1 Tensor Products

Let \( A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \):

1. Compute \( A \otimes B \) and \( B \otimes A \).

2. Let \( A, B, C, D \) matrices, \( x, y \) vectors and \( \lambda \) a complex number. For the following statements, say if it is true or false and prove it:

   (a) \( A \otimes (\lambda B) = (\lambda A) \otimes B \)
   (b) \( A \otimes (BC) = (AB) \otimes C \)
   (c) \( (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \)
   (d) If \( A \) and \( B \) are diagonalizable, then \( A \otimes B \) is diagonalizable
   (e) \( (A \otimes B)^\dagger = B^\dagger \otimes A^\dagger \)
   (f) \( (A + B) \otimes C = A \otimes C + B \otimes C \)
   (g) \( (A \otimes B)(x \otimes y) = (Ax) \otimes (By) \)

2 Measurements and Probabilities

For all \( |\varphi\rangle \) and \( \mathcal{B} \), justify that \( |\varphi\rangle \) is a state, \( \mathcal{B} \) a basis, then measure \( |\varphi\rangle \) in \( \mathcal{B} \) and give the probability of each outcome:

1. \( |\varphi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \mathcal{B} = (|0\rangle, |1\rangle) \).

2. \( |\varphi\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \mathcal{B} = (|0\rangle, |1\rangle) \).

3. \( |\varphi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \mathcal{B} = \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \).

4. \( |\varphi\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \mathcal{B} = \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \).

5. \( |\varphi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \mathcal{B} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle) \).

6. \( |\varphi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \mathcal{B} = (i|00\rangle, |01\rangle, -|10\rangle, e^{i\frac{\pi}{4}} |11\rangle) \).

7. \( |\varphi\rangle = \frac{1}{2} (|00\rangle + i|01\rangle + |10\rangle + |11\rangle), \mathcal{B} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle) \).

8. \( |\varphi\rangle = \frac{1}{2} (|00\rangle + i|01\rangle + |10\rangle + |11\rangle), \mathcal{B} = \left( |00\rangle, |01\rangle, \frac{|10\rangle + |11\rangle}{\sqrt{2}}, \frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) \).
3  Gates

1. Construct a SWAP-gate, that is to say a quantum circuit $U$ such that for any $a, b \in \{0, 1\}$ is such that $U \cdot |a\rangle|b\rangle \mapsto |b\rangle|a\rangle$, using CNOT gates.
   
   Hint: how do you swap two digits using only XOR gate in the classical case?

2. Suppose you are given access to a bit-query-oracle for a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, that is to say a gate $U_f$ such that $\forall a \in \{0, 1\}^n, \forall b \in \{0, 1\}, U_f|a\rangle|b\rangle = |a\rangle|b \oplus f(a)\rangle$.
   Construct a circuit giving access to a phase-query-oracle, that is to say construct a circuit $U'_f$ such that for any $(a, b) \in \{0, 1\}^n \times \{0, 1\}$, $U'_f \cdot |a\rangle|b\rangle = (-1)^b f(a) |a\rangle|b\rangle$.

4  Partial Measurements

Let $|\varphi\rangle, |\psi\rangle$ two normalized quantum states, let $|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle|\varphi\rangle + |1\rangle|\psi\rangle)$, suppose we apply $H$ to the first qubit, then measure that qubit in the computational basis. Give the probability of measurement 1 as a function of $|\varphi\rangle$ and $|\psi\rangle$. 