TUTORIAL 2

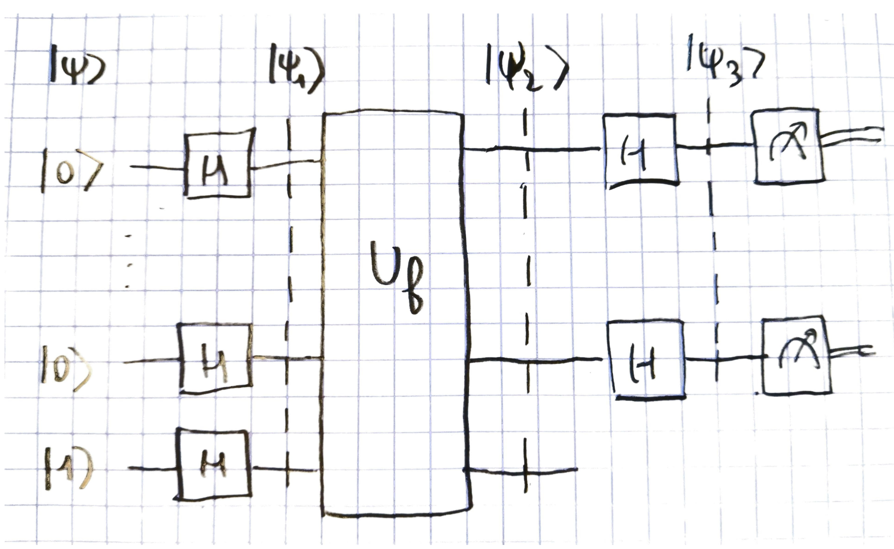
1 Partial Measurements

1. For all $|\varphi\rangle$ and $\mathcal{B}$, measure $|\varphi\rangle$ in $\mathcal{B}$ and determine the post-measurement states and their respective probabilities.
   (a) $|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and $\mathcal{B} = \{I \otimes |+\rangle\langle+|, I \otimes |-\rangle\langle-|\}$.
   (b) $|\varphi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$, and $\mathcal{B} = \{I \otimes |+\rangle\langle+|, I \otimes |-\rangle\langle-|\}$.
   (c) $|\varphi\rangle = 2^{-n/2} \sum_{a\in\{0,1\}^n} |a\rangle|a \text{ MOD } 2\rangle$, and $\mathcal{B}$ = measure the second register in the computational basis.
   (d) $|\varphi\rangle = 2^{-n/2} \sum_{a\in\{0,1\}^n} |a\rangle|a \text{ MOD } 4\rangle$, and $\mathcal{B}$ = measure the second register in the computational basis.
2. Let $|\varphi\rangle, |\psi\rangle$ two normalized quantum states, let $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi\rangle + |1\rangle|\psi\rangle)$, suppose we apply H to the first qubit, then measure that qubit in the computational basis. Give the probability of measurement 1 as a function of $|\varphi\rangle$ and $|\psi\rangle$.

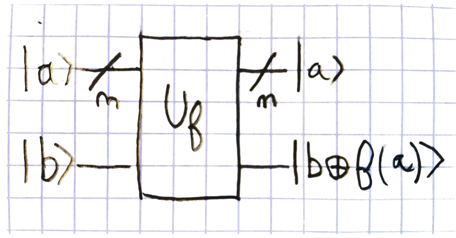
2 Bernstein-Vazirani Problem

The *Bernstein-Vazirani* problem is the following: given a function $f : \{0,1\}^n \to \{0,1\}$ which is of the form $f(x) = a \cdot x := \sum_{i=1}^n a_i \cdot x_i \text{ MOD } 2$ for some unknown $a \in \{0,1\}^n$, the objective is to retrieve the value of a.

Consider the following circuit:



with the gate U_f defined in the following way:



1. Compute the values of $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ (where $|\psi_3\rangle$ is the state corresponding to the first n qubits).
2. How efficiently, in terms of quantum query complexity, can we solve the Bernstein-Vaziran problem?

3 Quantum Random Access Code

We want to encode 2 bits in a single qubit in such a way that we should be able to recover the information about the first bit only or the second bit only with a good probability of success. Formally:

Definition 3.1 (QRAC). *A* quantum random access code *(QRAC) with success probability p is an encoding $f : \{0,1\}^2 \to \mathbb{C}^2$ (maps a pair of bits (x_1, x_2) into a quantum state $|f(x_1, x_2)\rangle$) and two unitaries U_1 and U_2 (U_i is the transformation we apply on the qubit $|f(x_1, x_2)\rangle$ to retrieve bit i) such that for all x_1, x_2:*

$$\mathbb{P}(\textit{Measure output } = x_1 | U_1 \textit{ was applied}) = |\langle x_1 | U_1 | f(x_1, x_2)\rangle|^2 \geq p$$
$$\mathbb{P}(\textit{Measure output } = x_2 | U_2 \textit{ was applied}) = |\langle x_2 | U_2 | f(x_1, x_2)\rangle|^2 \geq p$$

1. A classical random access code depicts the situation where you restrict yourself to encodings $f : \{0,1\}^2 \to \{0,1\}$, ie. encoding two bits into one bit. Then we aim to design a (probabilistic) strategy that, given $f(x_1, x_2)$, recover the information of either x_1 only or x_2 only. Show that such strategies cannot succeed with probability greater than 0.5.
2. How can you geometrically interpret a real unitary acting on real qubits?
3. Exhibit a quantum random access code with $p > 0.5$. What is the best you can achieve?
Hint : Plot the space of real qubits, and try to dispatch them the furthest appart, in such a way that two distinct unitaries can distinghuish them efficiently