TD 9: Distributions (II)

Exercise 1. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \rho \leq 1$, $\operatorname{supp} \varphi = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \rho = 1$. For all $\varepsilon > 0$, we set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$.

1. Prove that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{x \in \mathbb{R}^n} |(\rho_\varepsilon * \varphi)(x) - \varphi(x)| \underset{\varepsilon \to 0^+}{\to} 0.$$

2. Check that for all $f \in L^p(\mathbb{R}^n)$, $\lim_{\varepsilon \to 0^+} \|\rho_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} = 0$.

Exercise 2. Let Ω be an open subset of $\mathbb{R}^n$.

1. Let $\varphi \in C^\infty(\Omega \times \mathbb{R}^n)$ and $T \in \mathscr{D}'(\mathbb{R}^n)$. Assume that there exists a compact $K \subset \Omega$ such that

$$\forall y \in \mathbb{R}^n, \quad \operatorname{supp}(\varphi(\cdot, y)) \subset K.$$

Prove then that the function $y \in \mathbb{R}^n \mapsto T(\varphi(\cdot, y))$ is in $C^\infty(\mathbb{R}^n)$, with moreover

$$\forall \alpha \in \mathbb{N}^n, \quad \partial_y^\alpha(T(\varphi(\cdot, y)) = T(\partial_y^\alpha \varphi(\cdot, y)).$$

2. Let $\varphi \in C_0^\infty(\Omega \times \mathbb{R}^n)$ and $T \in \mathscr{D}'(\Omega)$. Prove that

$$\int_{\mathbb{R}^n} T(\varphi(\cdot, y))\,\mathrm{d}y = T\bigg(\int_{\mathbb{R}^n} \varphi(\cdot, y)\,\mathrm{d}y\bigg).$$

Exercise 3.

1. Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(0) = 1$. For all $\varphi \in C_0^\infty(\mathbb{R})$, prove that there exists $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\forall x \in \mathbb{R}, \quad \varphi(x) - \varphi(0)\theta(x) = x\psi(x).$$

2. Solve $xT = 0$ in $\mathscr{D}'(\mathbb{R})$.
3. Solve $xT = 1$ in $\mathscr{D}'(\mathbb{R})$.
4. Solve $(x-1)T = \delta_0$ and $(x-a)(x-b)T = 1$ with $a \neq b$ in $\mathscr{D}'(\mathbb{R})$.

Exercise 4. For all $x \in \mathbb{R}$ and $\varepsilon > 0$, we set

$$f_\varepsilon(x) = \log(x + i\varepsilon) = \log|x + i\varepsilon| + i\operatorname{Arg}(x + i\varepsilon),$$

the argument being taken in $(-\pi, \pi)$.

1. Prove that as ε goes to zero, the sequence (f_ε) converges in $\mathscr{D}'(\mathbb{R})$ to the locally integrable function $f_0 \in L^1_{loc}(\mathbb{R})$ defined by

$$f_0(x) = \begin{cases} \log(x) & \text{when } x > 0, \\ \log|x| + i\pi & \text{when } x < 0. \end{cases}$$

2. Compute f_0' in $\mathscr{D}'(\mathbb{R})$.
3. Deduce that the following equality holds in $\mathscr{D}'(\mathbb{R})$

$$\frac{1}{x+i0} := \lim_{\varepsilon\to 0^+} \frac{1}{x+i\varepsilon} = -i\pi\delta_0 + \mathrm{p.\,v.}(1/x).$$

4. Show similarly that

$$\frac{1}{x-i0} := \lim_{\varepsilon\to 0^+} \frac{1}{x-i\varepsilon} = i\pi\delta_0 + \mathrm{p.\,v.}(1/x).$$

Exercise 5.

1. What can be said about a distribution $T \in \mathscr{D}'(\mathbb{R})$ which satisfies $T' \in C^0(\mathbb{R})$?
2. Same question with a distribution $T \in \mathscr{D}'(\mathbb{R})$ such that $T^{(n)} = 0$ for some integer $n \in \mathbb{N}$.
3. Let Ω be a measurable subset of $\mathbb{R}^n$, $p \in [1, +\infty)$ and B_p be the unit ball of $L^p(\Omega)$. Prove that if a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is bounded on $B_p \cap \mathcal{D}(\Omega)$, then $T \in L^q(\Omega)$, where $q \in (1, +\infty]$ satisfies $1/p + 1/q = 1$.

Exercise 6.

1. Let $T \in \mathscr{D}'(\mathbb{R})$ and $f \in L^1_{loc}(\mathbb{R})$. For all $c \in \mathbb{R}$, we set

$$F_c(x) = c + \int_0^x f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

   Prove that $T' = f$ if and only if there exists $c \in \mathbb{R}$ such that $T = F_c$.
2. Check that for all $T \in \mathscr{D}'(\mathbb{R})$, the following convergence holds in $\mathscr{D}'(\mathbb{R})$

$$\frac{\tau_{-h}T - T}{h} \underset{h\to 0}{\rightarrow} T',$$

   where τ_{-h} denotes the translation operator.
3. Prove that a distribution $T \in \mathscr{D}'(\mathbb{R})$ is a Lipschitz function if and only if $T' \in L^\infty(\mathbb{R})$.
   Hint: Use the question 3 of the previous exercice.

Exercise 7. Let $E_n \in L^1_{loc}(\mathbb{R}^n)$ be the function defined by

$$E_n(x) = \begin{cases} \log(|x|) & \text{when } n = 2, \\ |x|^{2-n} & \text{when } n \geq 3. \end{cases}$$

1. Let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be a radial function, i.e. $u(x) = U(|x|)$ where $U \in C^2(\mathbb{R}^*)$. Prove that

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \quad (\Delta u)(x) = U''(|x|) + \frac{n-1}{|x|}U'(|x|).$$

2. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Justify that

$$(\Delta E_n)(\varphi) = \lim_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon} E_n(x)(\Delta\varphi)(x)\,\mathrm{d}x,$$

   where $\Omega_\varepsilon = \{x \in \mathbb{R}^n : |x| > \varepsilon\}$. By using Green's formula, conclude then that there exists a constant $c_n \in \mathbb{R}$ such that $\Delta E_n = c_n\delta_0$ in $\mathscr{D}'(\mathbb{R}^n)$