TUTORIAL 11

1 Homework 8

1. Let P, Q be two probability distributions over a set R. An element r is sampled from P with probability $1/2$ and from Q with probability $1/2$.
   (a) Proprose an algorithm which, on input r, distinguish between the case $r \leftarrow P$ and $r \leftarrow Q$ with probability $1/2 + 1/2 \cdot \Delta(P, Q)$.
   (b) Show that this success probability is optimal.
2. Compute $H(\rho)$ for:
   (a) $\rho = |+\rangle\langle+|$.
   (b) $\rho = I/2$.

2 Trace distance through a quantum channel

We recall that the trace distance is defined as follows:

Definition 2.1. $\Delta(\rho, \rho') = 1/2 \cdot \mathrm{Tr}\ (|\rho - \rho'|) = \max_\pi \mathrm{Tr}\ (\pi(\rho - \rho'))$ *where the maximum is taken among the orthogonal projectors.*

1. Prove that $\Delta(\rho, \rho') = \Delta(U\rho U^\dagger, U\rho' U^\dagger)$ for any ρ, ρ' density operator and U unitary.
2. Show that the trace distance follows the triangular inequality.
3. Let ρ, ρ' be two density operators. Let Φ a quantum channel. We are going to show that quantum channels can only make the trace distance decrease.
   (a) Let M a semidefinite positive hermitian operator and π an orthogonal projector. Show that $\mathrm{Tr}\ (\pi M) \leq \mathrm{Tr}\ (M)$.
   (b) Let M a hermitian operator. Show that $M = A - B$ with A, B positive hermitian. Show that in this case, $|M| = A + B$.
   (c) Prove that $\Delta(\rho, \rho') \geq \Delta(\Phi(\rho), \Phi(\rho'))$.

3 Information Theory Quantities

3.1 Conditional entropy

Definition 3.1. *The conditional entropy $H(X|Y)$ is defined by*

$$H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y)\ ,$$

where $H(X|Y = y)$ is the entropy of the conditional distribution $P_{X|Y=y}$. Note that elements $y \in \mathcal{Y}$ with $P_Y(y) = 0$ do not participate to the sum.

1. What is the value of $H(X|X)$?
2. If X and Y are independent random variables, what is the value of $H(X|Y)$?
3. Prove that $0 \leq H(X|Y) \leq \log_2(|\mathcal{X}|)$.
4. Prove that $H(X|Y) = H(XY) - H(Y)$, where $H(XY) := H((X,Y)) = -\sum_{x,y} P_{XY}(x,y) \log_2 P_{XY}(x,y)$.

3.2 Mutual Information

Definition 3.2. *The mutual information is defined by*

$$\begin{aligned} I(X:Y) &= H(X) - H(X|Y) \\ &= H(X) + H(Y) - H(XY) \,. \end{aligned}$$

Writing out the definitions, we get $I(X:Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2 \frac{P_{XY}(x,y)}{P_X(x) P_Y(y)}$.

1. What is the value of $I(X:X)$?
2. If X and Y are independent, what is the value of $I(X:Y)$?
3. For any pair of random variables, prove that $I(X:Y) \geq 0$.
   Hint: Use Jensen's inequality on $-I(X:Y)$.

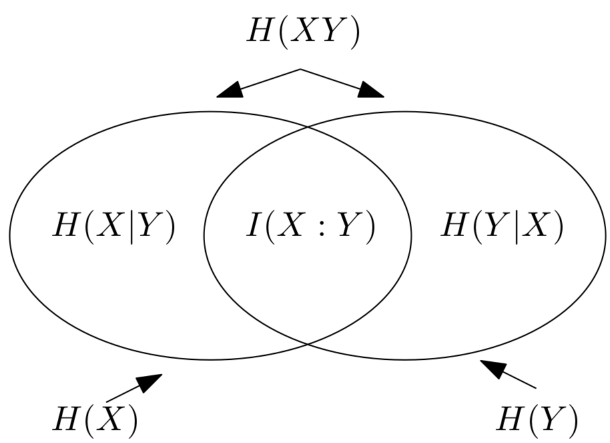

Figure 1: Relation between the entropic measures we have introduced