TD 7: Compactness in L^p spaces

Exercise 1 (Equi-integrability). Let $(X, \mathcal{A}, \mu)$ be a measured space and $\mathcal{F} \subset L^1(X)$ being bounded. Prove that the following assertions are equivalent:

1. For all $\varepsilon > 0$, there exists some $M > 0$ such that
$$\sup_{f \in \mathcal{F}} \int_{\{|f|>M\}} |f| \, \mathrm{d}\mu < \varepsilon.$$
2. For all $\varepsilon > 0$, there exists some $\eta > 0$ such that for any measurable set A,
$$\mu(A) < \eta \Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, \mathrm{d}\mu < \varepsilon.$$
3. There exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to \infty} \Phi(x)/x = \infty$ and
$$\sup_{f \in \mathcal{F}} \int_X \Phi(|f|) \, \mathrm{d}\mu < \infty.$$

When one of the above conditions is satisfied, the set $\mathcal{F}$ is said to be *equi-integrable.*
Hint: to show 2. ⇒ 3., consider the sequence $(M_n)_n$ such that
$$\sup_{f \in \mathcal{F}} \int_X |f| \mathbb{1}_{|f|>M_n} \, \mathrm{d}\mu < 2^{-n}.$$

In the following two exercices, the notion of equi-integrability introduced in the previous exercice will be considered. When $p \in [1, +\infty)$, a set $\mathcal{F} \subset L^p(X)$ will be said to be equi-integrable when the set $\{|f|^p : f \in \mathcal{F}\}$ is equi-integrable in $L^1(X)$.

Exercise 2 (Vitali's convergence theorem). We consider $(X, \mathcal{A}, \mu)$ a σ-finite measure space. Let $p \in [1, +\infty)$ and $(f_n)_n$ be a sequence in $L^p(X)$. Assume that

1. The sequence $(f_n)_n$ is a Cauchy sequence in measure, meaning that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that
$$\forall m, n \geq n_0, \quad \mu\big(|f_n - f_m| \geq \varepsilon\big) < \varepsilon.$$
2. The sequence $(f_n)_n$ is equi-integrable in $L^p(X)$,
3. For all $\varepsilon > 0$, there exists a measurable set $\Gamma \subset X$ of finite measure such that
$$\forall n \geq 0, \quad \|f_n \mathbb{1}_{X \setminus \Gamma}\|_{L^p(X)} \leq \varepsilon.$$

Prove that $(u_n)_n$ is a Cauchy sequence in $L^p(X)$ (and therefore converges in this space).

Exercise 3 (Dunford-Pettis' Theorem). The objective of the exercise is to prove Dunford-Pettis' theorem:

> Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$. Then, the set $\{f_n\}$ is sequentially compact for the weak topology $\sigma(L^1, L^\infty)$ if and only if the sequence $(f_n)_n$ is equi-integrable.

First we prove the reciprocal: let $(f_n)_n$ be a bounded and equi-integrable sequence in $L^1(\Omega)$.

1. Show that we can reduce to the case where the f_n are non-negative.
2. Let $f_n^k = \mathbb{1}_{f_n \le k} f_n$. Show that $\sup_n \|f_n - f_n^k\|_{L^1} \to 0$.
3. Show that there exists an extraction (n') such that for all $k \in \mathbb{N}$, $f_{n'}^k \rightharpoonup f^k$ in $L^1(\Omega)$.
4. Prove that $(f^k)_k$ is an increasing sequence and deduce that there exists some $f \in L^1(\Omega)$ such that $f^k \to f$ in $L^1(\Omega)$.
5. Conclude that $f_{n'} \rightharpoonup f$ in $L^1(\Omega)$.

Now we want to prove the direct implication. Let $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$ satisfying $f_n \rightharpoonup f \in L^1(\Omega)$. We consider $\mathcal{X}$ the set of indicator functions and, for a fixed $\varepsilon > 0$, we also consider the sets X_n defined for all $n \ge 0$ by:

$$X_n := \left\{ \mathbb{1}_A \in \mathcal{X} : \forall k \ge n, \ \left| \int_A (f_k - f) \, \mathrm{d}x \right| \le \varepsilon \right\}.$$

6. Show that $\mathcal{X}$ and X_n are closed in $L^1(\Omega)$.
7. Using a Baire's argument, show that the sequence $(f_n)_n$ is equi-integrable.
8. Conclude.