TD 10: Logarithmic Sobolev inequality

Let γ denote the centered normalized Gaussian measure

$$\mathrm{d}\gamma(y) = \frac{1}{(2\pi)^{d/2}} e^{-|y|^2/2} \mathrm{d}y.$$

For $f \in C_b(\mathbb{R}^d)$ (continuous and bounded), set

$$P_t f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1-e^{-2t}}y) \, \mathrm{d}\gamma(y). \tag{1}$$

The formula (1) has also the probabilistic representation

$$P_t f(x) = \mathbb{E} f(e^{-t}x + \sqrt{1-e^{-2t}}Y),$$

where Y is a real-valued random variable of normal law $\mathcal{N}(0,1)$. Note that $P_t \colon C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$. We recall the following property:

> If X, Y are two independent random variable of normal law $\mathcal{N}(0,1)$ and α, $\beta \in \mathbb{R}$ satisfy $\alpha^2 + \beta^2 = 1$, then the random variable $\alpha X + \beta Y$ follows the normal law $\mathcal{N}(0,1)$.

1. Show that $(P_t)_{t\geq 0}$ is a semigroup on $C_b(\mathbb{R}^d)$.
2. Prove the invariance property for all $f \in C_b(\mathbb{R}^d)$

$$\langle P_t f, \gamma \rangle_{L^2} = \langle f, \gamma \rangle_{L^2}.$$

3. Prove the following inequality for all $f \in C_b(\mathbb{R}^d)$

$$|P_t f|^2 \leq P_t(f^2).$$

4. Let $L^2(\gamma)$ be the set of measurable functions $f \colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{L^2(\gamma)} = \left(\int_{\mathbb{R}^d} |f(y)|^2 \, \mathrm{d}\gamma(y) \right)^{1/2} = \left(\mathbb{E}|f(Y)|^2 \right)^{1/2}$$

is finite (Y there has the normal law $\mathcal{N}(0,1)$). Prove that, for all $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$,

$$\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}. \tag{2}$$

Justify that we can then extend the semigroup $(P_t)_{t\geq 0}$ by density as a semigroup of contractions on $L^2(\gamma)$

5. * Prove that the semigroup $(P_t)_{t\geq 0}$ has the generator T given by

$$Tf(x) = -\Delta f(x) + x \cdot \nabla f(x).$$

6. Check that for all $f, g \in C_0^1(\mathbb{R}^d)$

$$\langle Tf, g\rangle_{L^2(\gamma)} = \langle \nabla f, \nabla g\rangle_{L^2(\gamma)}.$$

7. Show that for all $x \in \mathbb{R}^d$ and $f \in C_b(\mathbb{R}^d)$,

$$P_t f(x) \to \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f \, \mathrm{d}\gamma \quad \text{as } t \to +\infty.$$

8. Show that P_t satisfies for all $f \in C_b^1(\mathbb{R}^d)$ and $t \geq 0$ that

$$\nabla P_t f(x) = e^{-t} P_t \nabla f(x).$$

In the following questions, we will use an interpolation procedure, by means of $t \mapsto P_t f(x)$, between $f(x)$ and $\langle f, \gamma\rangle$, to show the two following results:

Poincaré inequality for the Gaussian measure:

$$\int_{\mathbb{R}^d} f^2 \, \mathrm{d}\gamma - \langle f, \gamma\rangle^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, \mathrm{d}\gamma. \tag{3}$$

Logarithmic Sobolev inequality for the Gaussian measure:

$$\int_{\mathbb{R}^d} f \ln f \, \mathrm{d}\gamma - \langle f, \gamma\rangle \ln\langle f, \gamma\rangle \leq \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} \, \mathrm{d}\gamma. \tag{4}$$

Both inequality are understood for smooth functions, positive in the case of (4).

9. Let D be an open convex subset of $\mathbb{R}$, $\Phi \colon D \to \mathbb{R}$ be a smooth function and $f \in C_0^1(\mathbb{R}^d)$ be a function taking values in D. Prove the successive identities

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi(f) \, \mathrm{d}\gamma - \Phi(\langle f, \gamma\rangle) &= -\int_0^\infty \int_{\mathbb{R}^d} \frac{\mathrm{d}}{\mathrm{d}t} \Phi(P_t f) \, \mathrm{d}\gamma \, \mathrm{d}t \\ &= \int_0^\infty \int_{\mathbb{R}^d} \Phi''(P_t f) |\nabla P_t f|^2 \, \mathrm{d}\gamma \, \mathrm{d}t. \end{aligned}$$

10. Establish (3) and (4).