École Normale Supérieure de Lyon **Année 2023 - 2024**
Département de Mathématiques **M1 - EDP**

TD 2: Sobolev spaces

Exercise 1.

1. Show that $u(x) = |x|$ belongs to $W^{1,2}(-1,1)$ but not to $W^{2,2}(-1,1)$.
2. Check that $v(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$ belongs to $L^2(\mathbb{R})$ but not to $W^{1,2}(\mathbb{R})$.

Exercise 2. Let $1 \leq p < d$ and $\alpha \in [1, \infty]$. By using a homogeneity argument, show that if there exists a continuous injection $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\alpha(\mathbb{R}^d)$, then necessarily $p \leq \alpha \leq \frac{dp}{d-p}$.

Exercise 3. Let $\Omega = (0,1)$.

1. Prove that the following continuous embeddings hold
$$W^{1,1}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \quad \text{and} \quad W^{1,p}(\Omega) \hookrightarrow C^{0,1-1/p}(\bar{\Omega}) \quad \text{when } p \in (1,\infty],$$
with the convention $1/\infty = 0$.
2. Prove that for all $1 \leq p < \infty$, the space $W_0^{1,p}(\Omega)$ is given by
$$W_0^{1,p}(\Omega) = \big\{u \in W^{1,p}(\Omega) : u(0) = u(1) = 0\big\}.$$

Exercise 4. Let $\Omega = (0,1)$. Prove Poincaré's inequality
$$\forall f \in H_0^1(\Omega), \quad \|f\|_{L^2(\Omega)} \leq \frac{1}{\pi}\|f'\|_{L^2(\Omega)},$$
and check that the constant $1/\pi$ is optimal.
Hint: Use Fourier series.

Exercise 5. The aim of this exercise is to give a characterization of the space $H^1(0,1)$.

1. (a) Prove that if $u \in C^1[0,1]$, then we have for any $\alpha \in (0,1/2)$, $x \in (\alpha, 1-\alpha)$ and $h \in \mathbb{R}$ such that $|h| < \alpha$,
$$|u(x+h) - u(x)|^2 \leq h^2 \int_0^1 |u'(x+sh)|^2 \, \mathrm{d}s.$$
   (b) Deduce that for any $u \in H^1(0,1)$, any $\alpha \in (0,1/2)$ and $h \in \mathbb{R}$ such that $|h| < \alpha$, we have
$$\|\tau_h u - u\|_{L^2(\alpha,1-\alpha)} \leq |h|\|u'\|_{L^2(0,1)}.$$
2. Conversely, we assume that $u \in L^2(0,1)$ is such that there exists a constant $C > 0$ such that for any $\alpha \in (0,1/2)$ and for any $h \in \mathbb{R}$ such that $|h| < \alpha$, we have
$$\|\tau_h u - u\|_{L^2(\alpha,1-\alpha)} \leq C|h|.$$

(a) Let $\phi \in C_0^1(0,1)$ and $\alpha > 0$ such that ϕ is supported in $(\alpha, 1-\alpha)$. Prove that for any $|h| < \alpha$, we have

$$\int_\alpha^{1-\alpha} (u(x+h) - u(x))\phi(x)\,\mathrm{d}x = \int_0^1 u(x)(\phi(x-h) - \phi(x))\,\mathrm{d}x.$$

Deduce that

$$\left| \int_0^1 u(x)\phi'(x)\,\mathrm{d}x \right| \leq C\|\phi\|_{L^2(0,1)}.$$

(b) Conclude that $u \in H^1(0,1)$.

Exercise 6. Let $p \in [1, +\infty)$ and let Ω be an open subset of $\mathbb{R}^d$.

1. Assume that Ω is bounded in one direction, meaning that Ω is contained in the region between two parallel hyperplanes. Prove Poincaré's inequality: there exists $c > 0$ such that for every $f \in W_0^{1,p}(\Omega)$,

$$\|f\|_{L^p(\Omega)} \leq c\|\nabla f\|_{L^p(\Omega)}.$$

Hint: Consider first the case $\Omega \subset \mathbb{R}^{d-1} \times [-M, M]$.

2. Assume that Ω is bounded. Prove Poincaré-Wirtinger's inequality: there exists a constant $c > 0$ such that for any $f \in W^{1,p}(\Omega)$ satisfying $\int_\Omega f = 0$,

$$\|f\|_{L^p(\Omega)} \leq c\|\nabla f\|_{L^p(\Omega)}.$$

Exercise 7. Let Ω be an open subset of $\mathbb{R}^d$ and let $p \in (1, +\infty)$. Prove that for all $F \in W_0^{1,p}(\Omega)'$, there exist $f_0, f_1, \ldots, f_d \in L^q(\Omega)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that for all $g \in W_0^{1,p}(\Omega)$,

$$\langle F, g \rangle_{W_0^{1,p}(\Omega)', W_0^{1,p}(\Omega)} = \int_\Omega f_0 g\,\mathrm{d}x + \sum_{i=1}^d \int_\Omega f_i \partial_i g\,\mathrm{d}x.$$

Assuming that Ω is bounded, prove that we may take $f_0 = 0$.

Exercise 8. Let Ω and Ω' be two open subset of $\mathbb{R}^d$.

1. Let $H : \Omega' \to \Omega$ be a C^1-diffeomorphism such that the Jacobian $\mathrm{Jac}(H)$ and $\mathrm{Jac}(H^{-1})$ belong to L^∞. Prove that for all $u \in W^{1,p}(\Omega)$, we have $u \circ H \in W^{1,p}(\Omega')$ and that for all $1 \leq i \leq d$,

$$\partial_{y_i}(u \circ H) = \sum_{j=1}^n (\partial_{x_j} u \circ H)\partial_{y_i} H_j.$$

2. Let us now consider a function $G \in C_b^1(\mathbb{R})$ satisfying $G(0) = 0$. Show that for all $u \in W^{1,p}(\Omega)$, we have $G \circ u \in W^{1,p}(\Omega)$ and that for all $1 \leq j \leq n$,

$$\partial_{x_j}(G \circ u) = (G' \circ u)\partial_{x_j} u.$$

3. Do we need to assume that G' is bounded when $d = 1$?