{\mathbb{R}} \rho^2(x)\,\mathrm{d}x = 1.$$
Consider the sequence $(w_n)_n$, defined by
$$w_n(x) = \sqrt{n}\rho(nx), \quad x \in K,\, n \geq 1.$$
Prove that $(w_n)_n$ converges weakly to zero in $L^2(K)$ and that for all $f \in C^0(K)$,
$$\lim_{n\to\infty} \int_K f(x)|w_n(x)|^2\,\mathrm{d}x = f(0).$$
Is the sequence $(w_n)_n$ converging strongly ?
2. Let $(u_n)_n$ be a sequence in $L^2(K)$ which converges weakly to zero. Prove that there exists a sub-sequence $(u_{\varphi(n)})_n$ of $(u_n)_n$ and a continuous linear form $l \in C^0(K)^*$ such that for all $f \in C^0(K)$,
$$\lim_{n\to\infty} \int_K f(x)|u_{\varphi(n)}(x)|^2\,\mathrm{d}x = l(f).$$
Deduce that $(u_{\varphi(n)})_n$ converges strongly if and only if $l = 0$.
3. Consider the sequence $(v_n)_n$, defined by
$$v_n(x) = \sin(nx), \quad x \in K,\, n \geq 1.$$
Check that $(v_n)_n$ converges weakly to zero in $L^2(K)$. Prove then that there exists a continuous linear form $l \in C^0(K)^*$ such that for all $f \in C^0(K)$,
$$\lim_{n\to\infty} \int_K f(x)|v_n(x)|^2\,\mathrm{d}x = l(f).$$
Compute the numbers $l(1)$ and $l(\cos(5x))$.