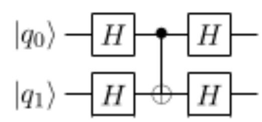
TUTORIAL 3

1 Assignment 1

1.1 Circuit Simplification

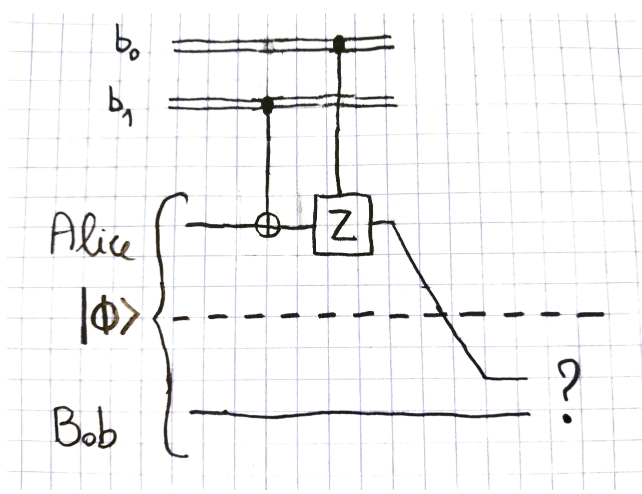
Simplify the following circuit:



1.2 Superdense Coding

In this exercise, we will have two actors, Alice and Bob. Alice has two classical bits of information $(b_0, b_1) \in \{0,1\}^2$ and she want to send them to Bob.

1. (informal) What is the minimum number of classical bits that Alice has to send to Bob in order to communicate him (b_0, b_1)?
2. Now suppose that Alice and Bob share an entangled **EPR pair** (or **Bell pair**), that is to say there is a quantum state $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ such that the first qubit is owned by Alice (she can only perform operations of the form $U \otimes I$ on $|\phi\rangle$) and the second qubit is owned by Bob.
   Alice is going to perform operations on her qubit and send it to Bob. If $b_1 = 1$, she applies X (bit flip) and then if $b_0 = 1$ she applies Z (phase flip), she send her part of the qubit back to Bob.
   Explain how Bob can recover the values of b_0 and b_1 using $|\phi\rangle$.



2 Hadamard $H^{\otimes n}$

For any $y \in \{0,1\}^n$, compute $H^{\otimes n}|y\rangle$.

3 Are Circuits Ambiguous?

1. Let $|\phi\rangle$ be a 2-qubits state. A gate A is applied to the first qubit, and then a gate B is applied to the second qubit. Show that the gate B could have been applied before the gate A.

2. Let $|\phi\rangle$ be a 2-qubits state. Check that measuring the first qubit, then the second one gives the same result as measuring the second qubit, then the first.

3. Let $|\phi\rangle$ be a $n+1$-qubits state. A gate U is applied to the first n qubits, and then the last qubit is measured. Show that measuring the last qubit and then applying the gate U on the first n qubits gives the same result.

4 Unitary Approximation

The goal of the exercise is to define what are approximations of unitaries, and show that it is relevant in the sense that it will give roughly the same outcomes when composed and measured.

Recall that the norm of an operator A (the so-called *operator norm*) is defined as:

$$\|A\| := \sup_{|\psi\rangle \text{NOT}=0} \frac{\|A|\psi\rangle\|_2}{\||\psi\rangle\|_2} .$$

Furthermore, we will say that the unitary $\tilde{U}$ approximates U with precision δ if:

$$\|\tilde{U} - U\| \leq \delta .$$

1. Show that $\|\cdot\|$ is indeed a norm, and furthermore that it satisfies:

$$\|AB\| \leq \|A\|\|B\| .$$

2. Show that if $\|\tilde{U} - U\| \leq \delta$, then $\|\tilde{U}^{-1} - U^{-1}\| \leq \delta$.

3. Show that if each U_i is approximated by $\tilde{U}_i$ with precision δ_i, then:

$$\|\tilde{U}_L\tilde{U}_{L-1}\ldots\tilde{U}_2\tilde{U}_1 - U_L U_{L-1}\ldots U_2 U_1\| \leq \sum_{j=1}^{L} \delta_j .$$

4. We say that U computes a binary function $F(x)$ with precision ε if:

$$\forall x \in \{0,1\}^n, |\langle F(x)|U|x\rangle|^2 \geq 1 - \varepsilon .$$

   Show that if that U is approximated by $\tilde{U}$ with precision δ, then $\tilde{U}$ computes $F(x)$ with precision $\epsilon + 2\delta$.
   Hint: Note that for an operator A, you have $|\langle x|A|x\rangle| \leq \|A\|$.

5 Quantum 1-Machine

You have a device that outputs only $|0\rangle$. You can use this device several times, and measure in any basis. How many calls to the device do you need to get a $|1\rangle$?