Duality methods for variational inequalities and Non-Newtonian fluid mechanics

Enrique Fernández-Nieto[⋆], Paul Vigneaux[•]

[⋆]Dpto. Matemática Aplicada I, Universidad de Sevilla
[•]UMPA, Ecole Normale Supérieure de Lyon – France

PhD Course IMUS
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This PhD Course at IMUS is made of two parts :

October 2013 – Paul Vigneaux

- Introduction and Course's motivation :
- Non-Newtonian Fluid Mechanics. Bingham flows.
- Formulations with variational inequalities
- Minimization, Lagrangians and duality

February 2014 – Enrique Fernández-Nieto

- Augmented Lagrangian methods for numerical resolution
- Bermudez-Moreno algorithm for numerical resolution
- Theoretical derivation of optimal parameters
- Space discretization. Effective implementation.

Following slides are part of the material used for the Course's motivation.

Course's motivation and introduction

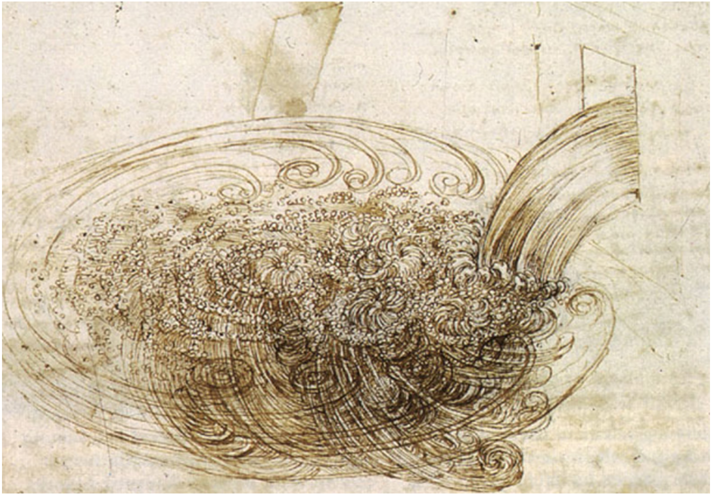
1. Motivations from natural/physical phenomena
2. Reminder : Newtonian case & Navier-Stokes
3. Generalization : Non-Newtonian
4. The Bingham case in more details

Course's motivation and introduction

The "first" flow :

When one thinks about fluid flows,

first example which usually comes to mind is **flow of water**



Leonardo da Vinci. In *Codex Atlanticus*, drawings and writings from $\sim$1478 to $\sim$1518.

The “first” flow :

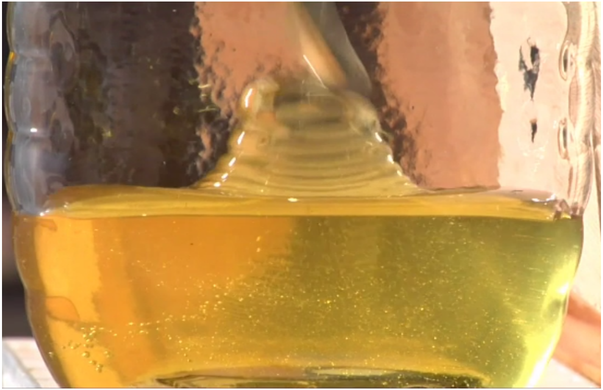
Though **liquids such water** can exhibit very complex flows :

- e.g. turbulence,
- topological changes (free surface flows), etc

these liquids can be described by what is known as the most **simple behaviour law** : a *linear law* (explained later ☺)

Fluids more “complicated” than water : 1

For instance, let us have a look at this flow of honey : ▸ Movie



one can “feel” that this material is more “viscous” (again, a concept to be defined later ☺) and that some **flow features** (actually linked to ***non linear behaviour laws***) can not be encountered with water

Fluids more “complicated” than water : 2a

In other cases, “viscosity” can depend on “shear” applied on the fluid :

- Corn flour (Maizena) ▸ Movie
- mix gently : very “fluid”
- mix strong : very “viscous”

Source : Blkutter 2009, Wikimedia.

Same kind of feeling with sand/water at low tide on a beach : do not strike too hard, it can hurt ☺

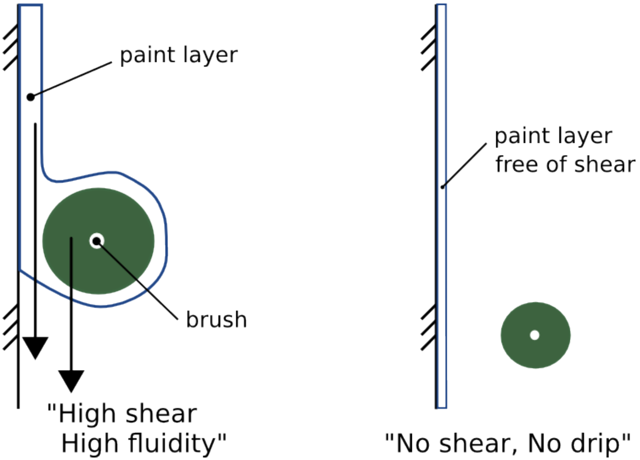
“viscosity” ↗ with “shear”

Fluids more “complicated” than water : 2b

More commonly, one encounters : **“viscosity”** ↘ **with “shear”**

Very useful for wall paint :

- high shear when spread paint on wall : easy painting
- no shear when pulling out the brush : no drip



Fluids more "complicated" than water : 3

Another striking effect is **"the ability to NOT flow"** :

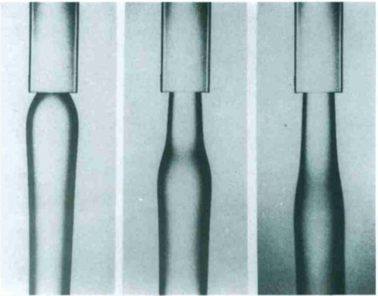
- toothpaste, mayonnaise, etc
- If you gently turn an *open tube* upside down
- ... material stays in the tube.
- $\rightarrow$ **Need to shake/squeeze the tube to let it flow**

Conversely, some geophysical materials flow and then stop with a shape different from water surface at rest :

- Snow avalanches Movie
- Mud flows (after heavy rains)
- Lava flows (+ thermal effects) Movie

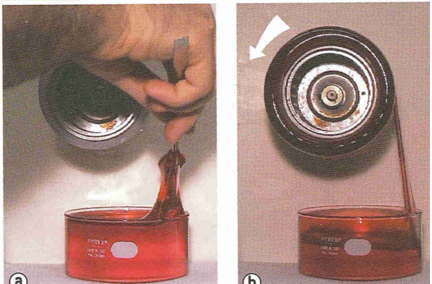
Fluids more “complicated” than water : etc

∃ other “exotic” situations not described here, including



H. Gieskus. Rheologica Acta 8, 1968, 411-421

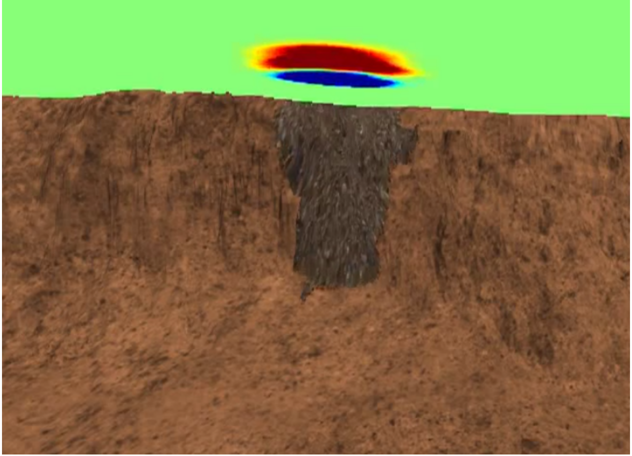
G. McKinley et al. NNFDRG @ MIT. 2005



A local example : Alboran Sea

Submarine landslide triggers a tsunami,

▸ Movie by EDANYA team, Malaga :



Non-Newtonian models are of interest for these materials.

Course's motivation and introduction

Newtonian case & Navier-Stokes

Recall the (Eulerian, incompressible) Navier-Stokes equations :

- $u(x_1, x_2, x_3; t)$: velocity, $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$
- $p(x_1, x_2, x_3; t)$: pressure, $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$
- $f(x_1, x_2, x_3; t)$: body forces, $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$

$$0 = \nabla . u = div(u) = \sum_i \frac{\partial u_i}{\partial x_i} \quad \text{: mass conservation} \tag{1}$$

$$u.\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$D(u) = \frac{\nabla u + (\nabla u)^t}{2} \ : \ \text{rate of deformation tensor}$$

$$\underbrace{\rho\left(\partial_t u + u.\nabla u\right)}_{inertia} - \underbrace{2\eta\ \nabla . D(u)}_{viscous} + \underbrace{\nabla p}_{pressure} = \underbrace{f}_{body\ force} \tag{2}$$

Rk : Eulerian vs Lagrangian and Convected derivative $D_t u$

Navier-Stokes asymptotics

Depending on the Reynolds number $Re = \frac{\rho U L}{\eta} \equiv \frac{\textit{inertia effects}}{\textit{viscous effects}}$ we have formally

- $Re \to \infty$: Euler equations

$$\rho \left(\partial_t u + u.\nabla u\right) + \nabla p = f \tag{3}$$

- $Re \to 0$: Stokes equations

$$-2\eta \,\nabla . D(u) + \nabla p = f \tag{4}$$

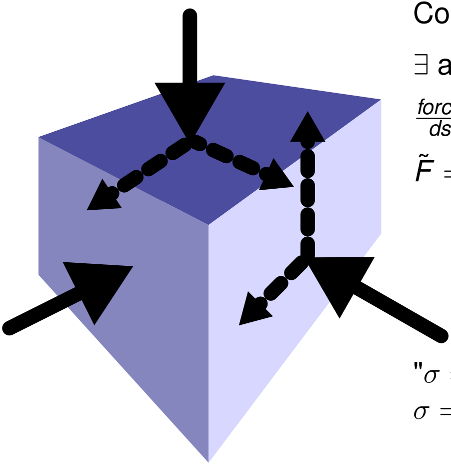
Rk : transit^{o} to turbulence : laminar, recirculation, Von Karman
Rk : No compressible effects here
Rk : No energy equation here taken into account

More on the momentum equations (say, Stokes)

We need to recall a bit the origin of viscous/pressure

Newton's 2nd law : acceleration $= \sum$ *forces*

To write it, we consider an elementary volume *dx dy dz* (à la Batchelor) and we look for all the external forces :



Continuum Mechanics ensures :

$\exists$ a **stress tensor** σ s. t.

$\frac{\textit{force}}{\textit{ds}}$ *on a surface* of normal *n* is

$\tilde{F} = \sigma . n$ with Cauchy tensor

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

"$\sigma =$ tangential $+$ normal stress"

$\sigma = \underbrace{\sigma'}_{\textit{viscous}} \underbrace{-p\,\mathrm{Id}}_{\textit{pressure}}$, $\sigma' = \sigma'(D(u))$

The viscous stress tensor (Newtonian version)

Invoking again Continuum Mechanics :

$$\sigma' = 2\eta \, D(u) \tag{5}$$

:= **constitutive (or behaviour) equation.** It's **linear** ! Previous

Viscosity := link between stress and rate-of-deformation

Go back to momentum conservation (following a REV $\mathcal{V}$)

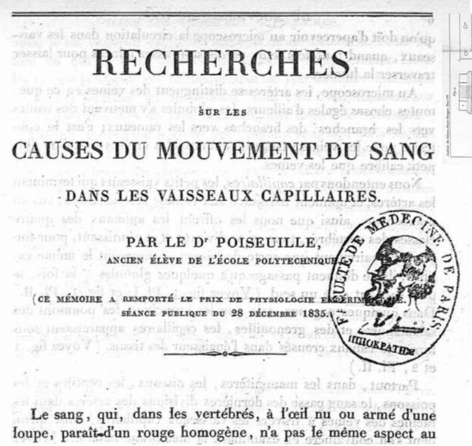
$$\frac{D}{Dt}\left[\int_{\mathcal{V}} \rho u \, dv\right] = \int_{\mathcal{V}} \rho f \, dv + \int_{\partial\mathcal{V}} \sigma.n \, ds, \quad \text{and use Div Thm} \tag{6}$$

$$\int_{\mathcal{V}} \rho \frac{Du}{Dt} \, dv = \int_{\mathcal{V}} \rho f \, dv + \int_{\mathcal{V}} \nabla.\sigma \, dv, \;\; \text{for every } \mathcal{V}, \tag{7}$$

which leads to the previous Navier-Stokes equations, since :

$$\nabla.\sigma = \nabla.(\sigma' - p\,\mathrm{Id}) = 2\eta \, \nabla.D(u) - \nabla p. \tag{8}$$

Blackboard interlude : a "1D case", Poiseuille flow (1835)



RECHERCHES

SUR LES

CAUSES DU MOUVEMENT DU SANG

DANS LES VAISSEAUX CAPILLAIRES.

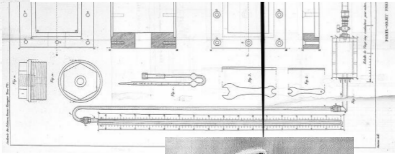
PAR LE D^r^ POISEUILLE,

ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE.

(CE MÉMOIRE A REMPORTÉ LE PRIX DE PHYSIOLOGIE EXPÉRIMENTALE.

SÉANCE PUBLIQUE DU 28 DÉCEMBRE 1835.)

Le sang, qui, dans les vertébrés, à l'œil nu ou armé d'une loupe, paraît d'un rouge homogène, n'a pas le même aspect au

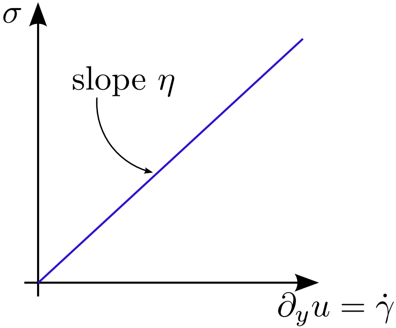


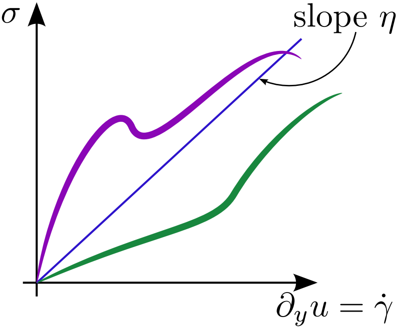
Course’s motivation and introduction

Why linear ?

We know that σ' is a function of $D(u)$ [convenience $\sigma' \rightsquigarrow \sigma$]

But is this only linear ? Can it be more general ?





Actually, physical examples of Part 1 are possible due to this type of non linearity, in particular $\eta = \eta(D(u))$ **is variable** ! ! !

$\Rightarrow$ these are so called **Generalized Newtonian fluids**

A bit of history : rheology

Study of deviation (from the linear law) σ' as a function of $D(u)$

belongs to the field called **Rheology**, studying *complex fluids*,

more precisely the **deformation and flow of matter.**

Term. due to E. C. **Bingham in 1929**, from greek "$\rho\varepsilon\omega$-"to flow"

Years "around" 1900's saw a significant increase of activity on these subjects, including :

Maxwell (1868), Boltzmann (1877), Bingham, Blair, Reiner, Herschel-Bulkley, Weissenberg (all between 1900–1930) ...

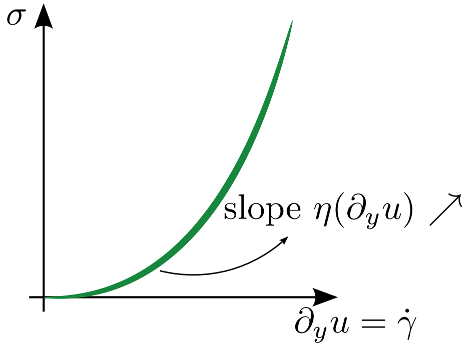
Then became a full field of activity, more vibrant today than ever

Revisiting examples : corn flour

Corn flour is a material s.t.

constitutive eq. is like $\rightarrow$

:= **shear-thickening**

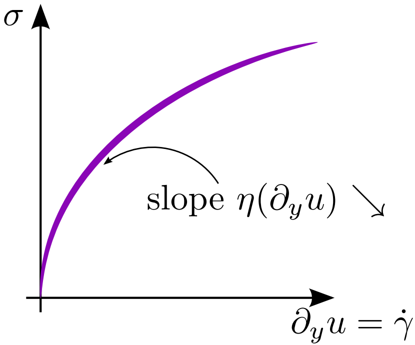


Revisiting examples : paint

Paint is a material s.t.

constitutive eq. is like $\rightarrow$

:= **shear-thinning**



More precisely, such shear-thinning/shear-thickening fluids

are called **power-law fluids** (indeed).

Revisiting examples : "threshold" fluids

And what about **mayonnaise**, snow, lava, etc ? ? ?

Revisiting examples : "threshold" fluids

And what about **mayonnaise**, snow, lava, etc ? ? ?

Below a stress τ_c,

nothing happens : $\dot{\gamma} = 0$,
rigid/no deformation.

Above τ_c,

material begins to flow ! ! !
$\dot{\gamma} > 0$,

Revisiting examples : "threshold" fluids

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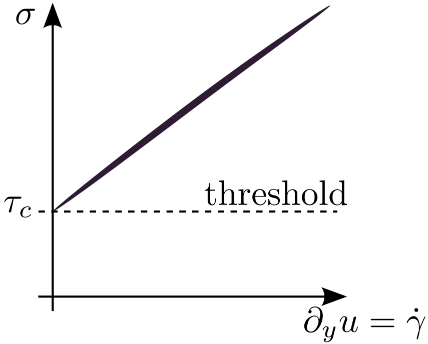


Constitutive equation :

$$\begin{cases} |\sigma| < \tau_c & \text{if} \quad |D(u)| = 0 \\ \sigma = 2\eta D(u) + \tau_c \frac{D(u)}{|D(u)|} & \text{if} \quad |D(u)| \neq 0 \end{cases}$$

"threshold" fluids : a bit of history

1916 ; 1922 $\rightarrow$

FLUIDITY AND PLASTICITY

BY

EUGENE C. BINGHAM, PH.D.

PROFESSOR OF CHEMISTRY AT LAFAYETTE COLLEGE, EASTON, PENNSYLVANIA

Plasticity - The origin

*"We may now define **plasticity** as a property of solids in virtue of which they hold their shape permanently under the action of small shearing stresses but they are readily deformed, worked or molded, under somewhat larger stresses. [...]"*

for a series of fluids a family of straight lines passing out from the origin as illustrated in Fig. 76.

In a plastic solid, a certain portion of the shearing force is used up in overcoming the *internal friction* of the material. If the stress is just equal to the friction or yield value, the material may be said to be at its elastic limit. If the stress is greater than the friction f, the excess, $F - f$, will be used up in producing plastic flow according to the formula

$$dv = \mu \,(F - f)\; dr \tag{73}$$

where μ is a constant which we will call the *coefficient of mobility*

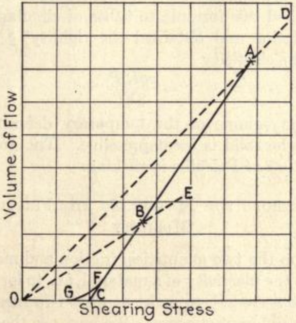
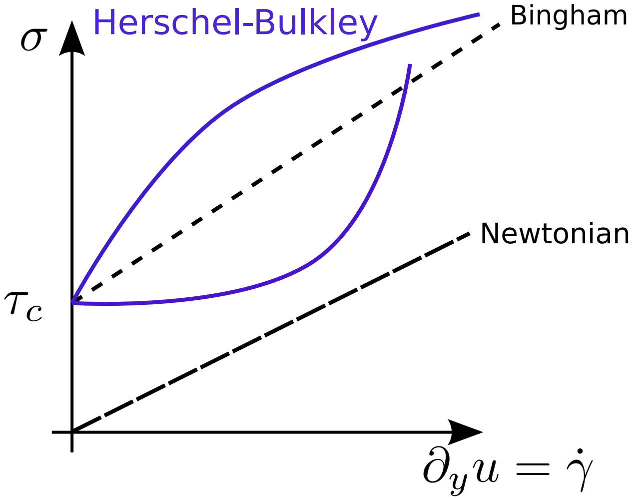


FIG. 77.—Flow-shear diagram of a plastic solid.

The general case ...

... combines **power-law and plasticity**

:= **Herschel-Bulkley** constitutive equation



Course's motivation and introduction

Reminder : Poiseuille flow

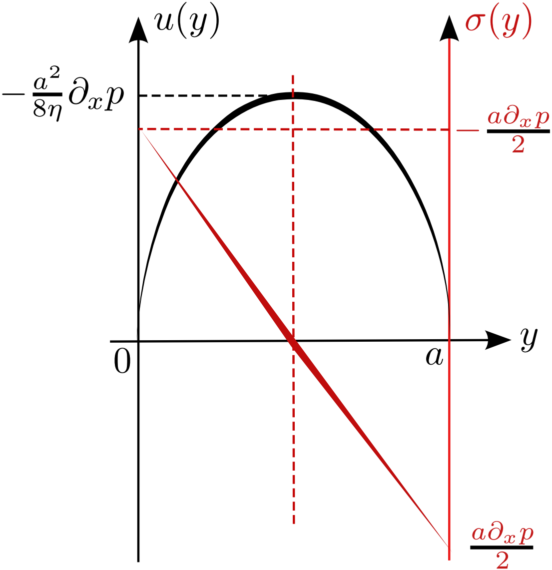
Stress goes to zero

at channel's center :

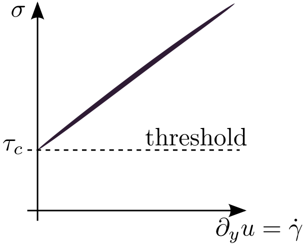
opens the door

for a central **plug** :=

moving plastic zone



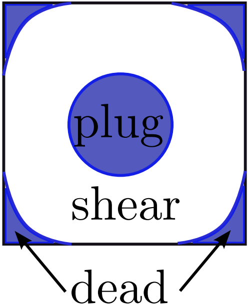
Blackboard interlude : 1D, Poiseuille-Bingham flow

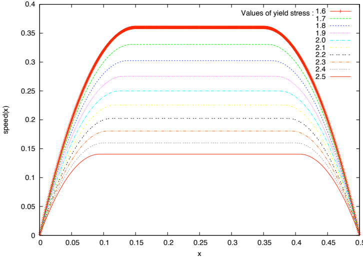


The square channel case

Phenomenology : presence of

- “plug” zones : motion with constant velocity
- “dead” zones : no motion
- “shear” zones : material deforms





Let's roll on' !

- Variational inequalities
- Minimization with constraints
- Duality Methods
- Spatial discretization
- ...

References

Papers and Books

- G.K. Batchelor. An introduction to fluid dynamics. Cambridge University Press (2000)
- P. Oswald. Rheophysics. The Deformation and Flow of Matter. Cambridge University Press (2009)
- R.I. Tanner, K. Walters. Rheology : an historical perspective. Elsevier (1998)

References
Videos

- Honey extraction :
  http ://www.youtube.com/watch ?v=5XO1FEZhESk
- Corn flour :
  http ://www.youtube.com/watch ?v=XEEs4eoRPGw
- Snow avalanche (France) :
  https ://www.youtube.com/watch ?v=IlzT6VjXw-E
- Lava Flow (Hawaii) :
  http ://www.youtube.com/watch ?v=HpNWLCmXyTE
- Alboran (Spain) :
  http ://www.youtube.com/user/grupoedanya