1 Homework 8

1. Let $P, Q$ be two probability distributions over a set $R$. An element $r$ is sampled from $P$ with probability $1/2$ and from $Q$ with probability $1/2$.

   (a) Propose an algorithm which, on input $r$, distinguish between the case $r ← P$ and $r ← Q$ with probability $1/2 + 1/2 \cdot \Delta(P, Q)$.

   (b) Show that this success probability is optimal.

2. Compute $H(\rho)$ for:

   (a) $\rho = |+\rangle\langle+|$.

   (b) $\rho = I/2$.

2 Trace distance through a quantum channel

We recall that the trace distance is defined as follows:

**Definition 2.1.** $\Delta(\rho, \rho') = 1/2 \cdot \text{Tr}(|\rho - \rho'|) = \max_\pi \text{Tr}(\pi(\rho - \rho'))$ where the maximum is taken among the orthogonal projectors.

1. Prove that $\Delta(\rho, \rho') = \Delta(U\rho U^\dagger, U\rho' U^\dagger)$ for any $\rho, \rho'$ density operator and $U$ unitary.

2. Show that the trace distance follows the triangular inequality.

3. Let $\rho, \rho'$ be two density operators. Let $\Phi$ a quantum channel. We are going to show that quantum channels can only make the trace distance decrease.

   (a) Let $M$ a semidefinite positive hermitian operator and $\pi$ an orthogonal projector. Show that $\text{Tr}(\pi M) \leq \text{Tr}(M)$.

   (b) Let $M$ a hermitian operator. Show that $M = A - B$ with $A, B$ positive hermitian. Show that in this case, $|M| = A + B$.

   (c) Prove that $\Delta(\rho, \rho') \geq \Delta(\Phi(\rho), \Phi(\rho'))$.

3 Information Theory Quantities

3.1 Conditional entropy

**Definition 3.1.** The conditional entropy $H(X|Y)$ is defined by

$$H(X|Y) = \sum_{y \in Y} P_Y(y) H(X|Y = y),$$

where $H(X|Y = y)$ is the entropy of the conditional distribution $P_{X|Y=y}$. Note that elements $y \in Y$ with $P_Y(y) = 0$ do not participate to the sum.
1. What is the value of $H(X|X)$?

2. If $X$ and $Y$ are independent random variables, what is the value of $H(X|Y)$?

3. Prove that $0 \leq H(X|Y) \leq \log_2(|X|)$.

4. Prove that $H(X|Y) = H(XY) - H(Y)$, where $H(XY) := H((X,Y)) = -\sum_{x,y} P_{XY}(x,y) \log_2 P_{XY}(x,y)$.

3.2 Mutual Information

**Definition 3.2.** The mutual information is defined by

$$I(X:Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(XY).$$

Writing out the definitions, we get $I(X:Y) = \sum_{x\in X, y\in Y} P_{XY}(x,y) \log_2 \frac{P_{XY}(x,y)}{P_X(x)P_Y(y)}$.

1. What is the value of $I(X:X)$?

2. If $X$ and $Y$ are independent, what is the value of $I(X:Y)$?

3. For any pair of random variables, prove that $I(X:Y) \geq 0$.

*Hint: Use Jensen’s inequality on $-I(X:Y)$.*

![Figure 1: Relation between the entropic measures we have introduced](image)