TUTORIAL 14

1 Homework 11

1. Let $C_n = \text{span}\{|0^n\rangle = |0,\ldots,0\rangle, |1^n\rangle\}$ the n quantum repetition code. Propose an error acting on 1 qubit that C_n cannot correct.
2. Show that $C_S = \{|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} : \forall g \in S, g \cdot |\psi\rangle = |\psi\rangle\}$ is a vector space.

2 Pauli Basis

Recall the definition of the Pauli matrices: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

1. Show that the function $\phi_n : M, N \mapsto \text{Tr}\,(M \cdot N^\dagger)$ is an hermitian inner product over $M_n(\mathbb{C})$. What is its associated norm?
2. Show that for any $A, A' \in M_m(\mathbb{C}), B, B' \in M_n(\mathbb{C})$, we have that
$$\phi_{nm}(A \otimes B, A' \otimes B') = \phi_m(A, A') \cdot \phi_n(B, B').$$
3. Show that the Pauli matrices are a basis of $M_2(\mathbb{C})$.
4. For any word $w \in \{I, X, Y, Z\}^n$, we define the associated operator $\sigma(w) = w_1 \otimes w_2 \otimes \ldots \otimes w_n$. Show that $\sigma(\{I, X, Y, Z\}^n)$ is a basis of $M_{2^n}(\mathbb{C})$.
5. For any $w \in \{I, X, Y, Z\}^n$, let $|w| = |\{w_i \neq I\}|$. We recall that for any $A \subset \{1, \ldots, n\}$, $\mathcal{E}[A]$ is the set of unitaries acting on the qubits A, and that
$$\mathcal{E}(n, t) = \sum_{A \subset \{1,\ldots,n\}, |A| \leq t} \mathcal{E}[A].$$
Show that $\sigma(\{w \in \{I, X, Y, Z\}^n, |w| \leq t\})$ is a basis of $\mathcal{E}(n, t)$.

3 Stabilizer Codes

Recall the definition of a stabilizer code:

Definition 3.1. *Let S be a subgroup of the Pauli group $G_n := \{A_1 \otimes \ldots \otimes A_n, A_i \in G_1\}$, $G_1 := \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$. We define the stabilizer code $\mathcal{C}_S$ to be the vector subspace of $(\mathbb{C}^2)^{\otimes n}$ stabilized by S, i.e.:*
$$|\psi\rangle \in \mathcal{C}_S \Leftrightarrow \forall h \in S, h \cdot |\psi\rangle = |\psi\rangle\ .$$

1. First, let us consider the example where $n = 3$ and $S = \{I, Z_1Z_2, Z_1Z_3, Z_2Z_3\}$, with $Z_i := Z[i]$. Give a basis of $\mathcal{C}_S$.

2. Recall that a group G is generated by H if $G = \{h_1h_2 \dots h_k : k \in \mathbb{N} \text{ and } \forall i \in [k], h_i \in H\}$, which we will denote by $G = \langle H \rangle$. Show that if S is a subgroup of G_n generated by H:

$$|\psi\rangle \in \mathcal{C}_S \Leftrightarrow \forall h \in H, h \cdot |\psi\rangle = |\psi\rangle \ .$$

   Thus, one can define a stabilizer code $\mathcal{C}_S$ only by considering a set of generators $\{g_1, \dots, g_\ell\}$ of the group S.

3. Show that if $-I \in S$, then $\mathcal{C}_S = \{0\}$.

4. Show that if there exists a non-commuting pair of elements in $\{g_1, \dots, g_\ell\}$, then $\mathcal{C}_{\langle g_1,\dots,g_\ell\rangle} = \{0\}$.

   Remark. In fact, these necessary conditions of nontriviality are also sufficient: if $S = \langle g_1, \dots, g_{n-k}\rangle$ with $n - k$ independent[1] commuting elements from G_n such that $-I \notin S$, then $\mathcal{C}_{\mathcal{S}}$ is a subspace of dimension 2^k. □

5. Recall that the Shor code is defined by $|\overline{0}\rangle := \frac{1}{2\sqrt{2}} (|000\rangle + |111\rangle)^{\otimes 3}$ and $|\overline{1}\rangle := \frac{1}{2\sqrt{2}} (|000\rangle - |111\rangle)^{\otimes 3}$. Show that the Shor code is a stabilizer code, i.e. there exists $g_1, \dots g_\ell$ such that $\mathcal{C}_{\langle g_1,\dots g_\ell\rangle} = \text{span}\{|\overline{0}\rangle, |\overline{1}\rangle\}$ (you can use the previous remark without proving it).

[1] $\forall i \in [n-k], g_i \notin \langle g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{n-k}\rangle$.