École Normale Supérieure de Lyon **Année 2023 - 2024**

Département de Mathématiques **M1 - EDP**

TD 1: Sobolev spaces and elliptic equations on the whole space

Exercise 1. Let us consider the function

$$\gamma_0 : \varphi(x', x_n) \in C_0^\infty(\mathbb{R}^n) \mapsto \varphi(x', x_n = 0) \in C_0^\infty(\mathbb{R}^{n-1}).$$

Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^n)$ to $H^{s-1/2}(\mathbb{R}^{n-1})$.
Hint: For all $\varphi \in C_0^\infty(\mathbb{R}^n)$, begin by computing the Fourier transform of the function $\gamma_0\phi$.

Exercise 2.

1. Show that $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$.
   Hint: Consider a function of the form $x \mapsto \chi(|x|)\,|\log|x||^{1/3}$.
2. Show that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^2)$,

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq c\Big(1 + \|u\|_{H^1(\mathbb{R}^2)}\sqrt{\log(1 + \|u\|_{H^2(\mathbb{R}^2)})}\Big).$$

Exercise 3. Let $\lambda > 0$. Prove that for any $s > 0$, the differential operator $-\Delta + \lambda$ is an isomorphism from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$.

Exercise 4. Let ρ be a compactly supported C^∞ function on $\mathbb{R}^3$. We are looking for a function $u \in C^2(\mathbb{R}^3)$ satisfying

$$-\Delta u = \rho, \tag{1}$$

under the following decreasing conditions at infinity

$$x \mapsto |x|u(x) \text{ is bounded}, \quad x \mapsto |x|^2\nabla u(x) \text{ is bounded}. \tag{2}$$

1. Check that the function $x \mapsto 1/|x|$ is of class C^2 on $\mathbb{R}^3 \setminus \{0\}$ and compute its Laplacian.

2. Let Ω be a smooth open subset of $\mathbb{R}^3$. We denote by $n(x)$ the unit normal vector exiting at $x \in \partial\Omega$ and $\mathrm{d}\sigma$ the measure surface on $\partial\Omega$. We consider two functions u, v of class C^2 on $\overline{\Omega}$. By using Stokes' formula, prove Green's formula for the Laplacian:

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\mathrm{d}\sigma(x).$$

3. For $0 < \alpha < \beta$, we define the following sphere and annulus

$$S_\alpha = \big\{x \in \mathbb{R}^3 : |x| = \alpha\big\} \quad \text{and} \quad A_{\alpha,\beta} = \big\{x \in \mathbb{R}^3 : \alpha \leq |x| \leq \beta\big\}.$$

   Let $0 < \varepsilon < R$. We consider $u \in C^2(\mathbb{R}^3)$ satisfying (2). For all $x \in \mathbb{R}^3$, prove the following identity

$$\frac{1}{\varepsilon^2}\int_{S_\varepsilon} u(x+y)\,\mathrm{d}\sigma(y) = \int_{A_{\varepsilon,R}} \frac{(-\Delta u)(x+y)}{|y|}\,\mathrm{d}y + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}(\varepsilon).$$

4. Prove that the unique solution of (1) satisfying (2) is given for all $x \in \mathbb{R}^3$ by

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} \, \mathrm{d}y.$$

5. Let $p \in [1, 3)$. Check that there exists a constant C_p independent of ρ, such that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le C_p \|\rho\|_{L^p(\mathbb{R}^3)}^{p/3} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{1-p/3}.$$

Hint: Consider the domains $\{|x-y| \le r\}$ *and* $\{|x-y| > r\}$*, and optimize with respect to* r*.*

6. Prove the following formula:

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y.$$

Exercise 5. The purpose of this exercice is to prove with a variational method that given a positive real number $p > 1$ and a function $f \in L^2(\mathbb{R})$, there exists a unique function $u \in H^2(\mathbb{R})$ satisfying

$$-u'' + u + |u|^{p-1}u = f \quad \text{in } L^2(\mathbb{R}). \tag{3}$$

1. In this question, we prove that there exists a unique $u \in H^1(\mathbb{R})$ such that

$$\forall v \in H^1(\mathbb{R}), \quad \int_0^1 (u'(x)v'(x) + u(x)v(x) + |u(x)|^{p-1}u(x)v(x) - f(x)v(x)) \, \mathrm{d}x = 0. \tag{4}$$

To that end, we introduce the functional $J : H^1(\mathbb{R}) \to \mathbb{R}$ defined for all $u \in H^1(\mathbb{R})$ by

$$J(u) = \frac{1}{2} \int_0^1 (|u(x)|^2 + |u'(x)|^2) \, \mathrm{d}x + \frac{1}{p+1} \int_{\mathbb{R}} |u(x)|^{p+1} \, \mathrm{d}x - \int_{\mathbb{R}} f(x)u(x) \, \mathrm{d}x.$$

   a) Check that the functional J is well-defined, strictly convex and coercive.

   b) Prove that the functional J is differentiable on $H^1(\mathbb{R})$ and give the expression of its derivative.

   c) Deduce from the preliminary question that the variational problem (4) admits a unique solution $u \in H^1(\mathbb{R})$.

2. Prove that the unique function $u \in H^1(\mathbb{R})$ satisfying (4) belongs to $H^2(\mathbb{R})$ and is also the unique function that satisfies (3).

3. When the function f is continuous on $\mathbb{R}$, check that $u \in C^2(\mathbb{R})$ is a strong solution of (3), in the sense that

$$\forall x \in \mathbb{R}, \quad -u''(x) + u(x) + |u(x)|^{p-1}u(x) = f(x).$$