Multiscale Geometric Analysis: Theory, Applications, and Opportunities

Emmanuel Candès, California Institute of Technology

Wavelet And Multifractal Analysis 2004, Cargese, July 2004

Collaborators

David Donoho (Stanford) Laurent Demanet (Caltech)

Classical Multiscale Analysis

- Wavelets: Enormous impact
  - Theory
  - Applications
  - Many success stories
- Deep understanding of the fact that wavelets are not good for all purposes
- Consequent constructions of new systems lying beyond wavelets

Overview

- Other multiscale constructions
- Problems classical multiscale ideas do not address effectively

Fourier Analysis

- Diagonal representation of shift-invariant linear transformations; e.g. solution operator to the heat equation

$$\partial_t u = a^2 \Delta u$$

  Fourier, *Théorie Analytique de la Chaleur*, 1812.

- Truncated Fourier series provide very good approximations of smooth functions

$$\|f - S_n(f)\|_{L_2} \leq C \cdot n^{-k},$$

  if $f \in C^k$.

Limitations of Fourier Analysis

- Does not provide any sparse decomposition of differential equations with variable coefficients (sinusoids are no longer eigenfunctions)
- Provides poor representations of discontinuous objects (Gibbs phenomenon)

Wavelet Analysis

- Almost eigenfunctions of differential operators

$$(Lf)(x) = a(x)\partial_x f(x)$$

- Sparse representations of piecewise smooth functions

Wavelets and Piecewise Smooth Objects

- 1-dimensional example:

$$g(t) = 1_{\{t > t_0\}} e^{-(t-t_0)^2}.$$

- Fourier series:

$$g(t) = \sum c_k e^{ikt}$$

Fourier coefficients have slow decay:

$$|c|_{(n)} \geq c \cdot 1/n.$$

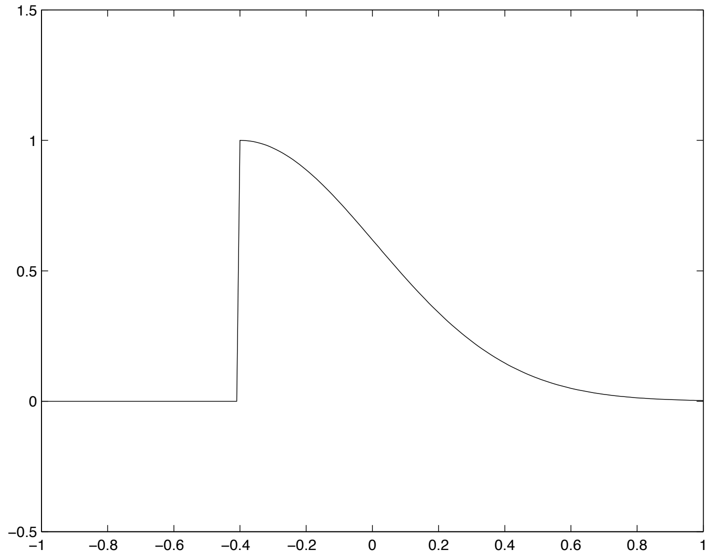
- Wavelet series:

$$g(t) = \sum \theta_\lambda \psi_\lambda(t)$$

Wavelet coefficients have fast decay:

$$|\theta|_{(n)} \leq c \cdot (1/n)^r \quad \text{for any } r > 0.$$

as if the object were non-singular



'Wavelets and Operators'

Calderòn-Zygmund Operator

$$(\boldsymbol{Kf})(\boldsymbol{x}) = \int \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\,\boldsymbol{dy}$$

- $\boldsymbol{K}$ singular along the diagonal
- $\boldsymbol{K}$ smooth away from the diagonal

- Sparse representation in wavelet bases (Calderòn, Meyer, etc.)
- Applications to numerical analysis and scientific computing (Beylkin, Coifman, Rokhlin)

Our Viewpoint

- Sparse representations of point-singularities
- Sparse representation of certain matrices
- *Simultaneously*
- Applications
  - Approximation theory
  - Data compression
  - Statistical estimation
  - Scientific computing
- More importantly: new mathematical architecture where information is organized by scale and location

New Challenges

- Intermittency in higher-dimensions
- Evolution problems
- CHA has not addressed these problems.

Agenda

- Limitations of existing image representations
- Curvelets: geometry and tilings in Phase-Space
- Representation of functions, signals
- Representation of operators, matrices

Three Anomalies

- Inefficiency of Existing Image Representations
- Limitations of Existing Pyramid Schemes
- Limitations of Existing Scaling Concepts

I: Inefficient Image Representations

Edge Model: Object $f(x_1, x_2)$ with discontinuity along generic C^2 smooth curve; smooth elsewhere.

Fourier is awful

Best m-term trigonometric approximation $\tilde{f}_m$

$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-1/2}, \quad m \to \infty$$

Wavelets are bad

Best m-term approximation by wavelets:

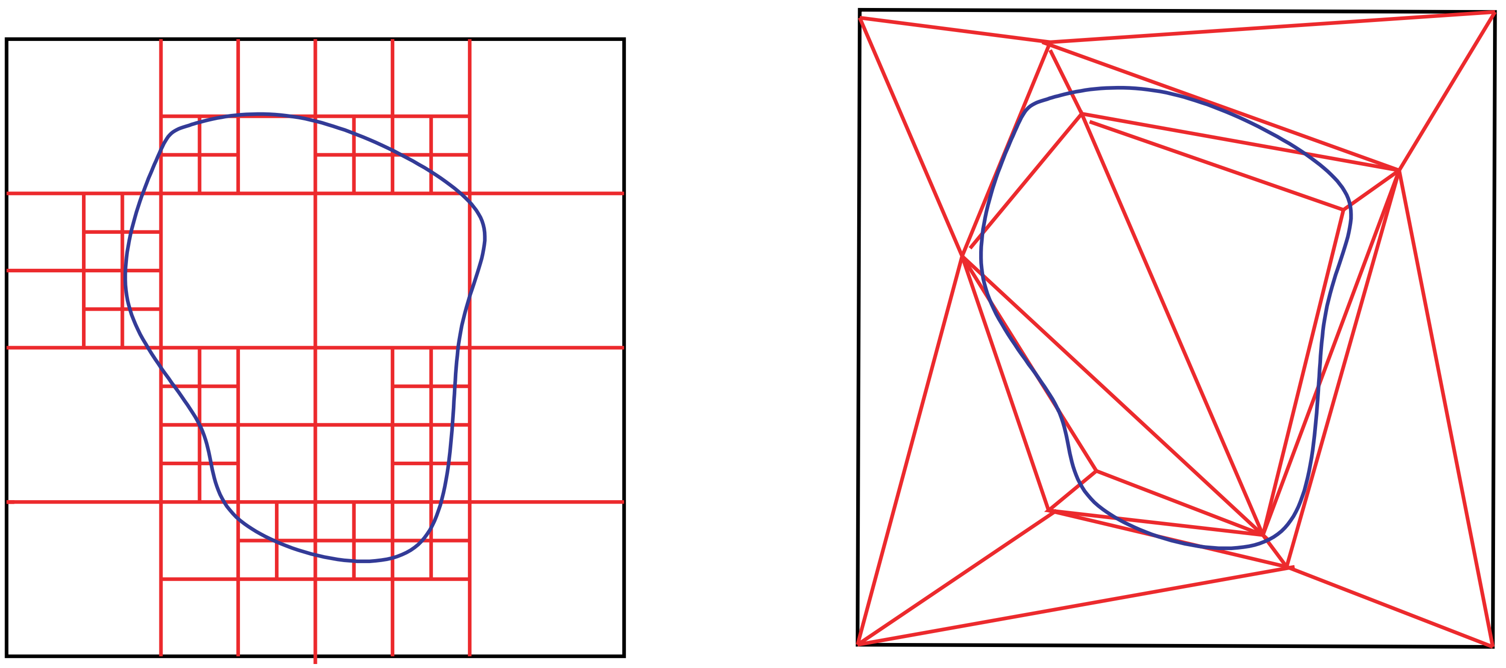
$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-1}, \quad m \to \infty$$

Optimal Behavior

There is a ‘dictionary’ of ‘atoms’ with best m-term approximant $\tilde{f}_m$

$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-2}, \quad m \to \infty$$

- No basis can do better than this.
- No depth-search limited dictionary can do better.
- No pre-existing basis does anything near this well.



(a) Wavelets (b) Triangulations

II. Limitations of Existing Pyramids

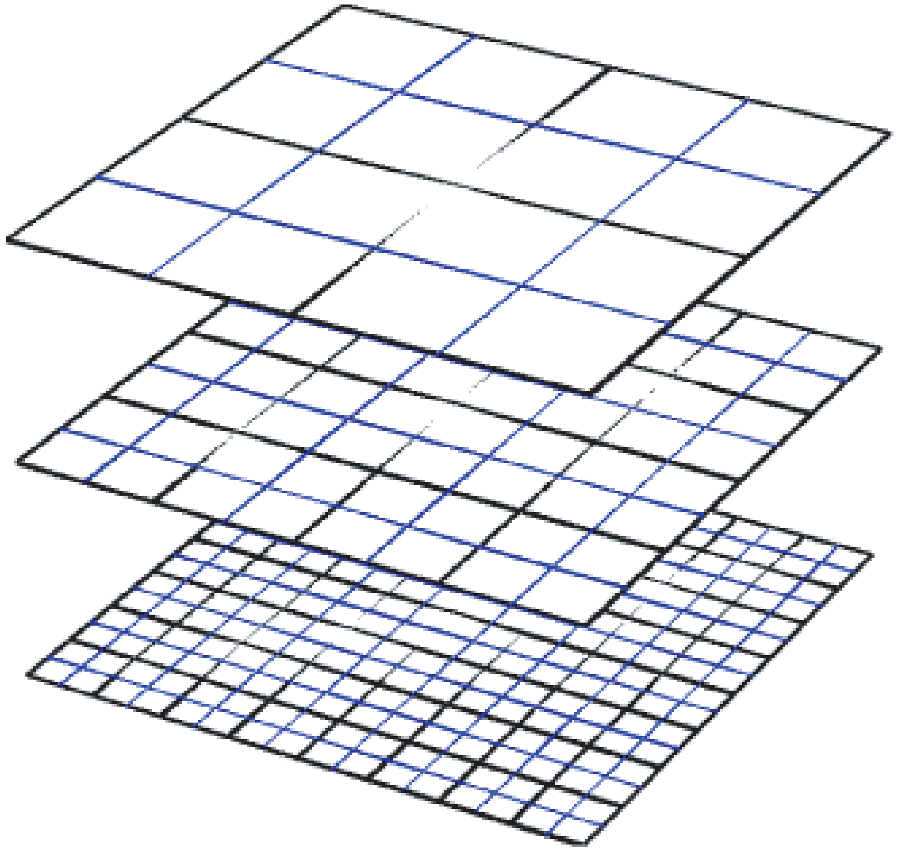
Canonical Pyramid Ideas (1980-present)

- Laplacian Pyramid (Adelson/Burt)
- Orthonormal Wavelet Pyramid (Mallat/Meyer)
- Steerable Pyramid (Adelson/Heeger/Simoncelli)
- Multiwavelets (Alpert/Beylkin/Coifman/Rokhlin)

Shared features

- Elements at dyadic scales/locations
- FIXED Number of elements at each scale/location

Wavelet Pyramid



Limitations of Existing Scaling Concepts

Traditional Scaling

$$f_a(x_1, x_2) = f(ax_1, ax_2), \qquad a > 0.$$

Curves exhibit different kinds of scaling

- Anisotropic
- Locally Adaptive

If $f(x_1, x_2) = 1_{\{y \geq x^2\}}$ then

$$f_a(x_1, x_2) = f(a \cdot x_1, a^2 x_2)$$

In Harmonic Analysis called Parabolic Scaling.

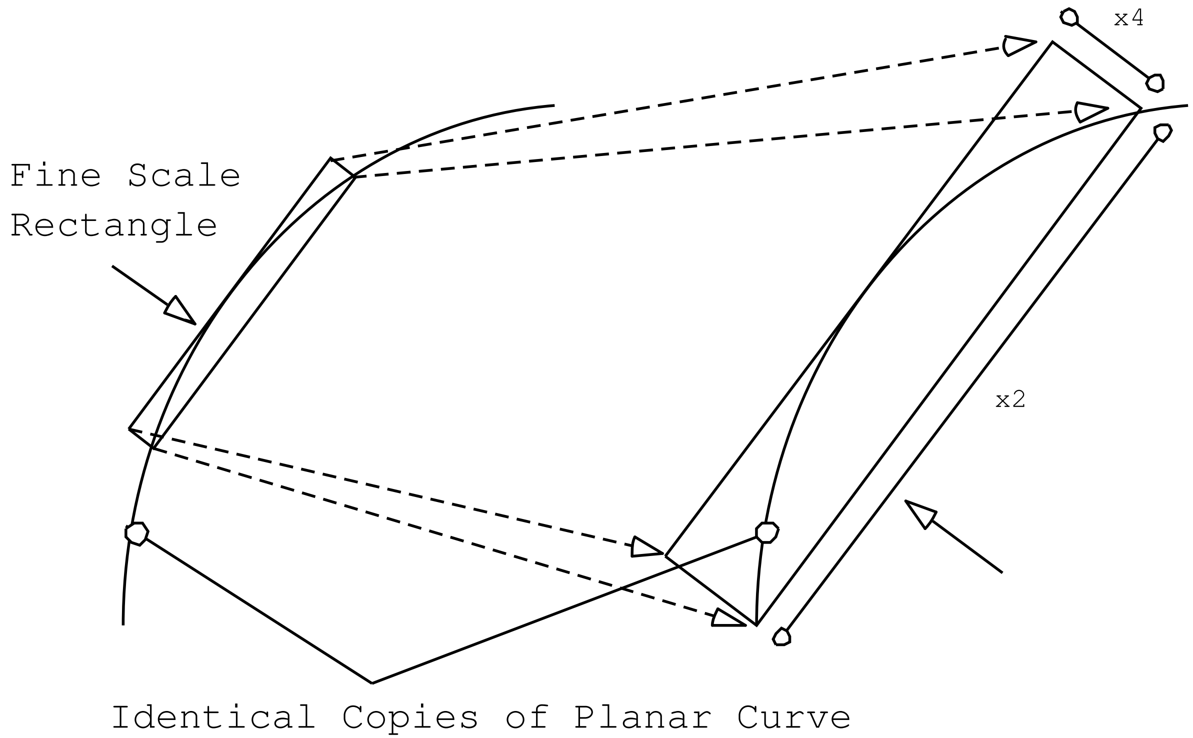


Figure 1: Curves are Invariant under Anisotropic Scaling

Curvelets

C. and Guo, 2002.

New multiscale pyramid:

- Multiscale
- Multi-orientations
- *Parabolic (anisotropy) scaling*

$$width \approx length^2$$

Earlier construction, C. and Donoho (2000)

Philosophy (Slightly Inaccurate)

- Start with a waveform $\varphi(x) = \varphi(x_1, x_2)$.
  - oscillatory in x_1
  - lowpass in x_2
- *Parabolic* rescaling

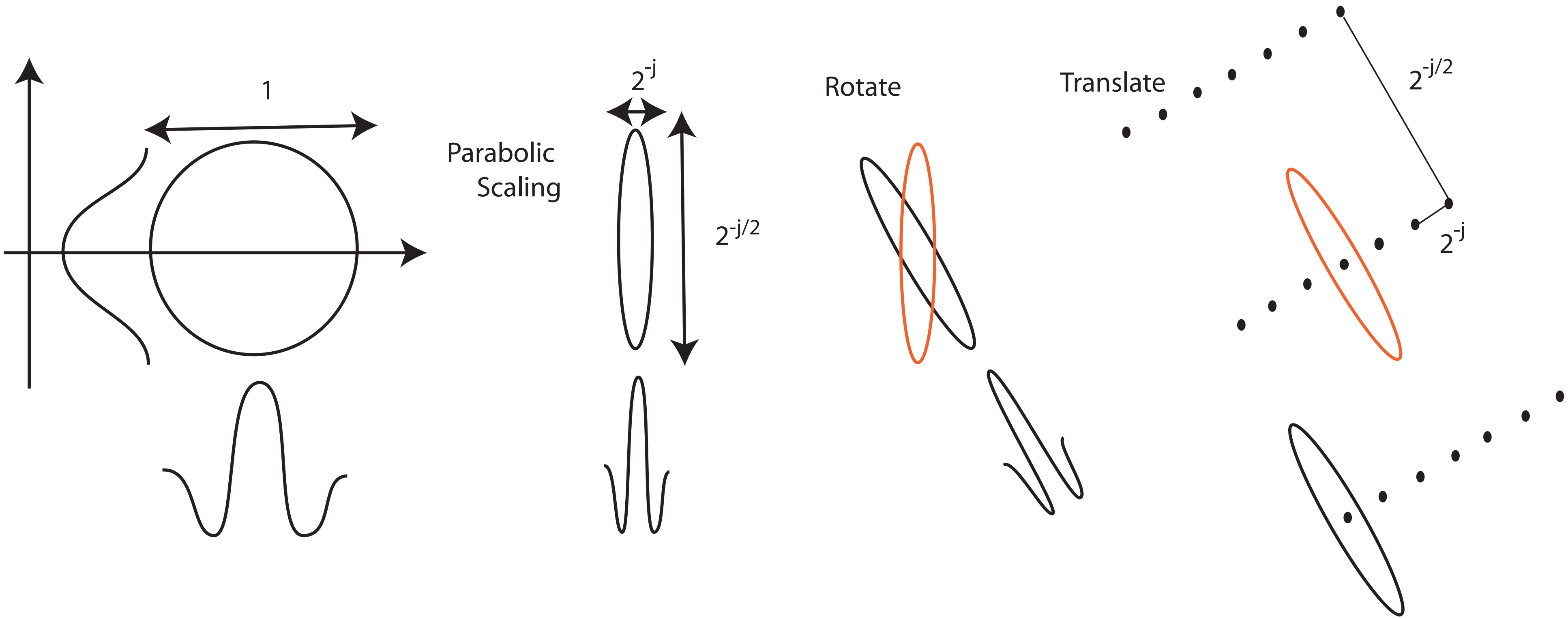
$$|D_j| \varphi(D_j x) = 2^{3j/4} \varphi(2^j x_1, 2^{j/2} x_2), \quad D_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \; j \geq 0$$

- Rotation (scale dependent)

$$2^{3j/4} \varphi(D_j R_{\theta_{j\ell}} x), \quad \theta_{j\ell} = 2\pi \cdot \ell 2^{-\lfloor j/2 \rfloor}$$

- Translation (oriented Cartesian grid with spacing $2^{-j} \times 2^{-j/2}$);

$$2^{3j/4} \varphi(D_j R_{\theta_{j\ell}} x - k), \quad k \in \mathbf{Z}^2$$


1
Parabolic
Scaling
2^{-j}
$2^{-j/2}$
Rotate
Translate
$2^{-j/2}$
2^{-j}

Curvelet Construction, I

Decomposition in space and angle

- 'Orthonormal' partition of unity (scale): smooth pair of windows (w_0, w_1)

$$w_0^2(r) + \sum_{j=0}^{\infty} w_1^2(2^{-j} r) = 1, \qquad r > 0.$$

- 'Orthonormal' partition of unity (angle):

$$\sum_{\ell=-\infty}^{\infty} v^2(t - \ell) = 1, \qquad t \in \mathbb{R}.$$

- For each pair $J = (j, \ell)$, define the two-dimensional window

$$W_J(\xi) = w_1(2^{-j}|\xi|) \cdot v(2^{\lfloor j/2 \rfloor}(\theta - \theta_{j,\ell})), \quad \theta_{j,\ell} = \pi \cdot \ell \cdot 2^{-\lfloor j/2 \rfloor}.$$

Note

$$W_{j,\ell}(\xi) = W_{j,0}(R_{\theta_{j,\ell}} \xi)$$

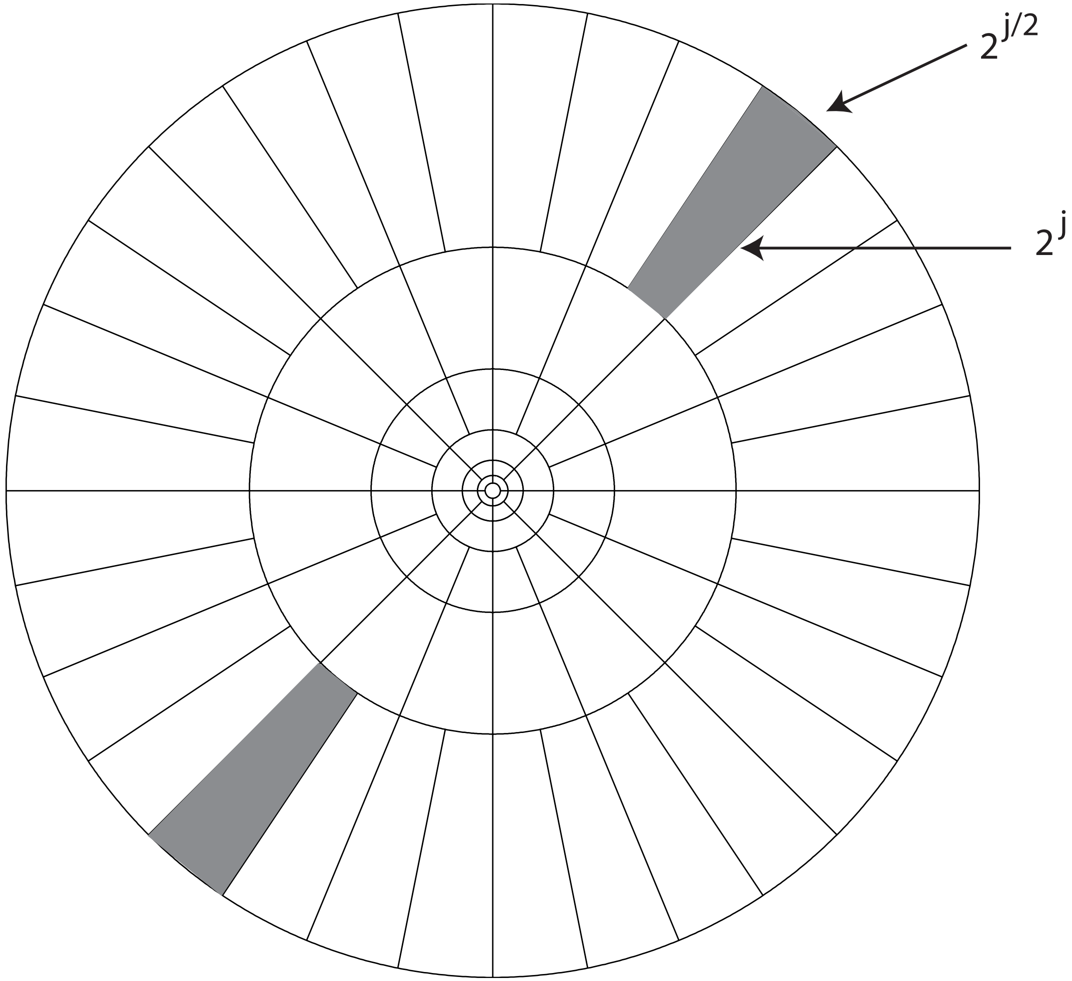
- W_J is an orthonormal partition of unity

$$\sum_J W_J^2(\xi) = 1.$$

Interpretation

- Divide frequency domain into annuli $|x| \in [2^j, 2^{j+1})$
- Subdivide Annuli into wedges. Split every other scale.
- Oriented local cosines on wedges

Partition: Frequency-side Picture



Curvelet Construction, II

Local bases

- Fix scale/angle pair (j, ℓ)

$$\text{supp} W_{j,\ell} \subset D_{j,\ell} = R_{\theta_{j,\ell}}^{-1} D_{j,0}$$

where $D_{j,0}$ is a rectangle of sidelength $2\pi 2^j \times 2\pi 2^{j/2}$, say.

- Rectangular grid Λ_j of step-size $2^{-j} \times 2^{-j/2}$

$$\Lambda_j = \{k : k = (k_1 2^{-j}, k_2 2^{-j/2}), k_1, k_2 \in \mathbb{Z}\}$$

- Local Fourier basis of $L_2(D_{j,0})$

$$e_{j,0,k}(\xi) = \frac{1}{2\pi 2^{3j/4}} e^{ik\xi}, \quad k \in \Lambda_j$$

and likewise local Fourier basis of $L_2(D_{j,\ell})$

$$e_{j,\ell,k} = e_{j,0,k}(R_{\theta_{j,\ell}} \xi)$$

- Curvelets (frequency-domain definition)

$$\hat{\varphi}_{j,\ell,k}(\xi) = W_{j,0}(R_{\theta_{j,\ell}}\xi)e_{j,0,k}(R_{\theta_{j,\ell}}\xi)$$

Properties

- Reproducing formula

$$f = \sum_{J,k} \langle f, \varphi_{J,k} \rangle \varphi_{J,k}$$

Why? Let g be the RHS and apply Parseval

$$\begin{aligned} \hat{g} &= \sum_{J,k} \langle \hat{f}, \hat{\varphi}_{J,k} \rangle \hat{\varphi}_{J,k} \\ &= \sum_{J,k} \langle \hat{f}, W_J e_{J,k} \rangle W_J e_{J,k} \\ &= \sum_J W_J \sum_k \langle \hat{f} W_J, e_{J,k} \rangle e_{J,k} \\ &= \sum_J \hat{f} W_J W_J = \hat{f} \sum_J W_J^2 = \hat{f} \end{aligned}$$

Conclusion $\hat{g} = \hat{f}$ and, therefore, $f = g$.

- Parseval relation (similar argument)

$$\|f\|^2 = \sum_{J,k} |\langle f, \varphi_{J,k} \rangle|^2$$

Curvelet: Space-side Viewpoint

In the frequency domain

$$\hat{\varphi}_{j,0,k}(\xi) = \frac{2^{-3j/4}}{2\pi} W_{j,0}(\xi) e^{i\langle k,\xi\rangle}, \quad k \in \Lambda_j$$

In the spatial domain

$$\varphi_{j,0,k}(x) = 2^{3j/4}\varphi_j(x-k), \quad W_{j,0} = 2\pi\hat{\varphi}_j$$

and more generally

$$\varphi_{j,\ell,k}(x) = 2^{3j/4}\varphi_j(R_{\theta_{j,\ell}}(x - R^{-1}_{\theta_{j,\ell}} k)),$$

All curvelets at a given scale are obtained by translating and rotating a single 'mother curvelet.'

Further Properties

- Tight frame

$$f = \sum_{j,\ell,k} \langle f, \varphi_{j,\ell,k} \rangle \varphi_{j,\ell,k} \qquad ||f||_2^2 = \sum_{j,\ell,k} \langle f, \varphi_{j,\ell,k} \rangle^2$$

- *Geometric* Pyramid structure
  - Dyadic scale
  - Dyadic location
  - Direction
- New tiling of phase space

- Needle-shaped
- Scaling laws
  - $length \sim 2^{-j/2}$
  - $width \sim 2^{-j}$

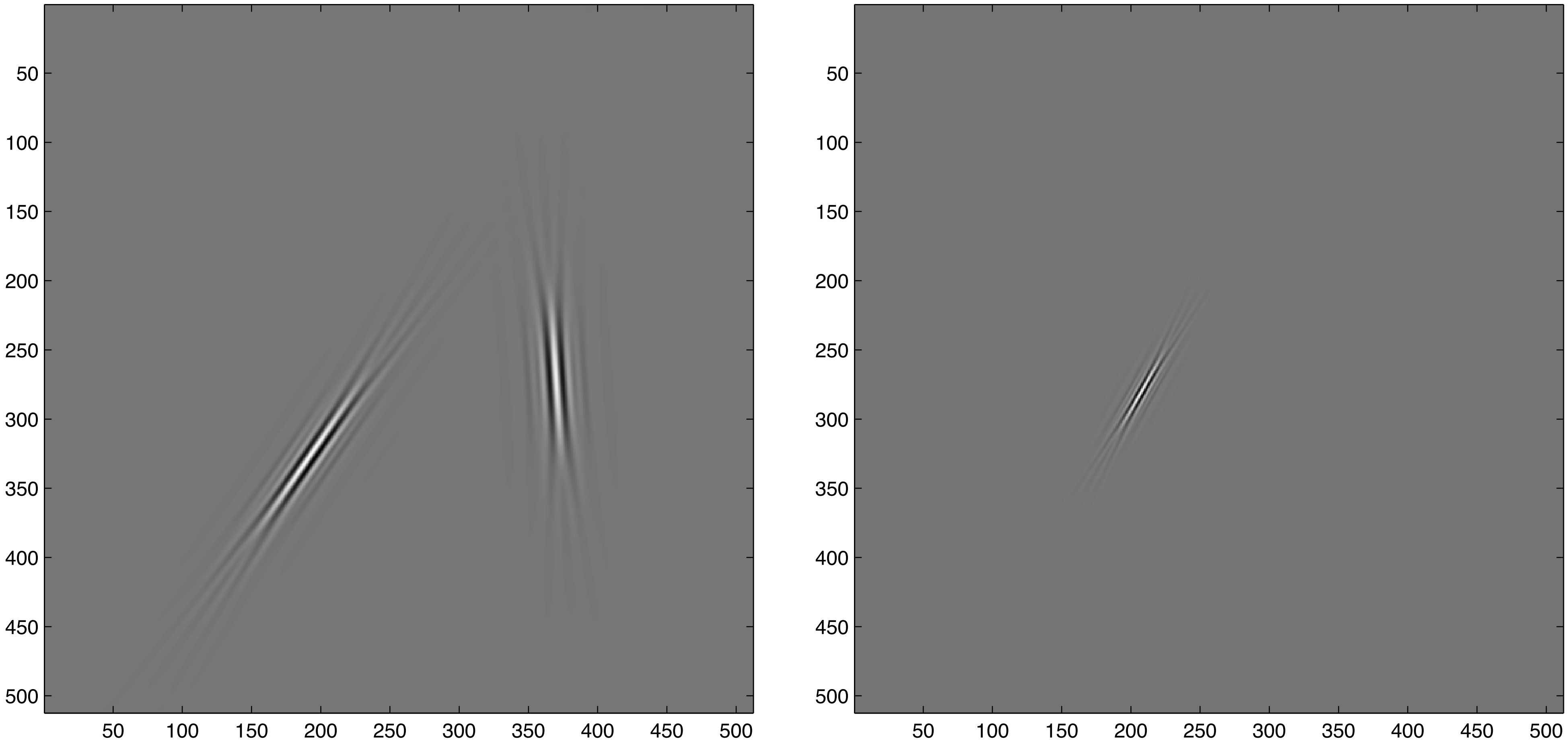
$$width \sim length^2$$

  - $\#$Directions $= 2^{\lfloor j/2 \rfloor}$
  - Doubles angular resolution at every other scale
- Unprecedented combination

Second Dyadic Decomposition

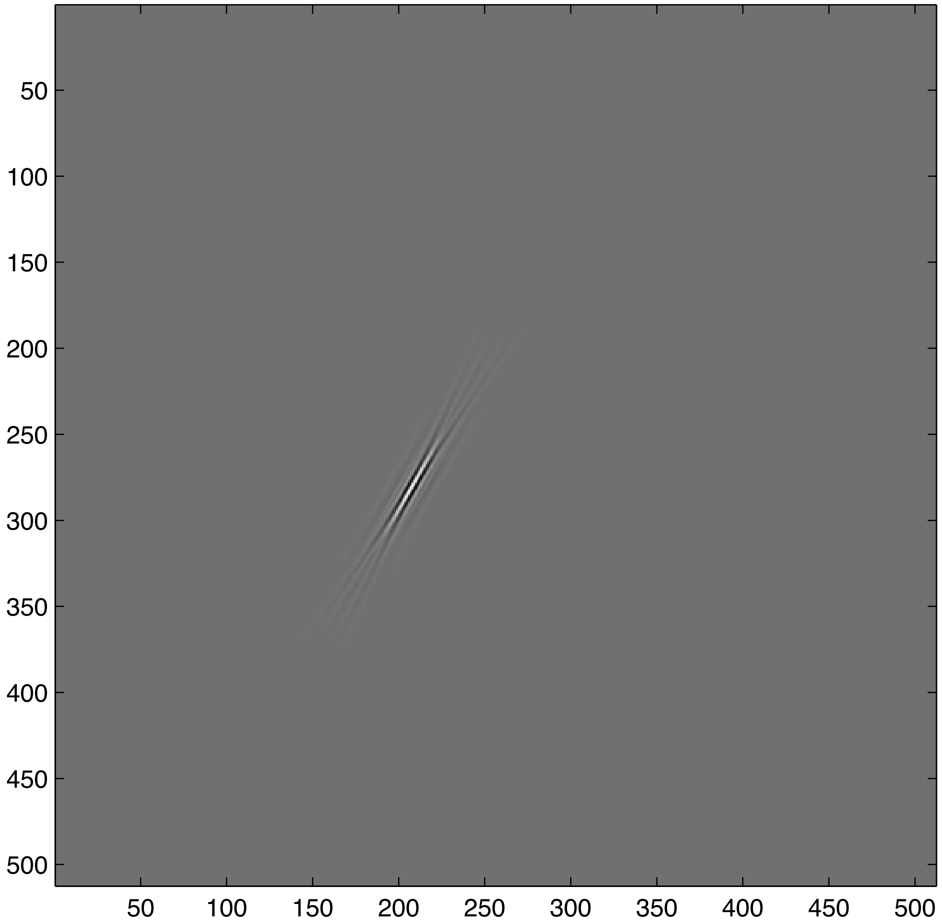
- Fefferman (70's), boundedness of Riesz spherical means
- Stein, and Seeger, Sogge and Stein (90's), L_p boundedness of Fourier Integral Operators.
- Smith (90's), atomic decomposition of Fourier Integral Operators.

Digital Curvelets

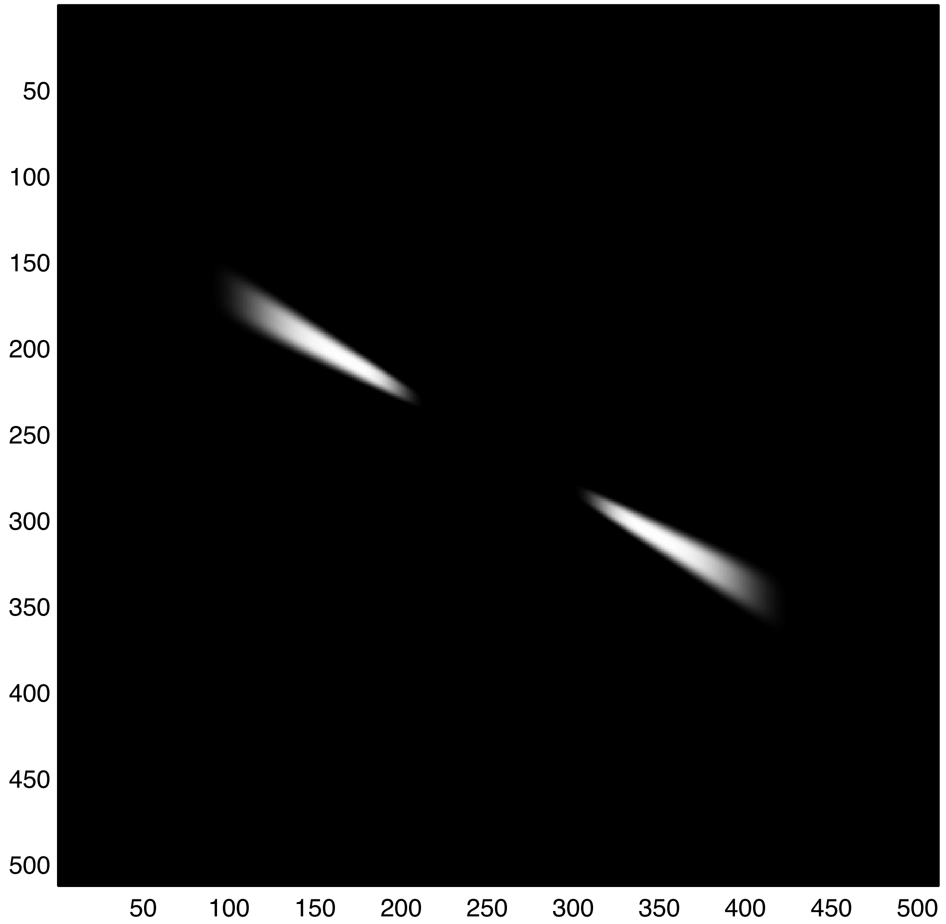


Source: DCTvUSFFT (Digital Curvelet Transform via USFFT's), C. and Donoho (2004).

Digital Curvelets: Frequency Localization



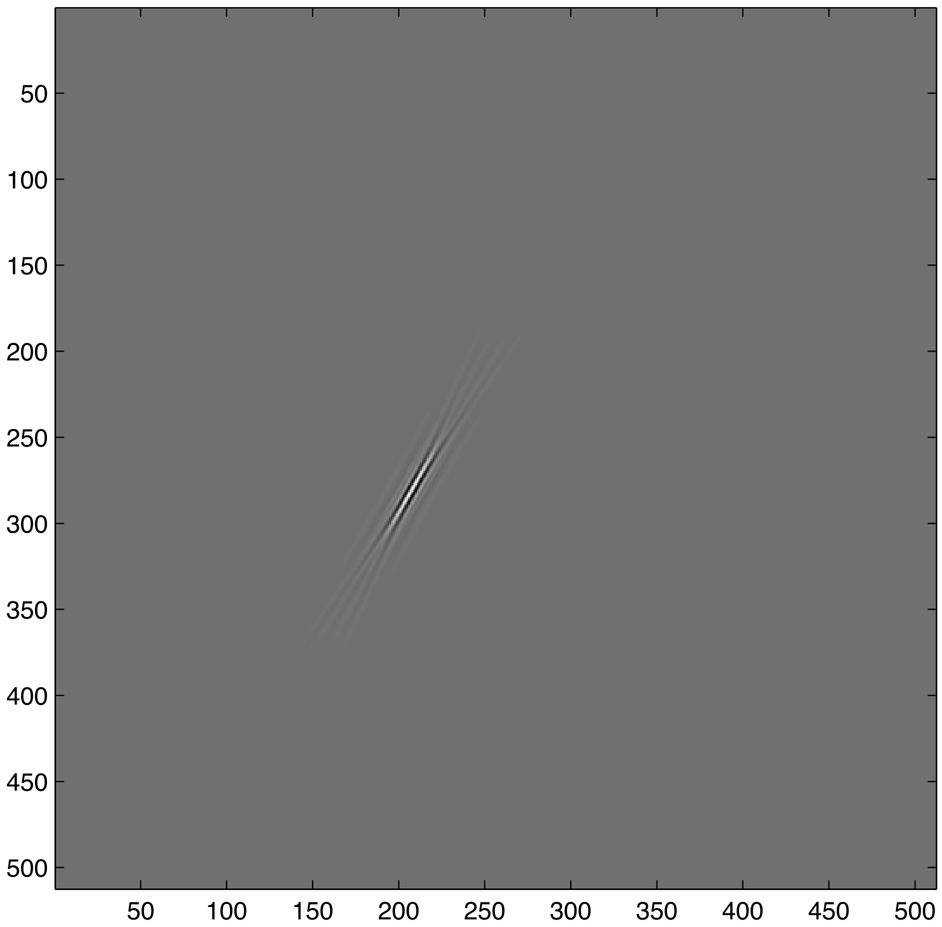
Time domain



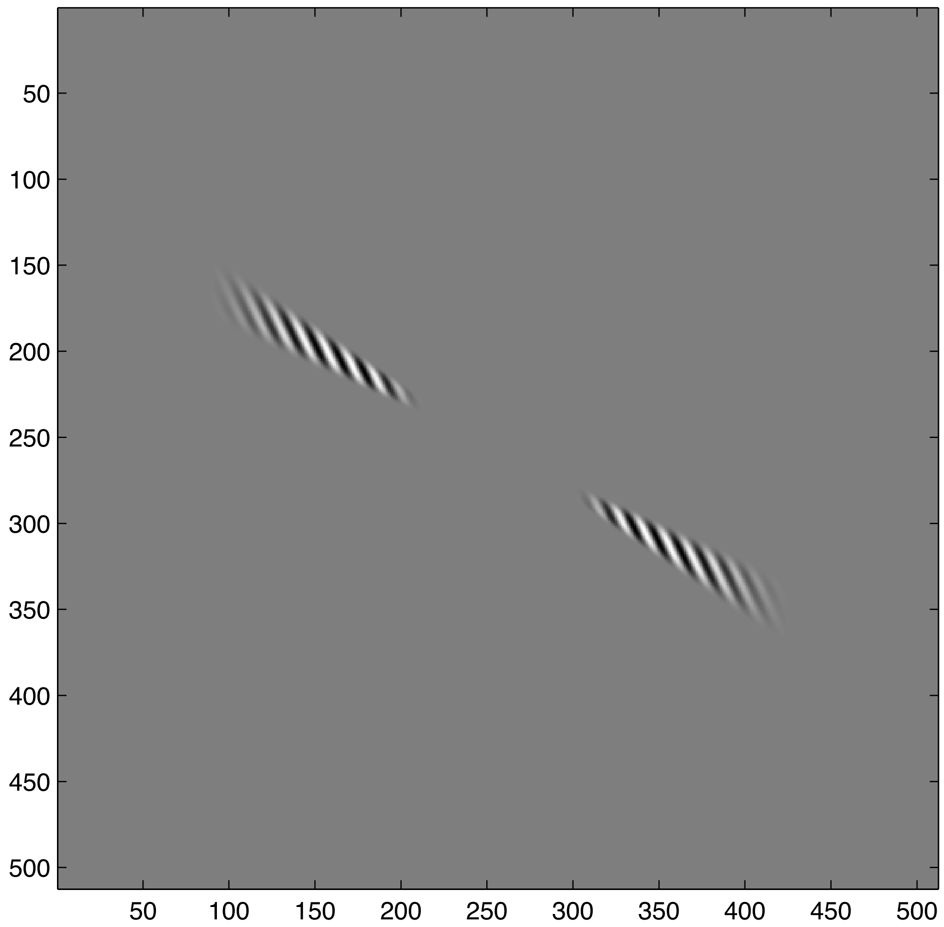
Frequency domain (modulus)

Source: DCTvUSFFT (Digital Curvelet Transform via USFFT's), C. and Donoho (2004).

Digital Curvelets: : Frequency Localization



Time domain



Frequency domain (real part)

Source: DCTvUSFFT (Digital Curvelet Transform via USFFT's), C. and Donoho (2004).

Curvelets and Edges

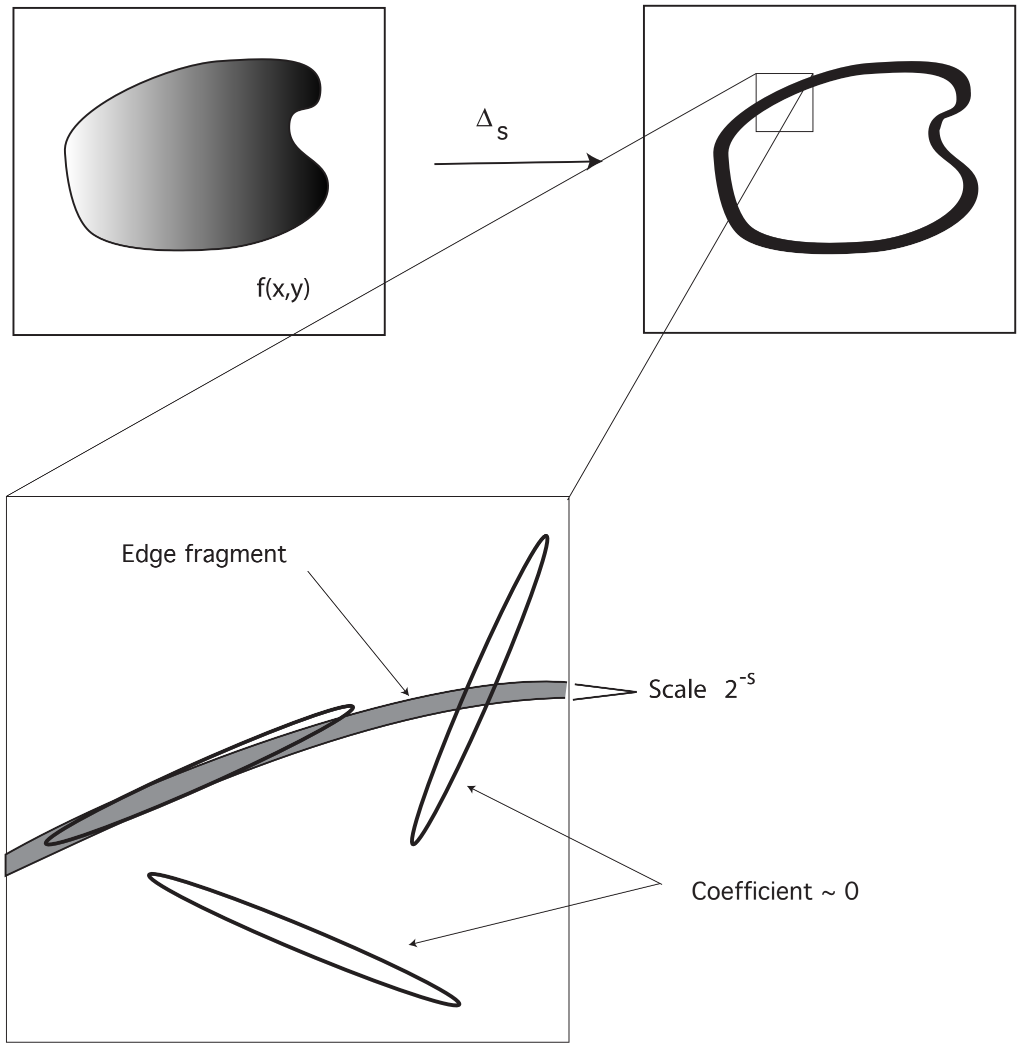
Optimality

- Suppose f is smooth except for discontinuity on C^2 curve
- Curvelet m-term approximations, naive thresholding

$$\|f - f_m^{curve}\|_2^2 \leq C\, m^{-2} (\log m)^3$$

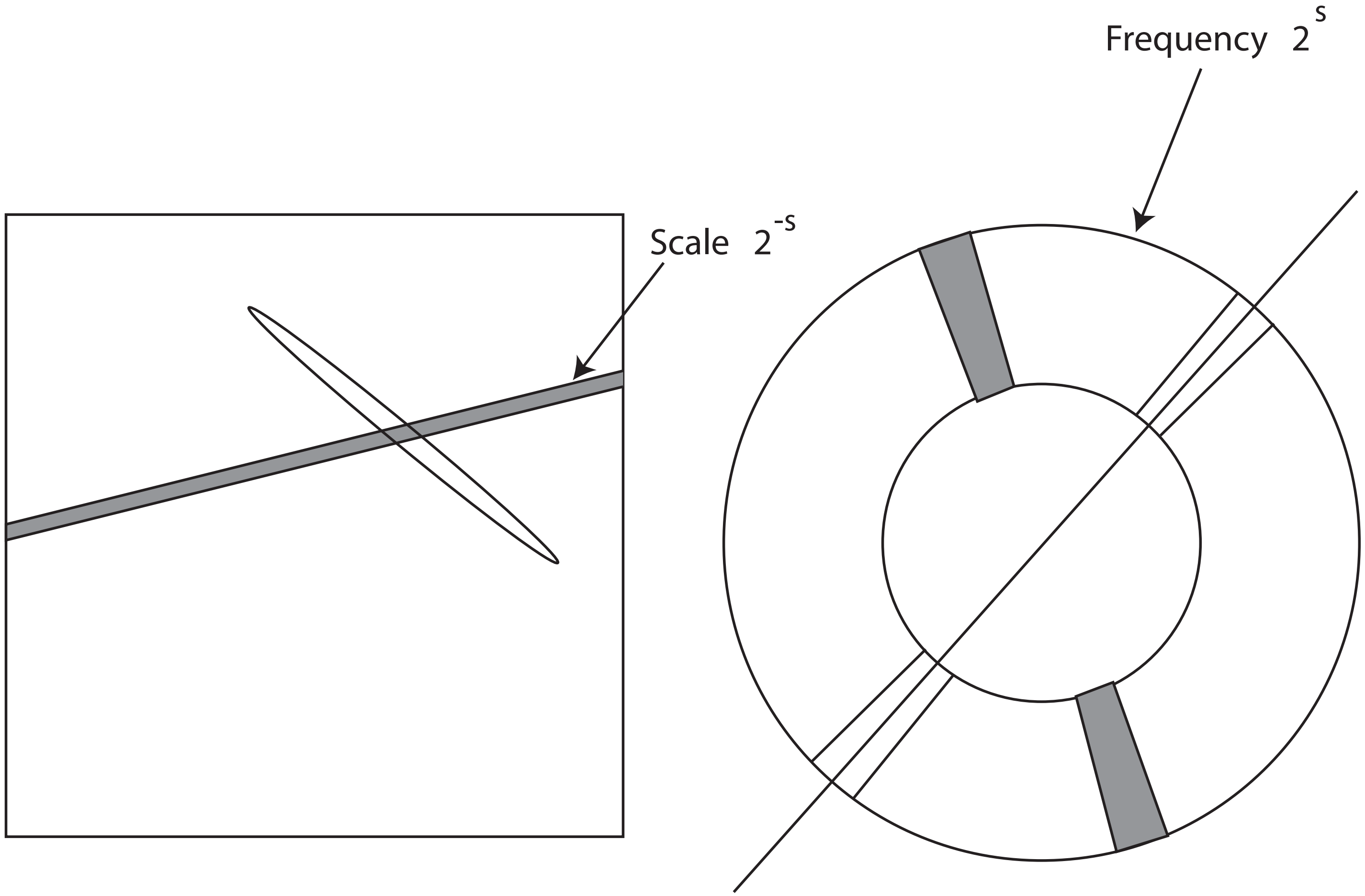
- Near-optimal rate of m-term approximation (wavelets $\sim m^{-1}$).

Idea of the Proof I



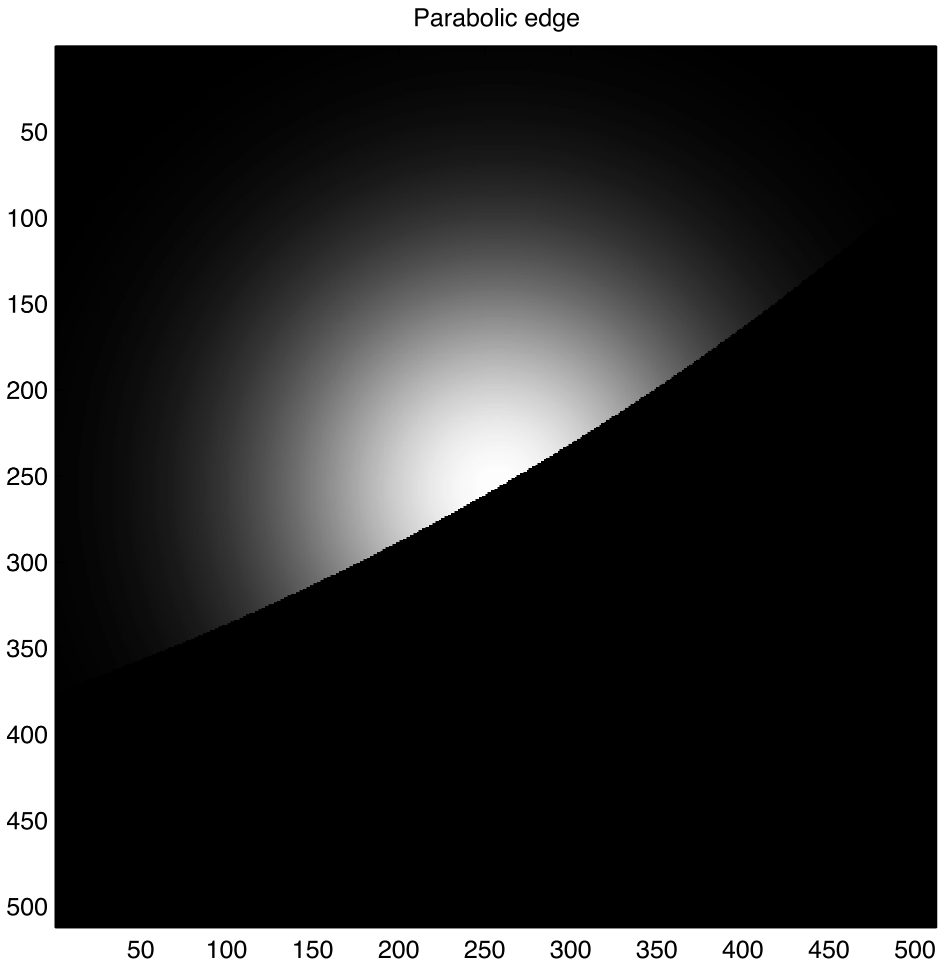
Decomposition of a Subband

Idea of the Proof II

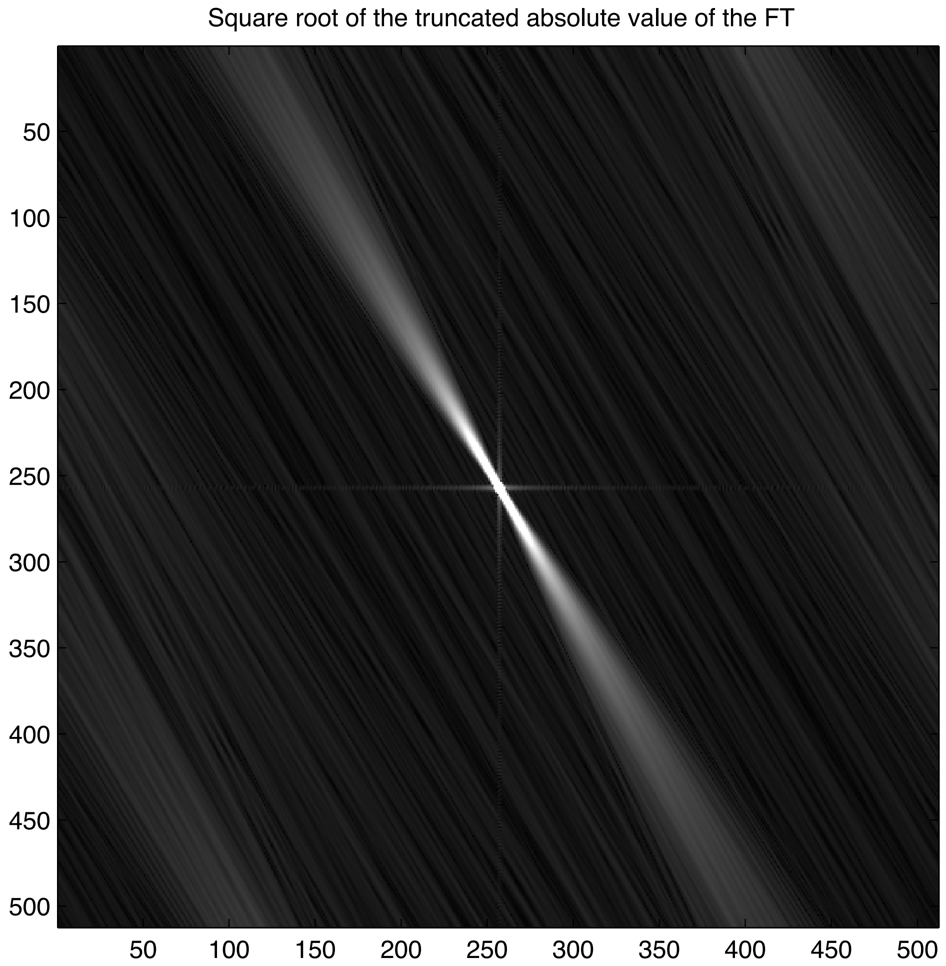


Microlocal behavior

Edge Fragments



Edge fragment



FT of an edge fragment (abs)

Curvelets and Warpings

C^2 change of coordinates preserves sparsity (C. 2002).

- Let $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ be a one to one C^2 function such that $\|J_\varphi\|_\infty$ is bounded away from zero and infinity.
- Curvelet expansion

$$f = \sum_\mu \theta_\mu(f)\gamma_\mu, \quad \theta_\mu(f) = \langle f, \gamma_\mu \rangle$$

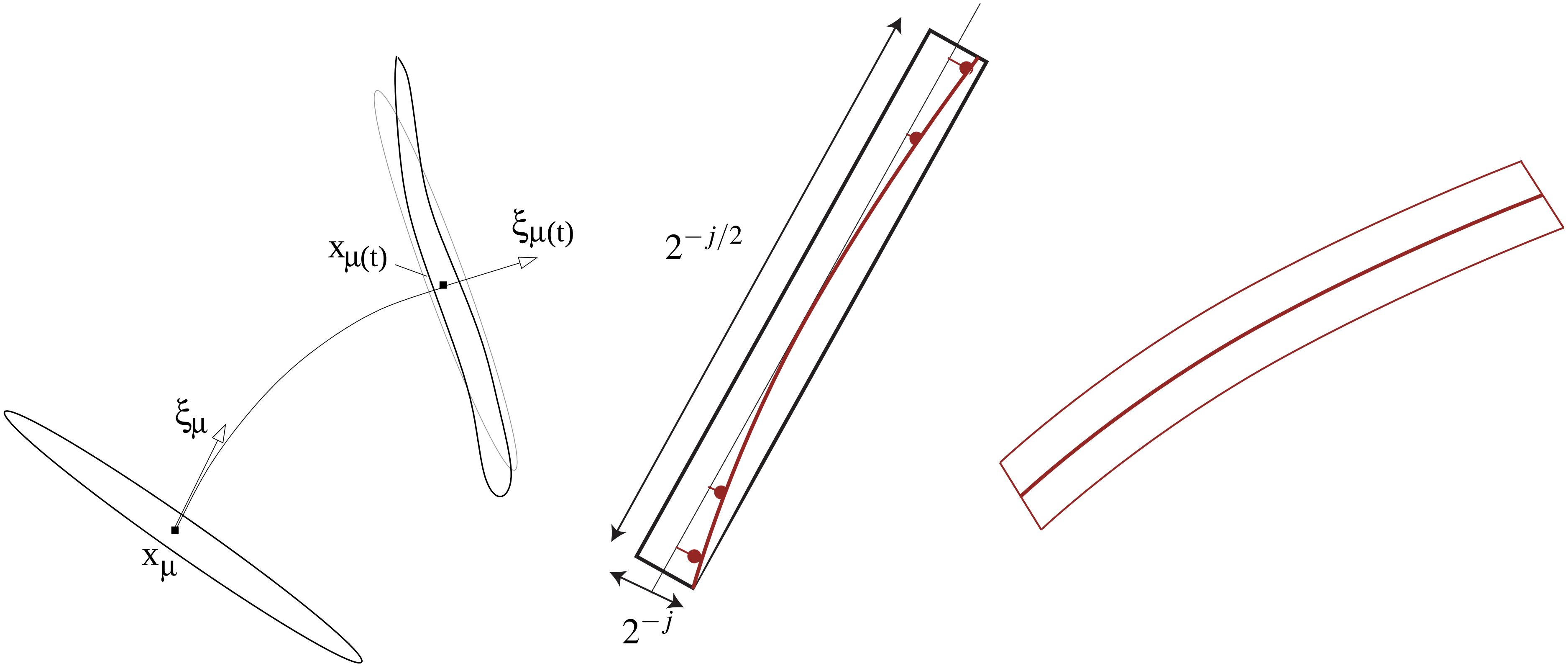
- Likewise,

$$f \circ \varphi = \sum_\mu \theta_\mu(f \circ \varphi)\gamma_\mu$$

The coefficient sequences of f and $f \circ \varphi$ are equally sparse.

Theorem 1 *Then, for each $p > 2/3$, we have*

$$\|\theta(f \circ \varphi)\|_{\ell_p} \leq C_p \cdot \|\theta(f)\|_{\ell_p}.$$



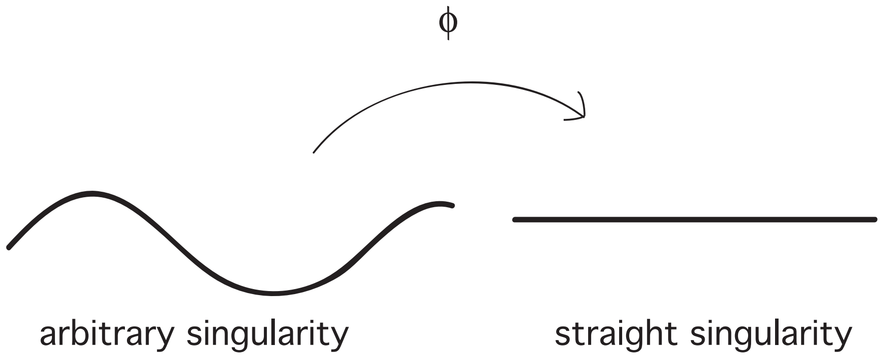
Curvelets are nearly invariant through a smooth change of coordinates

Curvelets and Curved Singularities

- f smooth except along a C^2 curve
- f_n, n-term approximation obtained by naive thresholding

$$\|f - f_n\|_{L_2}^2 \le C \cdot (\log n)^3 \cdot n^{-2}$$

- Why?
  1. True for a straight edge
  2. Deformation preserves sparsity
- Optimal



Curvelets and Hyperbolic Differential Equations

Representation of Evolution Operators

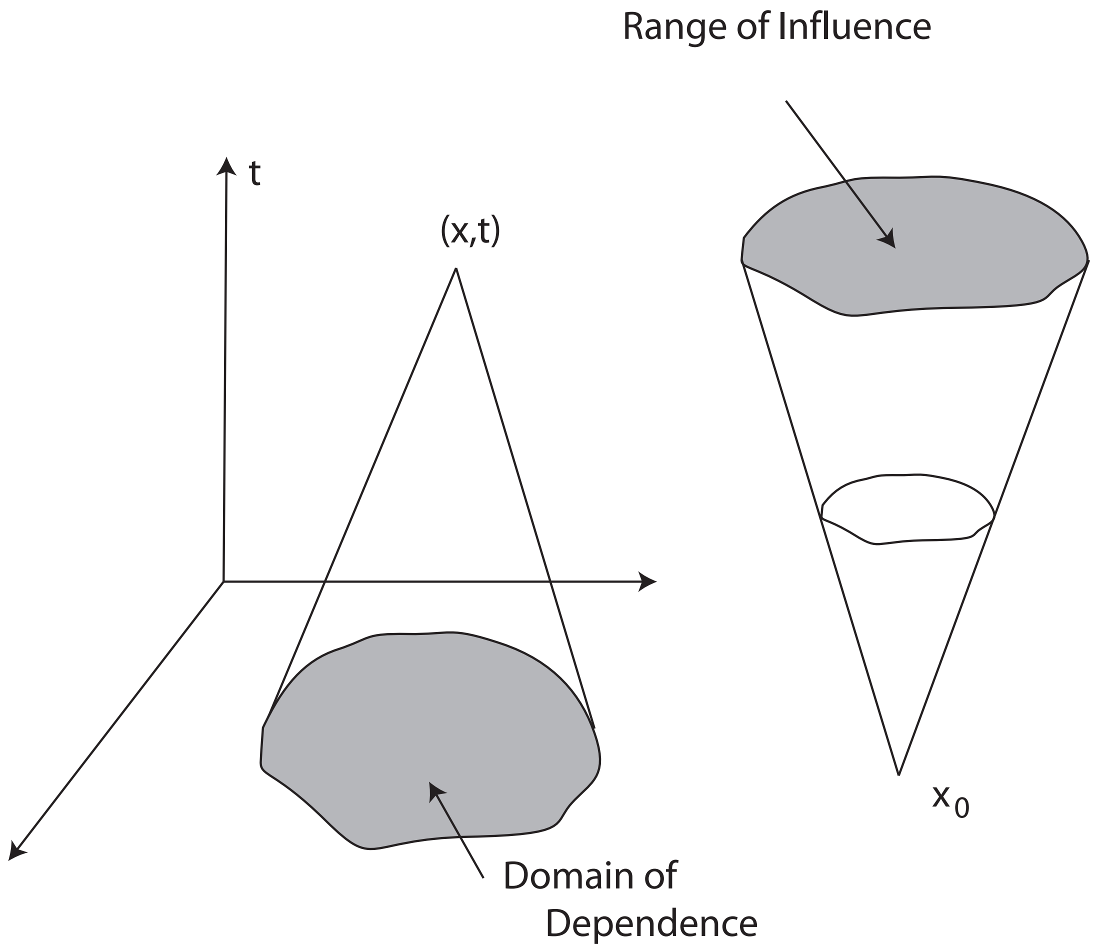
- Wave Equation

$$\partial_t^2 u = c^2(x)\Delta u, \quad x \in \mathrm{R}^2, \mathrm{R}^3, \ldots$$

- Evolution operator $S(t; x, y)$ is dense!
- Modern viewpoint
  - Basis φ_n of L_2.
  - Representation of S: $A(n, n') = \langle \varphi_n, S\varphi_{n'} \rangle$

$$\begin{tikzcd} u_0 \arrow[r, "S"] \arrow[d, "F"'] & u = Su_0 \arrow[d, "F"] \\ \theta(u_0) \arrow[r, "A"'] & \theta(u) \end{tikzcd}, \qquad \theta(u) = S\theta(u_0)$$

- Find a representation in which S is sparse



The solution operator S is a ‘dense’ integral.

Potential for Sparsity

If the matrix $\boldsymbol{A}$ is sparse, potential for

- fast multiplication
- fast inversion

Example: convolutions and Fourier transforms

Current Multiscale Thinking

- Traditional ideas:
  - Multigrids
  - Wavelets
  - Adaptive FEM's
  - etc.
- Traditional multiscale ideas are ill-adapted to wave problems:
  1. they fail to sparsify oscillatory integrals like the solution operator
  2. they fail to provide a sparse representation of oscillatory signals which are the solutions of those equations.

Peek at the Results

- Sparse representations of hyperbolic symmetric systems
- Connections with geometric optics
- Importance of parabolic scaling

Symmetric Systems of Differential Equations

$$\partial_t u_+ \sum_i A_i(x)\partial_i u + B(x)u = 0, \quad u(0,x) = u_0(x), \quad x \in \mathbf{R}^n,$$

u is m-dimensional and the A_i's are symmetric

Examples

- Maxwell
- Acoustic waves
- Linear elasticity

Example: Acoustic waves

System of hyperbolic equations: $\boldsymbol{v} = (\boldsymbol{u}, \boldsymbol{p})$

$$\begin{aligned} \partial_t \boldsymbol{u} + \nabla(c(x)p) &= 0 \\ \partial_t p + c(x)\nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

- Dispersion matrix

$$\sum_j A_j(x) k_j = c(x) \begin{pmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ k_1 & k_2 & 0 \end{pmatrix}$$

- Eigenvalues (λ_ν): $\lambda_\pm = \pm c(x)$, $\lambda_0 = 0$.
- Eigenvectors (R_ν)

$$R_0(k) = \begin{pmatrix} k^\perp/|k| \\ 0 \end{pmatrix}, \qquad R_\pm(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm k/|k| \\ 1 \end{pmatrix}.$$

Geometric Optics: Lax, 1957

- High-frequency wave-propagation approximation

$$v(x,t) = \sum_{\nu} e^{i\omega\Phi_{\nu}(x,t)} \left(a_{0\nu}(x,t) + \frac{a_{1\nu}(x,t)}{\omega} + \frac{a_{2\nu}(x,t)}{\omega^2} + \dots \right)$$

- Plug into wave equation
  - Eikonal equations

$$\partial_t \Phi_{\nu} + \lambda_{\nu}(x, \nabla_x \Phi) = 0.$$

  $\lambda_{\nu}(x,k)$ are the eigenvalues of the dispersion matrix $\sum_j A_j(x) k_j$
  - And 'transport' equations for amplitudes

Turns a linear equation into a nonlinear evolution equation!

Hamiltonian Flows

- Dispersion matrix

$$\sum_j A_j(x)k_j$$

- Eigenvalues $\lambda_\nu(x,k)$, eigenvectors $R_\nu(x,k)$
- Hamiltonian flows (in general, m of them) in *phase-space*

$$\begin{cases} \dot{x}(t) &= \nabla_k \lambda_\nu(x,k), & x(0) &= x_0, \\ \dot{k}(t) &= -\nabla_x \lambda_\nu(x,k), & k(0) &= k_0. \end{cases}$$

- Eikonal equations from geometric optics

$$\partial_t \Phi_\nu + \lambda_\nu(x, \nabla_x \Phi) = 0.$$

Φ is constant along the Hamiltonian flow $\Phi(t, x(t)) = Cste$. Problem: valid before caustics, i.e. before Φ becomes multi-valued (ray-crosses).

Hyper-curvelets

- Frequency definition

$$\hat{\Phi}_{\nu\mu}(k) = E_\nu \hat{\varphi}_\mu(k).$$

- Hyper-curvelets build-up a (vector-valued) tight-frame, namely

$$\|u\|_{L_2}^2 = \sum_{\nu,\mu} |[u, \Phi_{\nu\mu}]|^2,$$

and

$$u = \sum_{\nu,\mu} [u, \Phi_{\nu\mu}] \Phi_{\nu\mu}$$

- Other possibilities:

$$\hat{\Phi}_{\nu\mu}(k) = R_\nu(k)\hat{\varphi}_\mu(k), \quad \Phi_{\mu\nu}(x) = \int R_\nu(x,k)\hat{\varphi}_{\mu\nu}(k) e^{i\langle k,x\rangle}\, dk.$$

Approximation

The action of the wave propagator on a curvelet is well- approximated by a rigid motion.

- Initial data: $\Phi_{\nu\mu}(x)$
- Approximate solution

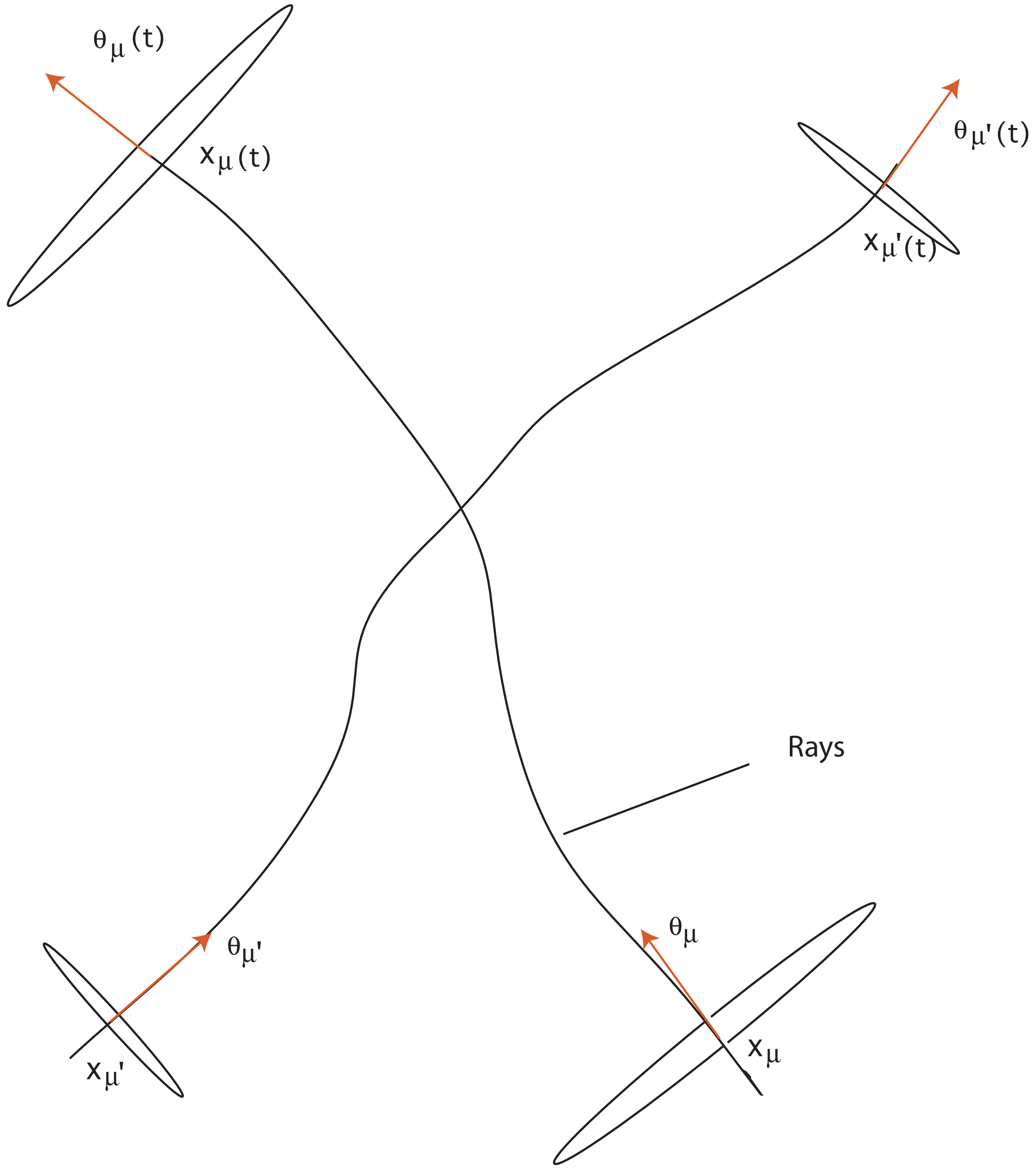
$$\hat{\Phi}_{\nu\mu}(t,k) = R_\nu(k)\hat{\varphi}_{\mu(t)}(k)$$

where

$$\varphi_{\mu(t)}(x) = \varphi_\mu\left(U_\mu(t)(x - x_\mu(t)) + x_\mu\right)$$

– $U_\mu(t)$ is a rotation

– $x_\mu(t)$ is a translation



$\theta_\mu (t)$
$x_\mu (t)$
$\theta_{\mu'} (t)$
$x_{\mu'} (t)$
Rays
$\theta_{\mu'}$
θ_μ
x_μ
$x_{\mu'}$

Main Result

Curvelet representation of the propagator $S(t)$:

$$A(\nu, \mu; \nu', \mu') = \langle \Phi_{\nu\mu}, S(t)\Phi_{\nu'\mu'} \rangle$$

Theorem 2 (C. and Demanet) *Suppose the coefficients of a general hyperbolic system are smooth, i.e.* C^∞.

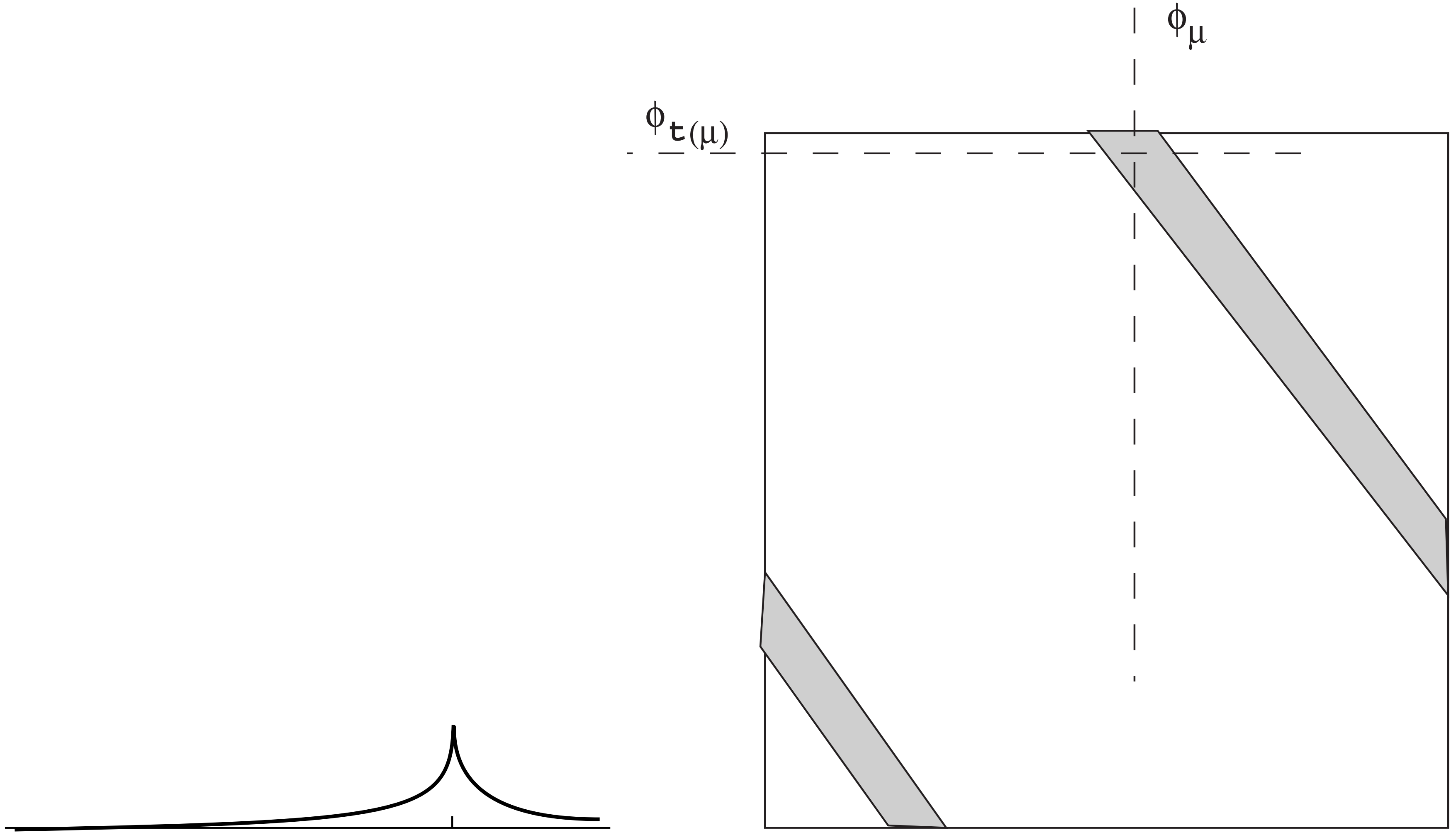
- *The matrix is sparse. Suppose* a *is either a row or a column of* A*, and let* $|a|_{(n)}$ *be the* n*-largest entry of the sequence* $|a|$*, then for each* $r > 0$*,* $|a|_{(n)}$ *obeys*

$$|a|_{(n)} \leq C_r n^{-r}.$$

- *The matrix is well-organized. There is a natural distance over the curvelet indices such that for each* M*,*

$$|a(\nu, \mu; \nu', \mu')| \leq C_{M,t}(\nu, \nu') \sum_{\nu''} (1 + d(\mu, \mu_{\nu''}(t)))^{-M}$$

The constant is growing at most like $C_1 e^{C_2 t}$*, and for* $\nu \neq \nu'$*,* $C_1 << 1$.



Sketch of the curvelet representation of the wave propagator

$$A(\nu', \mu'; \nu, \mu) = \langle \Phi_{\nu\mu}, S(t)\Phi_{\nu'\mu'} \rangle.$$

Our Claim

Curvelets provide an *optimally sparse representation of wave propagators*.

- Fourier matrix is dense
- FEM matrix is dense
- Wavelet matrix is dense

Second-Order Scalar Equations

$$\partial_t^2 u - \sum_{i,j} a_{ij}(x)\partial_i\partial_j u = 0$$

Sparsity comes for free (in the scalar curvelet system)

Why Does This Work?

"The purpose of calculations is insight, not numbers..."

(adapted from R. W. Hamming)

- Wave operator (constant coefficients)

$$\partial_t^2 u = c^2 \Delta u, \quad u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x).$$

- Fourier transform

$$\hat{u}(t,k) = \int u(t,x) e^{-ik\cdot x}\, dx.$$

- Solution

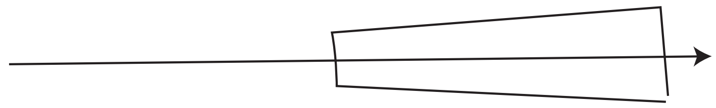
$$\begin{aligned} \hat{u}(t,k) &= \cos(c|k|t)\hat{u}_0(k) + \frac{\sin(c|k|t)}{|k|}\hat{u}_1(k) \\ \hat{u}(t,k) &= e^{\pm ic|k|t}\hat{u}_0(k) + \frac{e^{\pm ic|k|t}}{|k|}\hat{u}_1(k) \end{aligned}$$

- Set $t' = ct$

$$e^{i|k|t'} = e^{ik_1 t' + \delta(k)t'}, \quad \delta(k) = |k| - k_1.$$

- Because of parabolic scaling

$$\delta(k) = O(1)$$



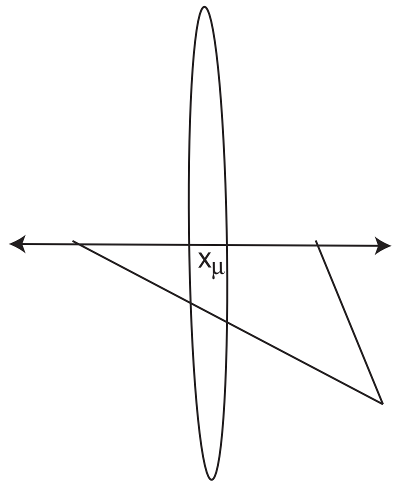
cos(|k| t') ~ cos(k_1 t')

- Frequency modulation is nearly linear

$$e^{i|k|t'} = \hat{h}(k) \cdot e^{ik_1 t'}.$$

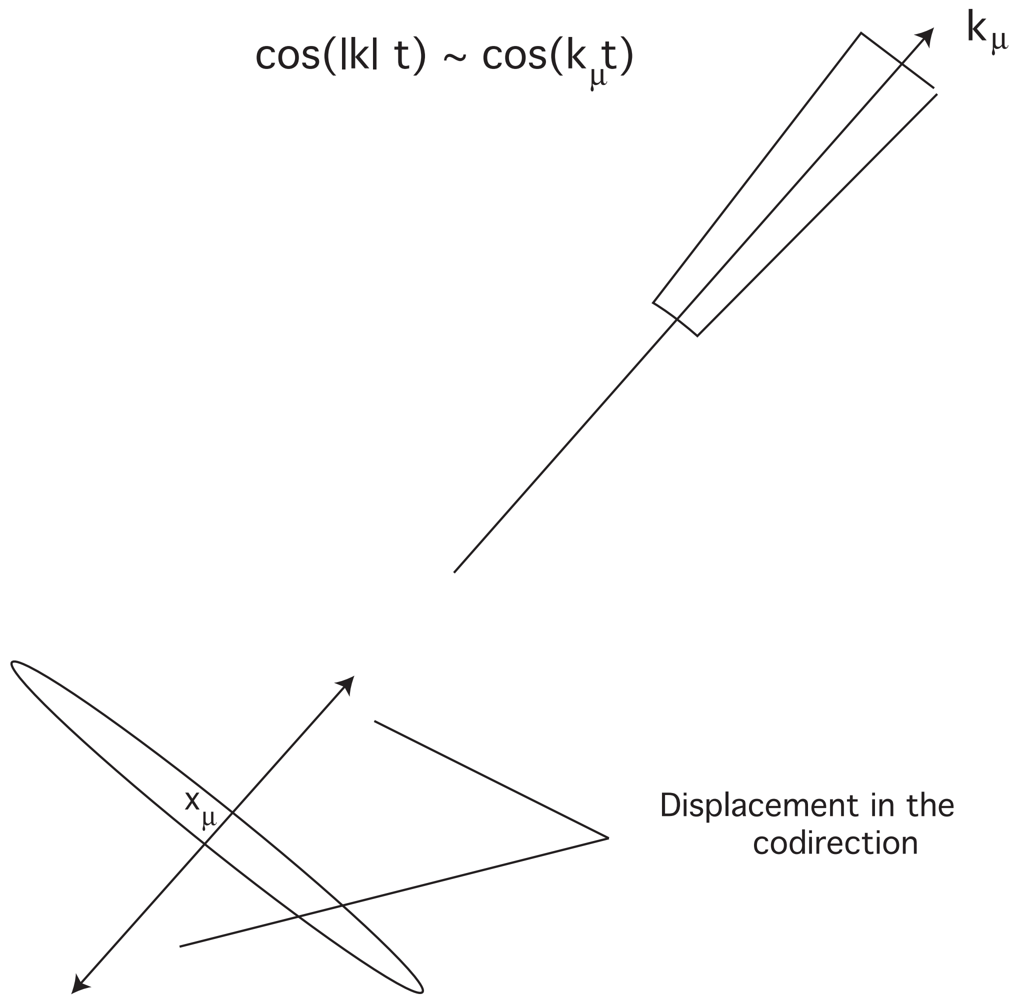
$\hat{h}$ smooth and non-oscillatory

- Space-side picture:
  1. modulation corresponds to a displacement of $\pm ct$ in the codirection.
  2. multiplication by $\hat{h}$: gentle convolution



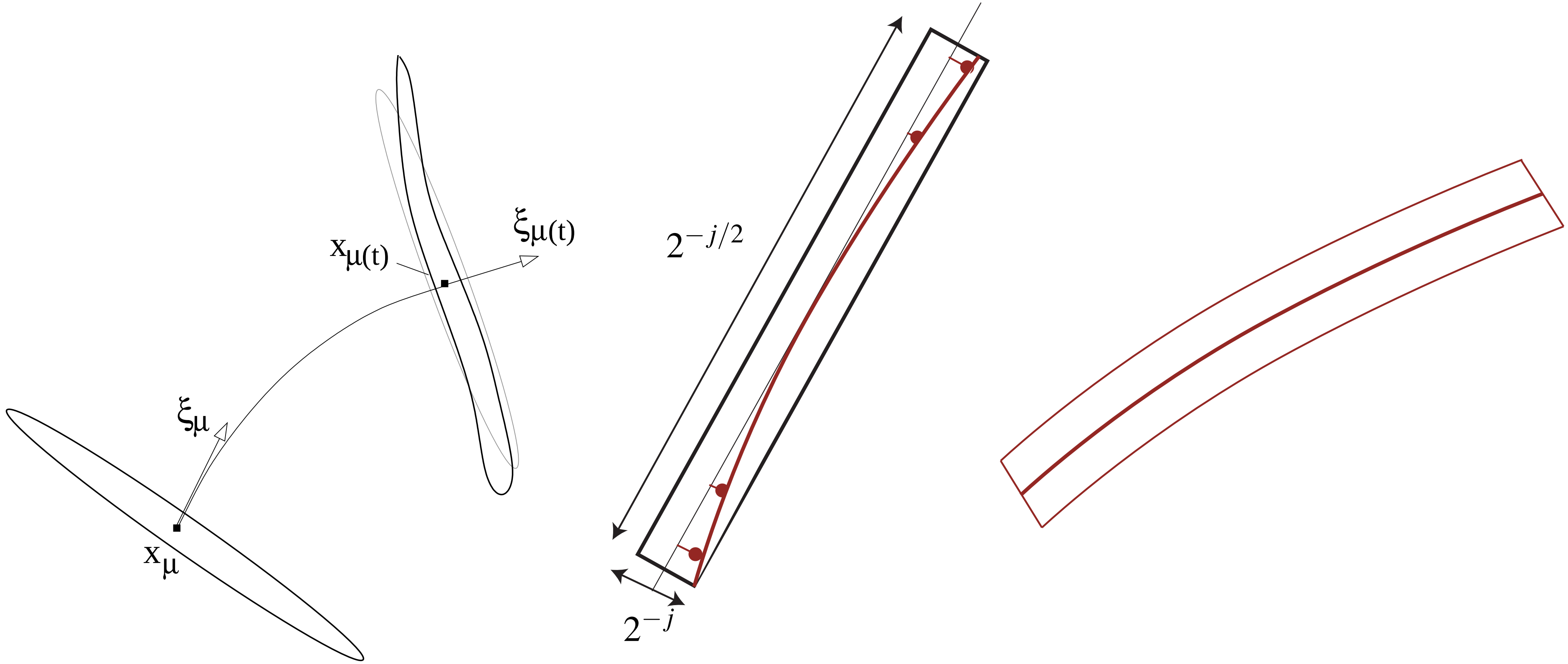
Displacement in the codirection

$$\cos(|k|\ t) \sim \cos(k_\mu t)$$



Warpings

- Variable coefficients: $u_{tt} = c^2(x)\Delta u$.
- At small scales, the velocity field varies varies little over the support of a curvelet.
- Model the effect of variable velocity field as a warping g: $\phi_\mu(g(x))$.
- Warpings do not distort the geometry of a curvelet.


ξμ
xμ
xμ(t)
ξμ(t)
2−j/2
2−j

Importance of Parabolic Scaling

- Consider arbitrary scaling (anisotropy incresases as α decreases)

$$width \sim 2^{-j},\ length \sim 2^{-j\alpha}, \quad 0 \leq \alpha \leq 1.$$

  - ridgelets $\alpha = 0$ (very anisotropic),
  - curvelets $\alpha = 1/2$ (parabolic anisotropy),
  - wavelets $\alpha = 1$ (roughly isotropic).

- For wave-like behavior, need

$$width \leq length^2$$

- For particle-like behavior, need

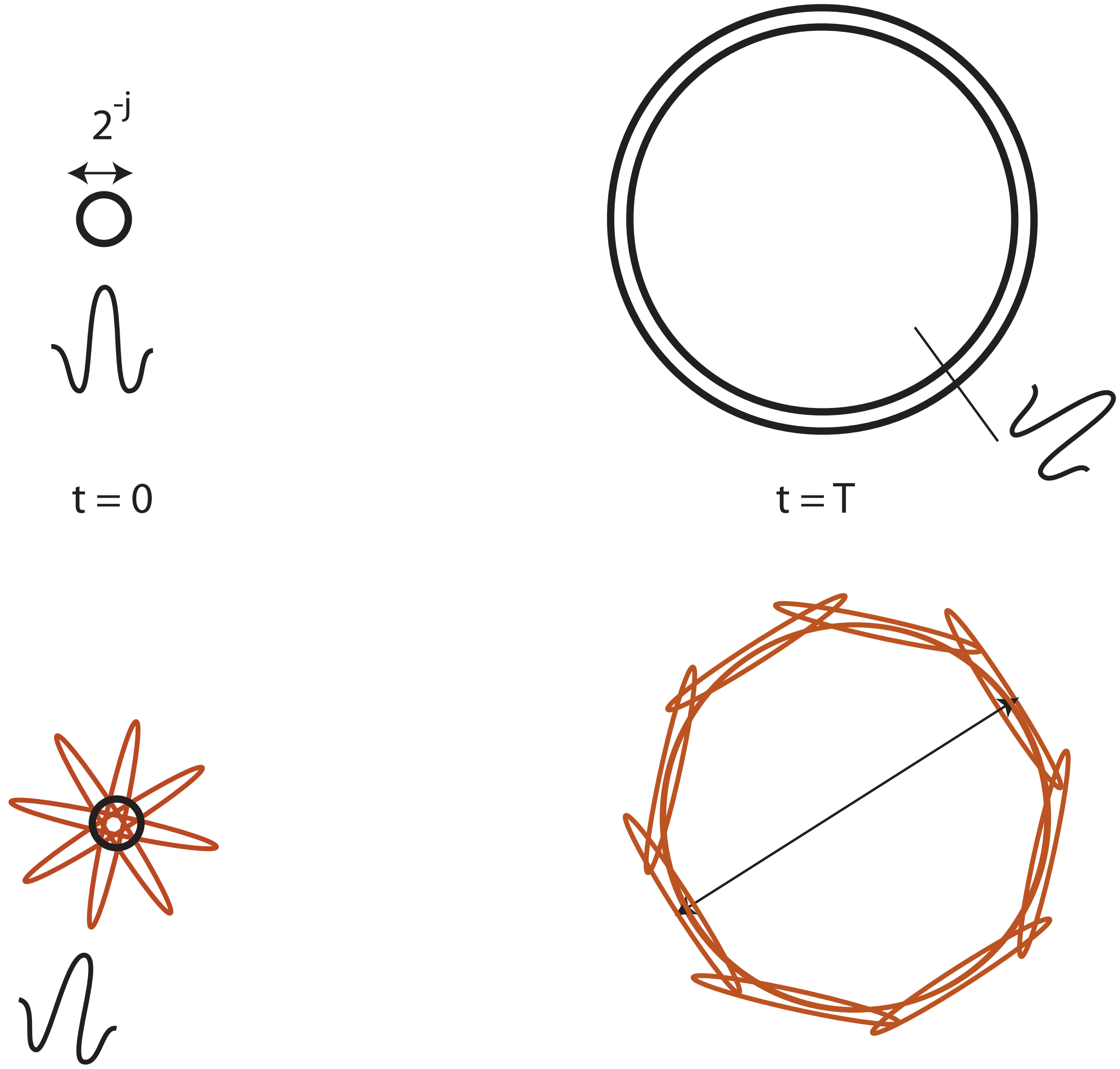
$$width \geq length^2$$

- For both (simultaneously), need

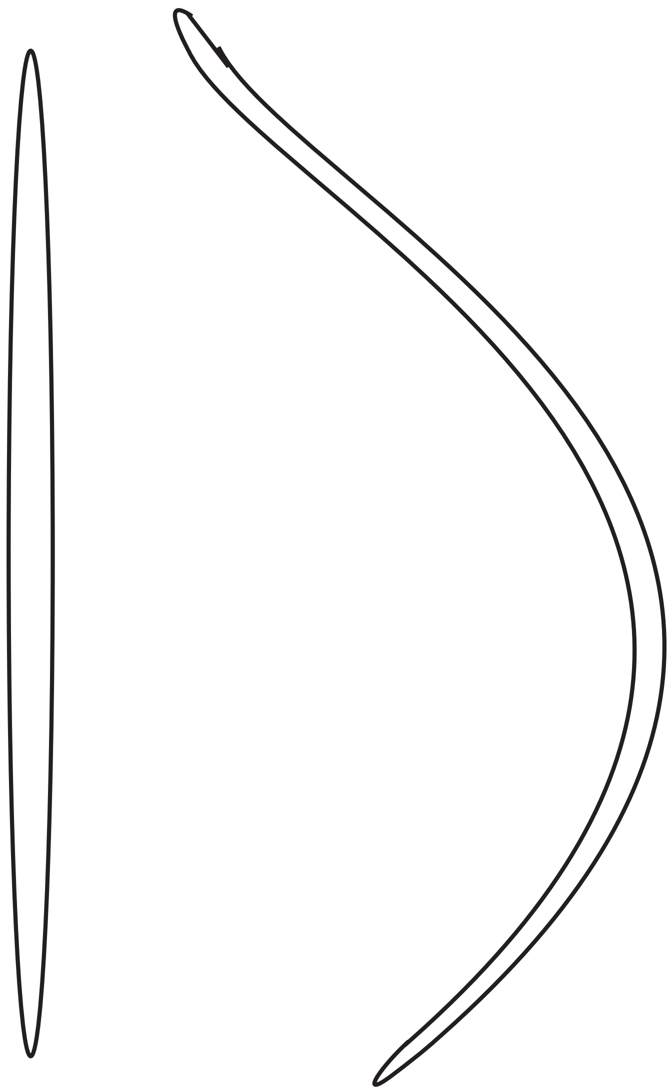
$$width \sim length^2$$

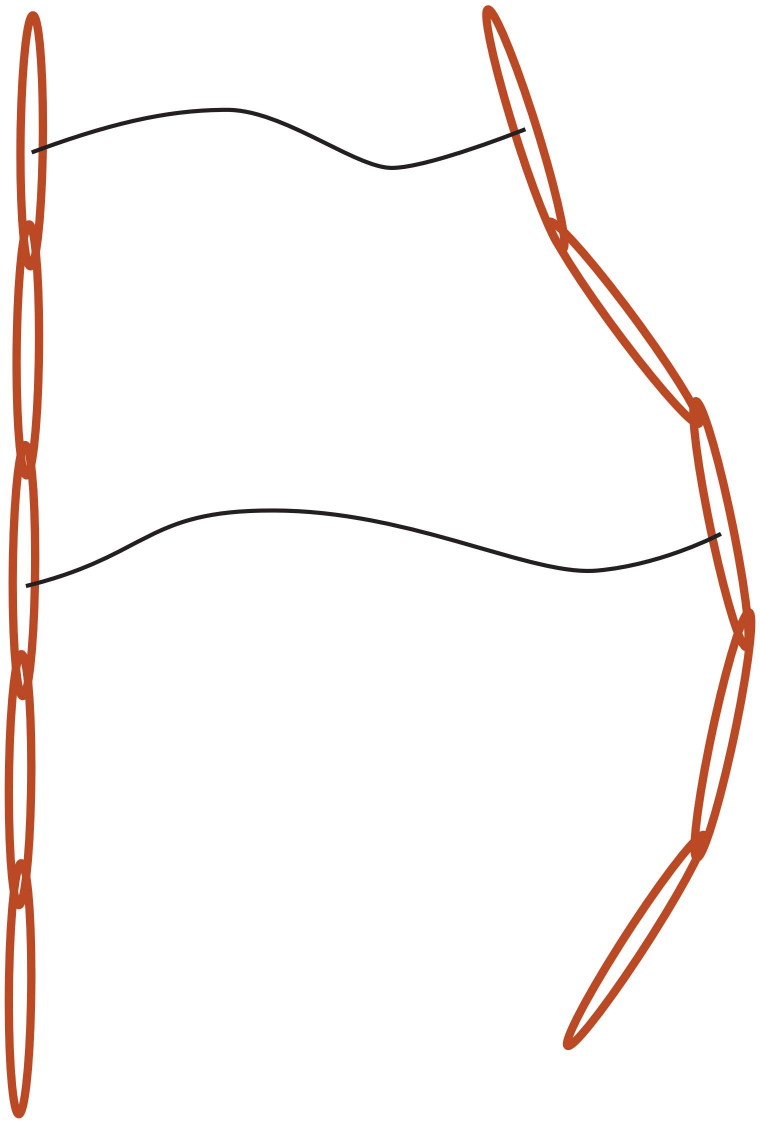
- *Only working scaling:* $\alpha = 1/2$

Examples I

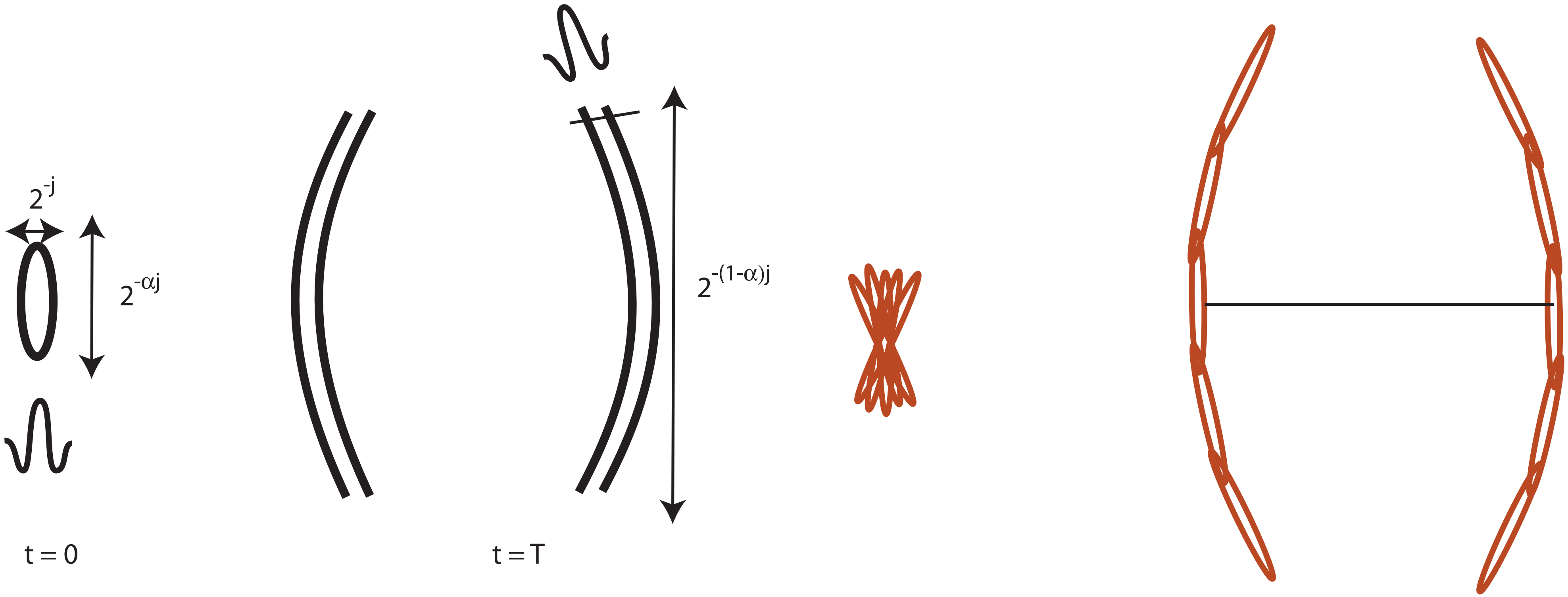


Examples II





Examples III



Architecture of the Argument

1. *Decoupling into polarized components:* Decompose the wavefield $u(t, x)$ into m one-way components $f_\nu(t, x)$

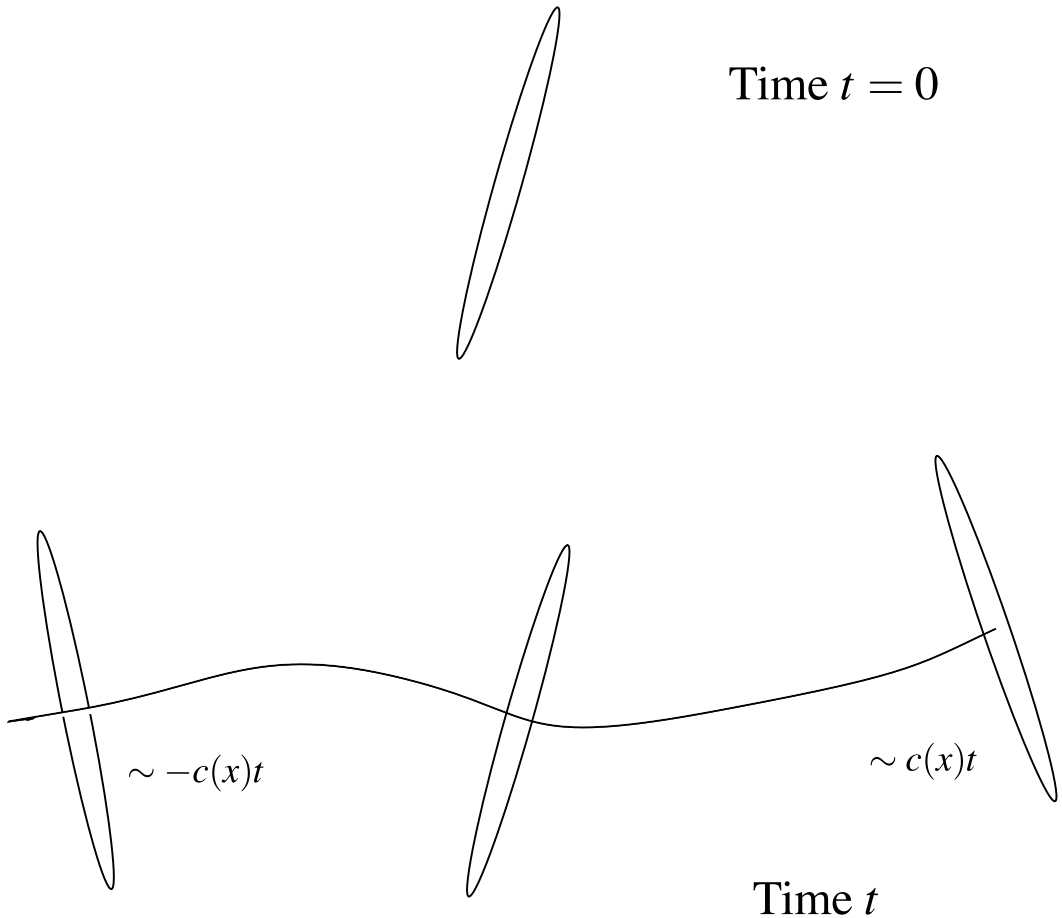
$$u(t, x) = \sum_{\nu=1}^{m} R_\nu f_\nu(t, x),$$

R_ν: pseudo-differential operator (independent of time), maps scalar fields into m-dimensional vector fields.

$$S(t) = \sum_{\nu=1}^{m} R_\nu e^{-it\Lambda_\nu} L_\nu + \text{negligible},$$

L_ν: pseudo-differential operator, maps m-dimensional vector fields into scalar fields.

Curvelet Splitting



$$S(t) \sim \sum_{\nu} S_{\nu}(t), \quad S_{\nu}(t) = R_{\nu} e^{-it\Lambda_{\nu}} L_{\nu}$$

2. *Fourier Integral Operator parametrix.* We then approximate for small times $t > 0$ each $e^{-it\Lambda_{\nu}}$, $\nu = 1, \ldots, m$, by an oscillatory integral or Fourier Integral Operator (FIO) $F_{\nu}(t)$

$$F_{\nu}(t) f(x) = \int e^{i\Phi_{\nu}(t,x,k)} \sigma_{\nu}(t,x,k) \hat{f}(k)\, dk$$

   Key issue: decoupling is crucial because this what makes it work for large times (not an automatic consequence of Lax's construction)

$$S_{\nu}(nt) = S_{\nu}(t)^n$$

3. *Sparsity of FIO's*. General FIO's $F(t)$ are sparse and well-structured when represented in tight frames of (scalar) curvelets φ_{μ}—a result of independent interest.

Potential for Scientific Computing

- Transform this theoretical insight into effective algorithms
- Sources of inspiration
  - Rokhlin
  - Engquist & Osher
  - Others

Our Viewpoint

$$u_t = e^{-Pt} u_0$$

$$\begin{array}{ccc} u_0 & \xrightarrow{e^{-Pt}} & u_t \\ F \big\downarrow & & \big\downarrow F \\ \theta_0 & \xrightarrow[A(t)]{} & \theta_t \end{array}$$

For any t, $A(t)$ is *sparse*

Background: Fast and accurate Digital Curvelet Transform is available (with D. Donoho).

Preliminary Work

- Grid size N
- Accuracy ϵ
- *Complexity*

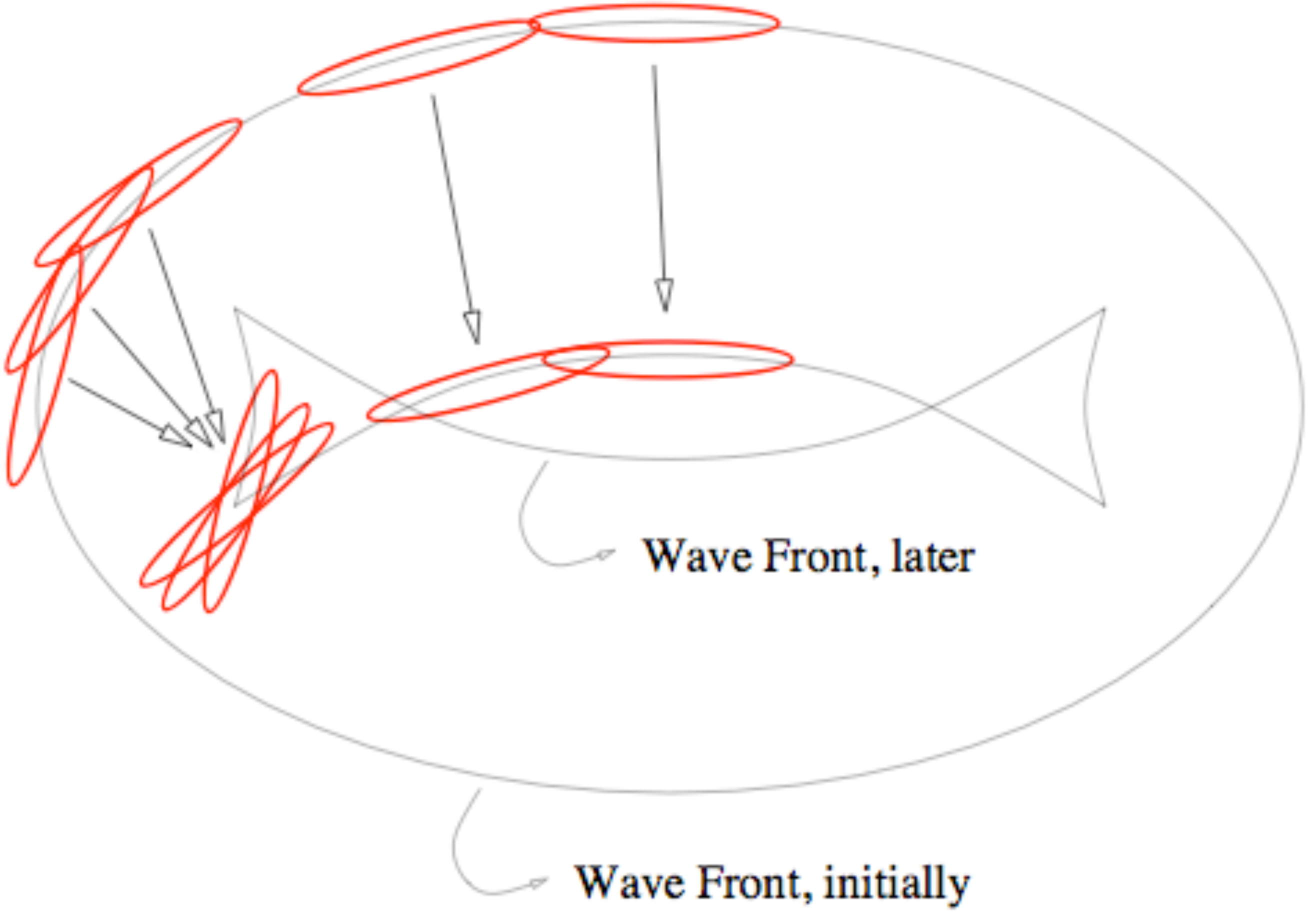
$$O(N^{1+\delta} \cdot \epsilon^{-\delta}), \quad \text{any } \delta > 0,$$

- *Conjecture*

$$O(N \log N \cdot \epsilon^{-\delta})$$

- Arbitrary initial data
- Works in any dimension
- General procedure

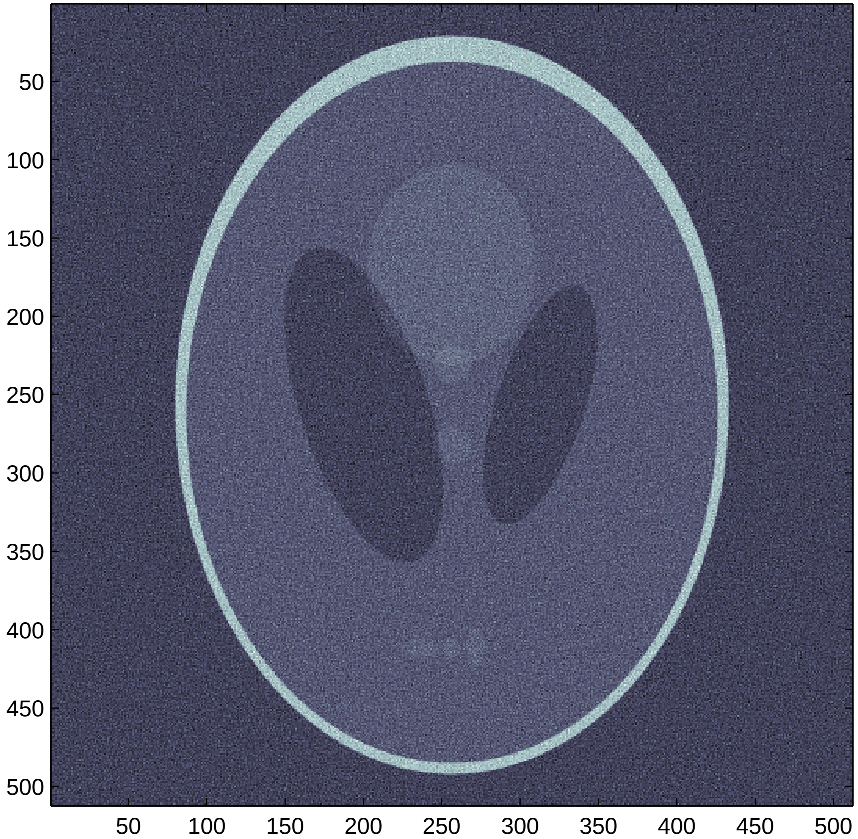
Propagating curvelets is not geometric optics!



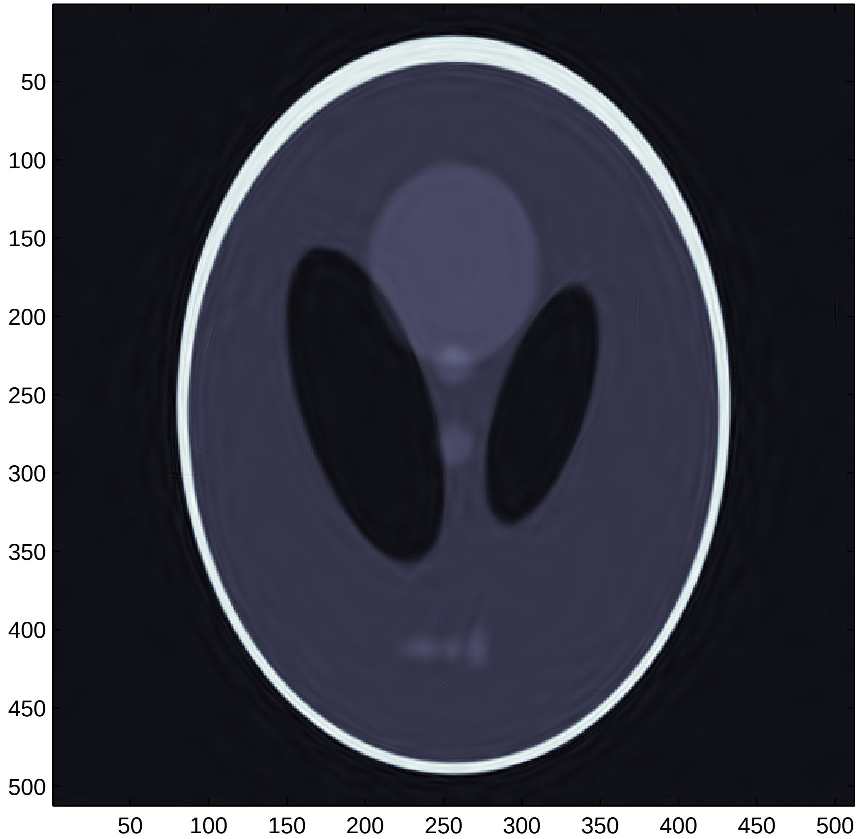
Summary

- New geometric multiscale ideas
- Key insight: geometry of Phase-Space
- New mathematical architecture
- Addresses new range of problems effectively
- Promising potential

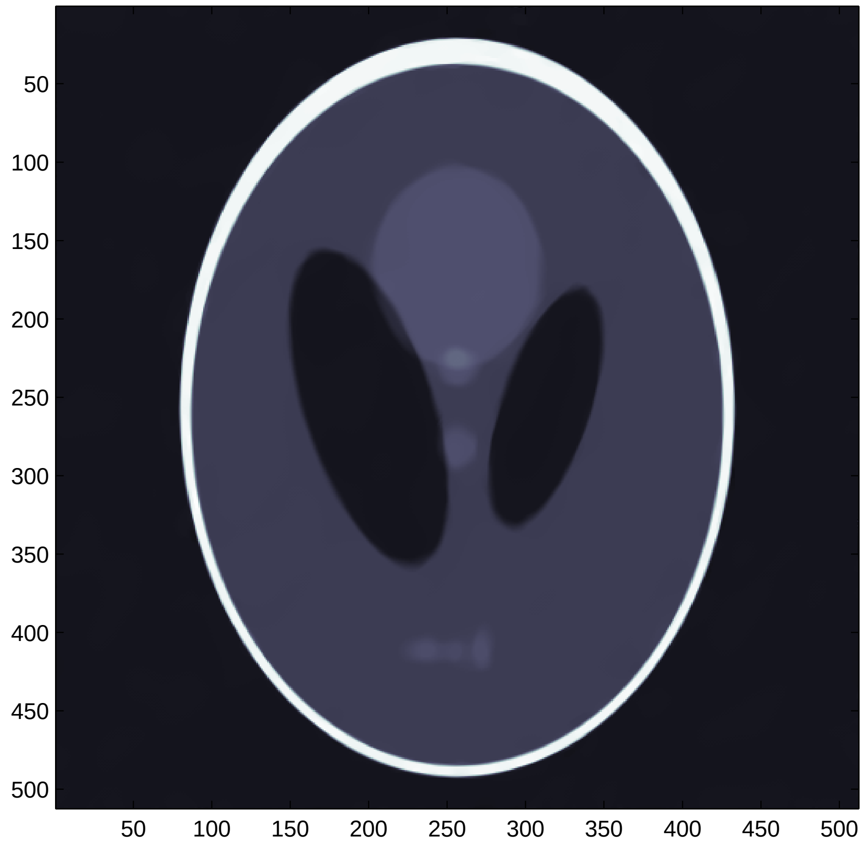
Numerical Experiments, I



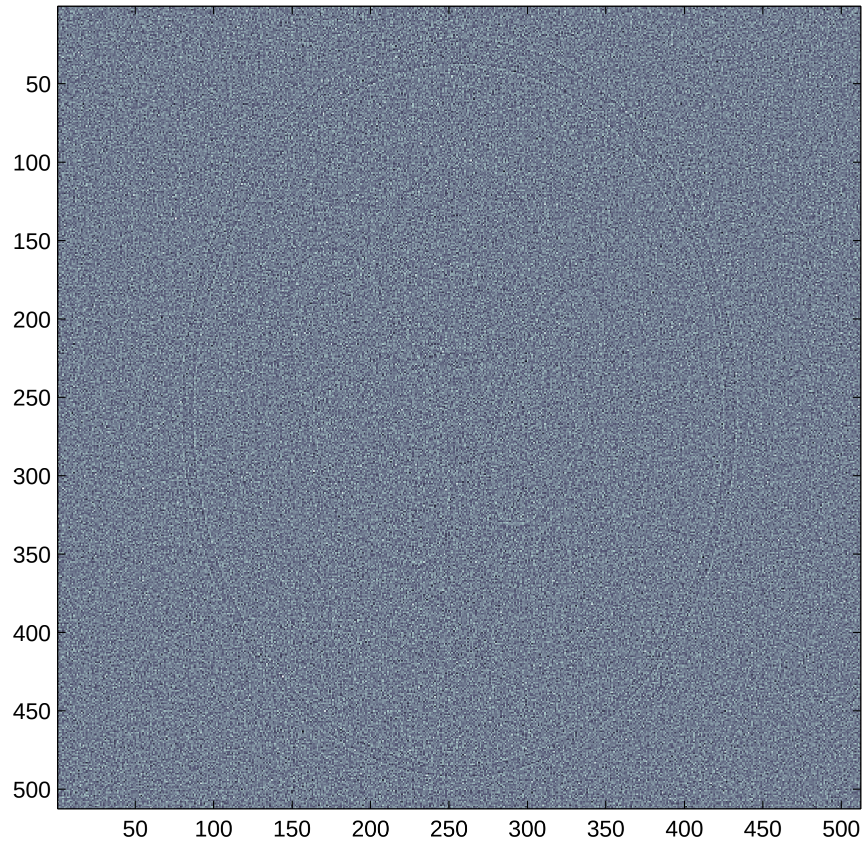
(a) Noisy Phantom



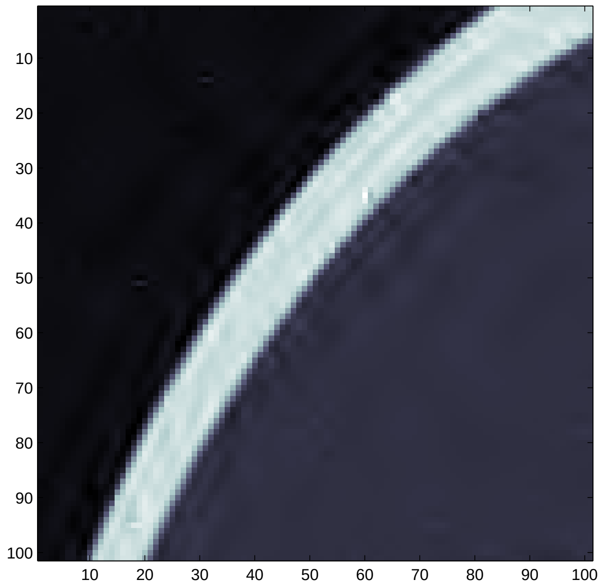
(b) Curvelets



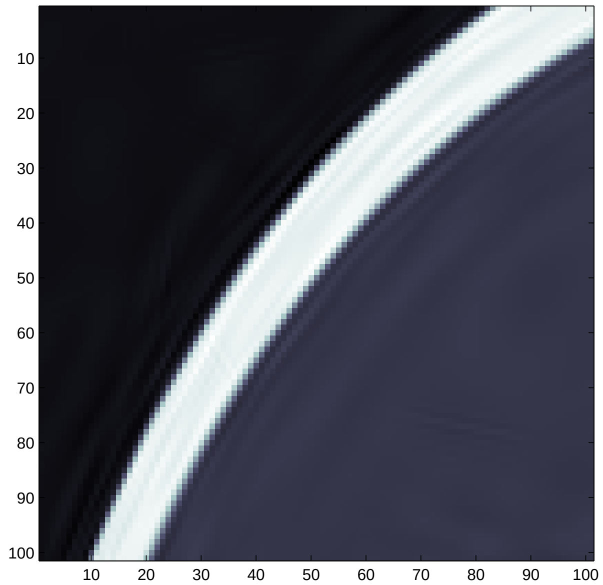
(c) Curvelets and TV



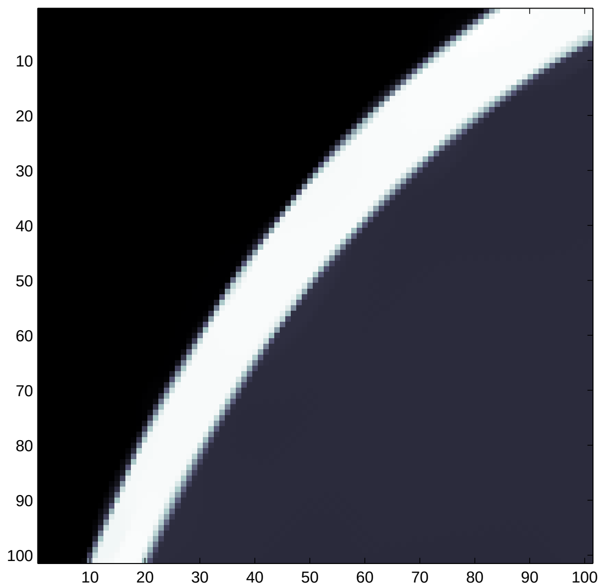
(d) Curvelets and TV: Residuals

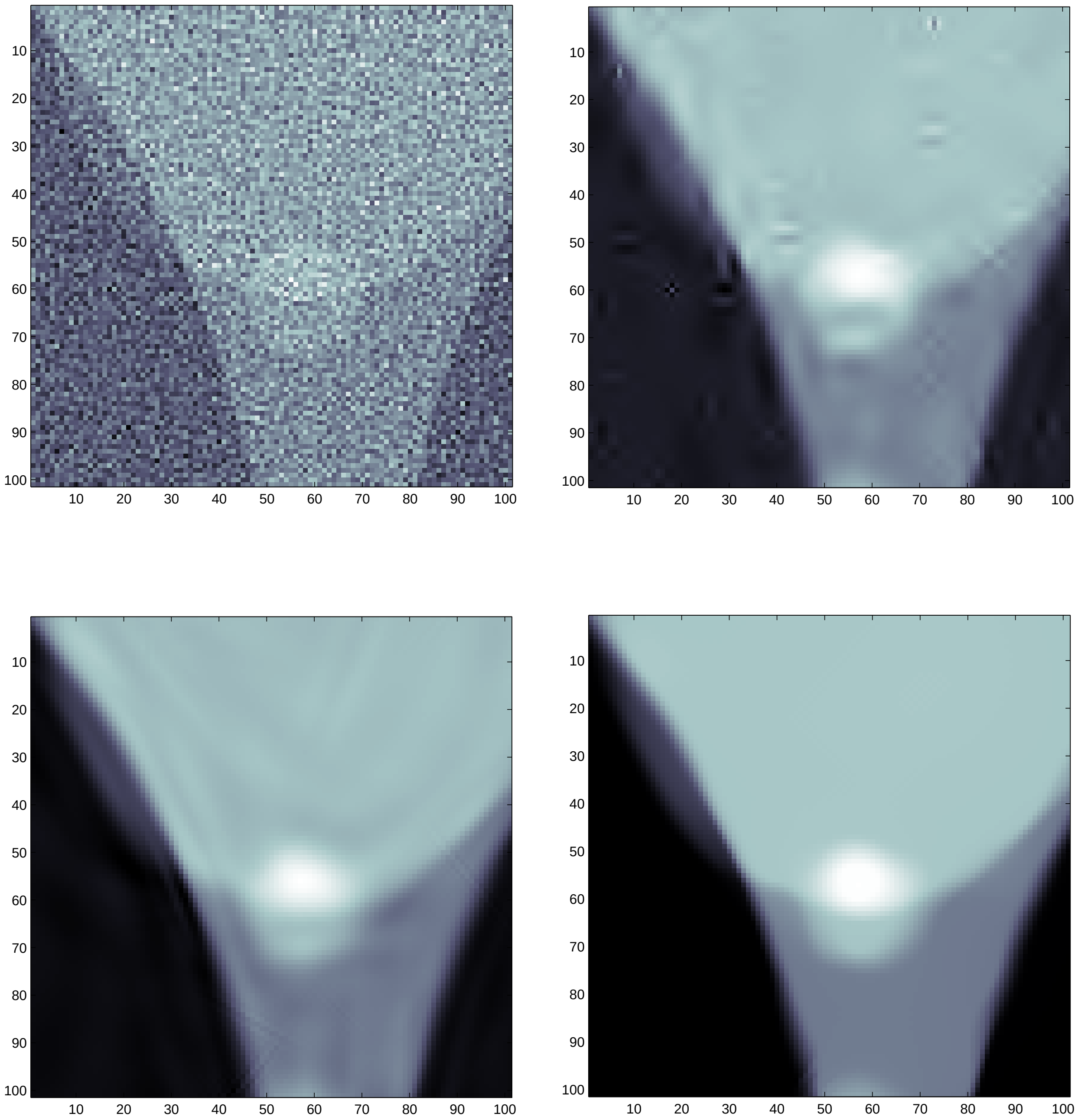


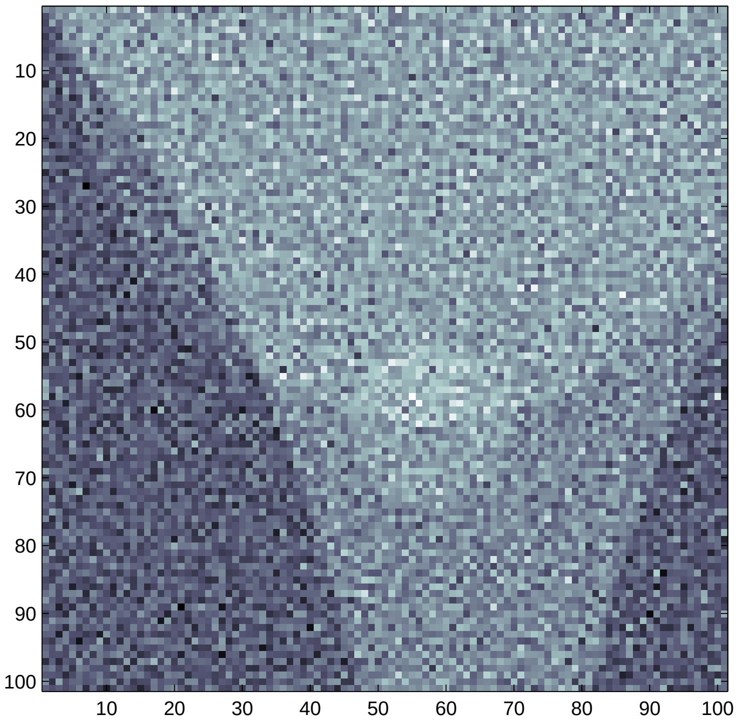
(e) Wavelets



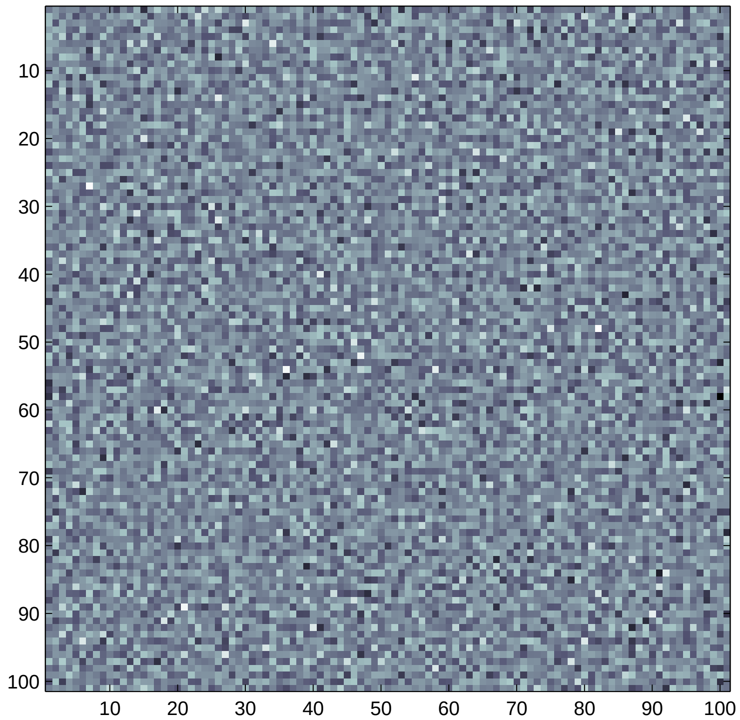
(f) Curvelets



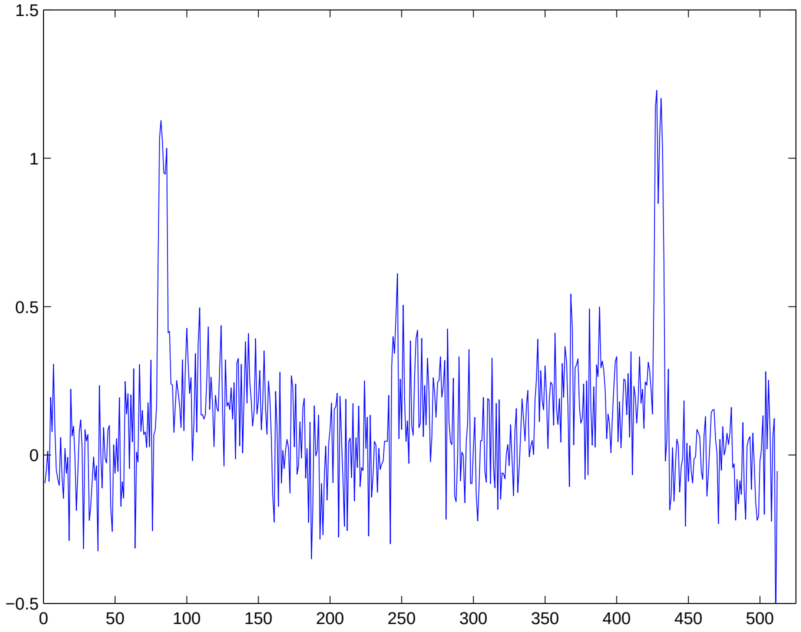

10 20 30 40 50 60 70 80 90 100



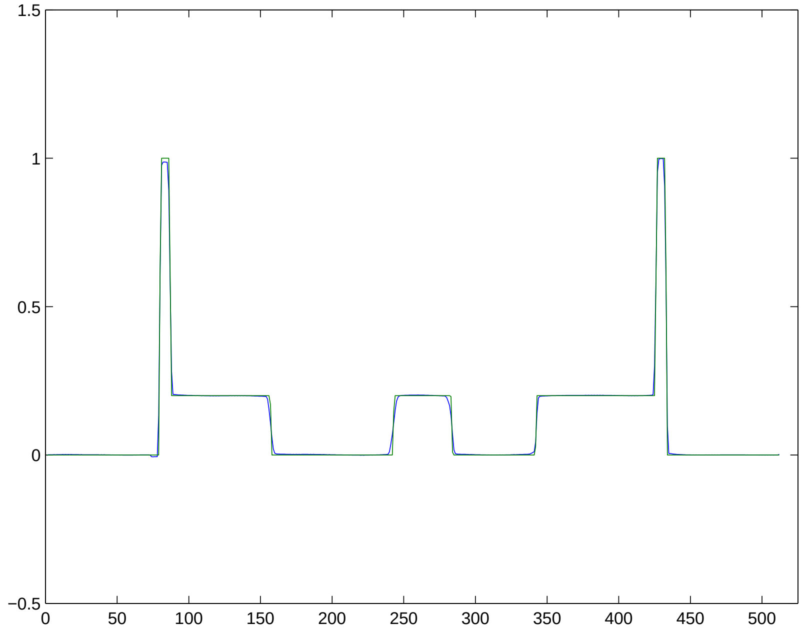
(l) Noisy



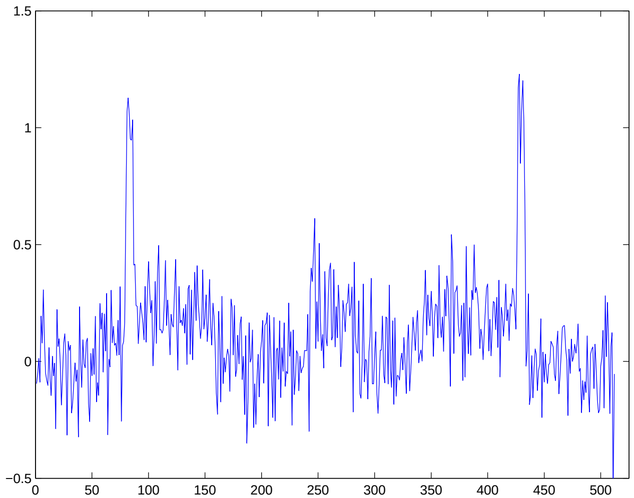
(m) Curvelets & TV: Residuals



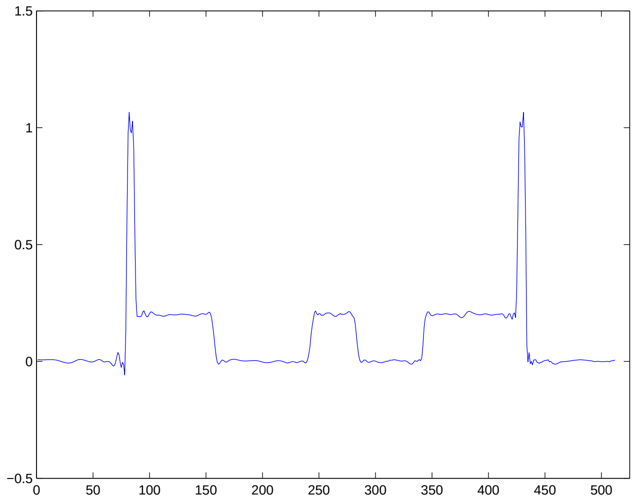
(n) Noisy Scanline



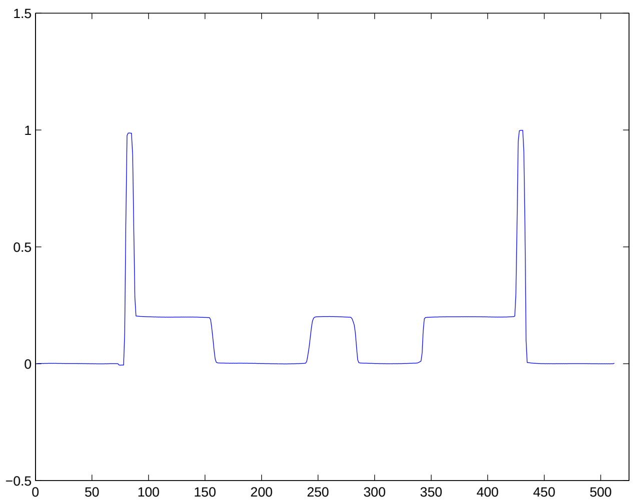
(o) True Scanline and Curvelets and TV Reconstruction



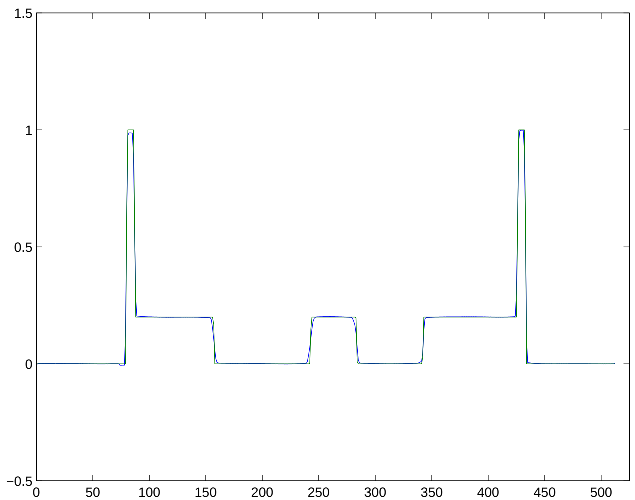
(a) Noisy Scanline



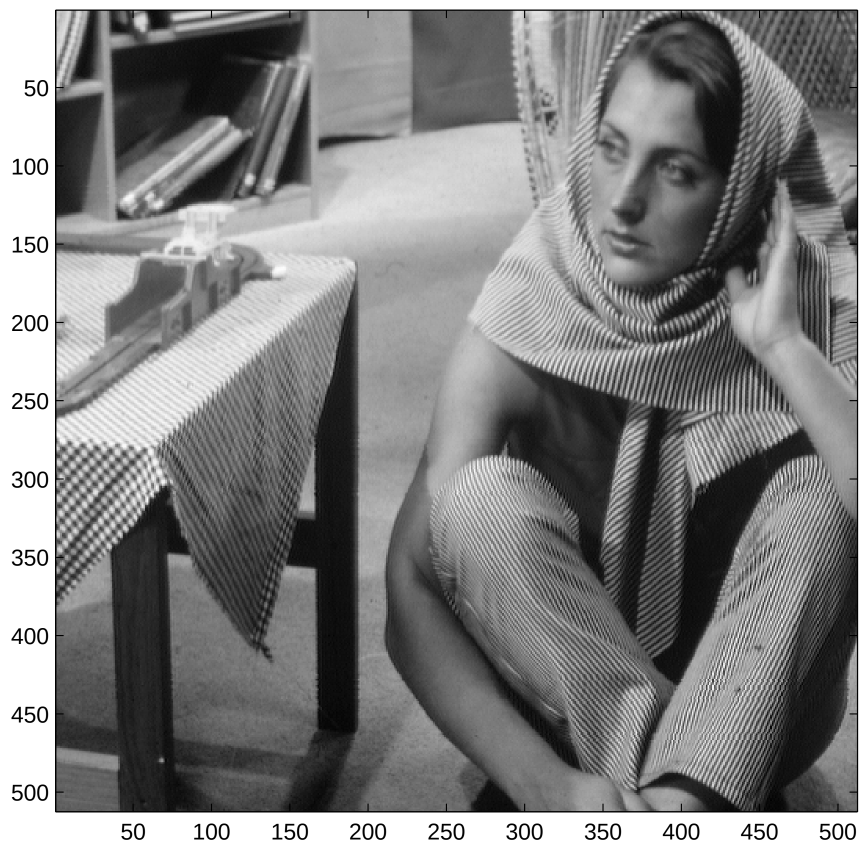
(b) Curvelets



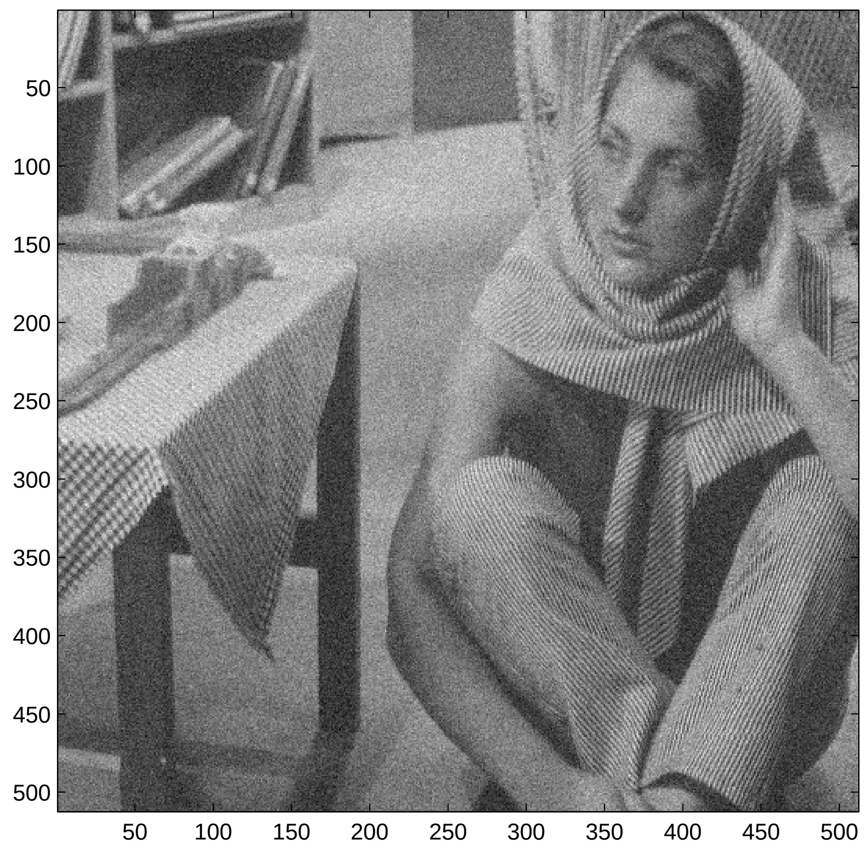
(c) Curvelets and TV



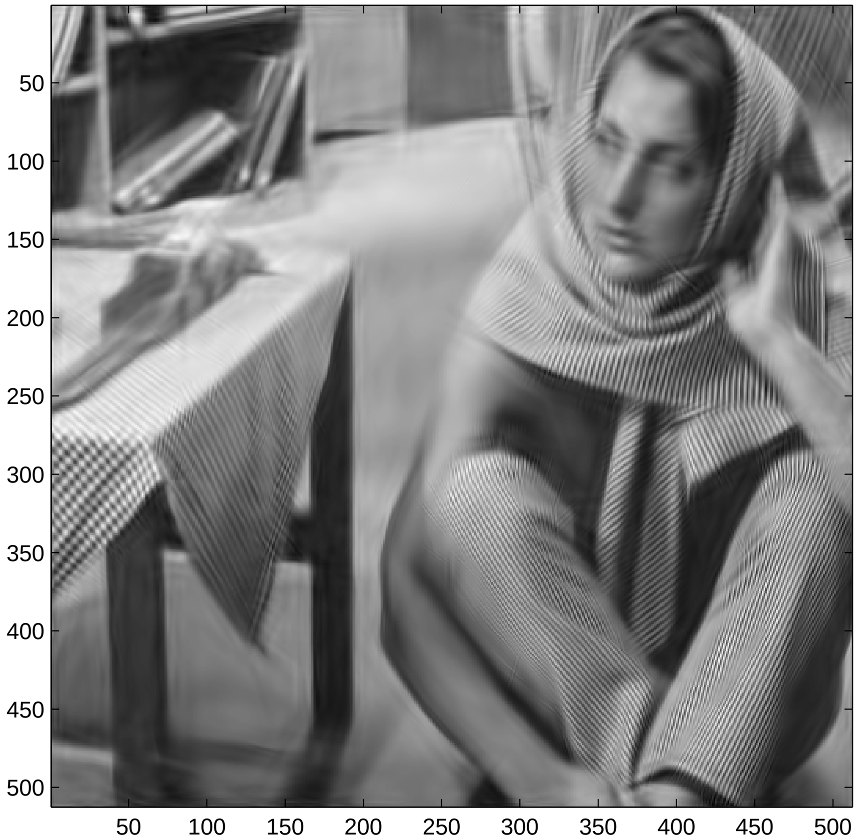
(d) True Scanline and Curvelets and TV Reconstruction



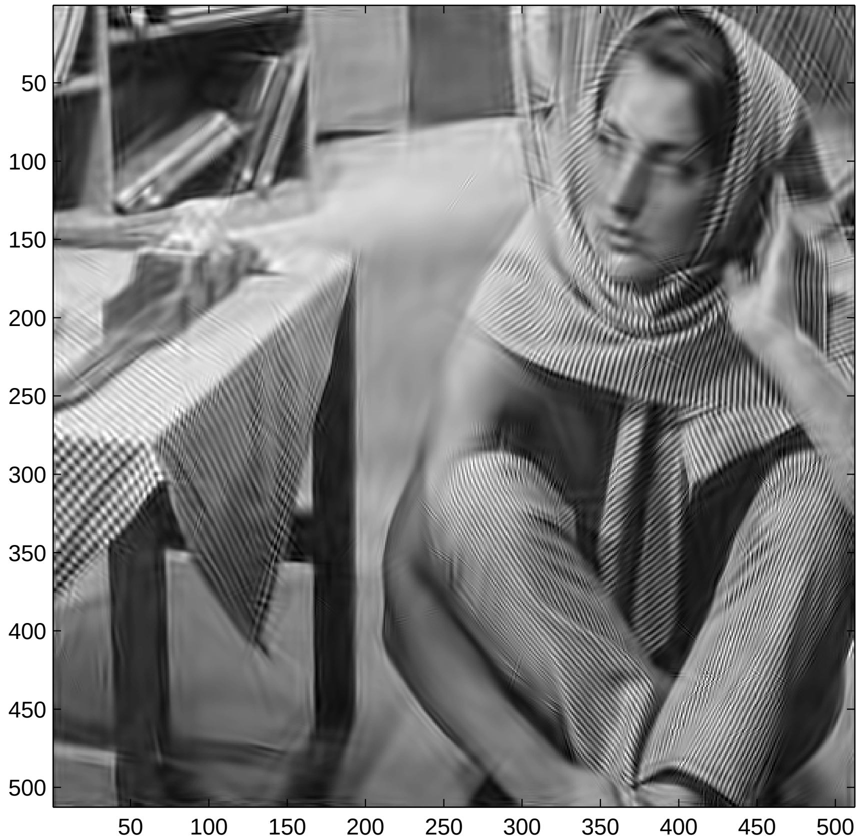
(p) Original



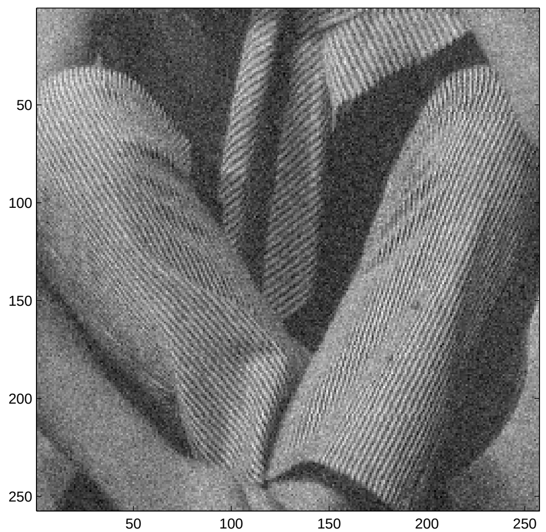
(q) Noisy



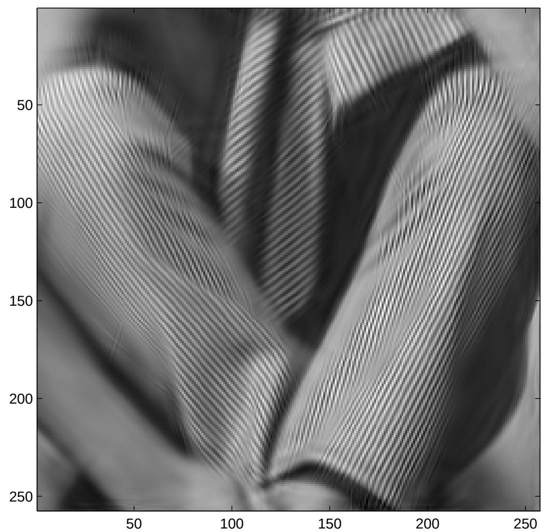
(r) Curvelets



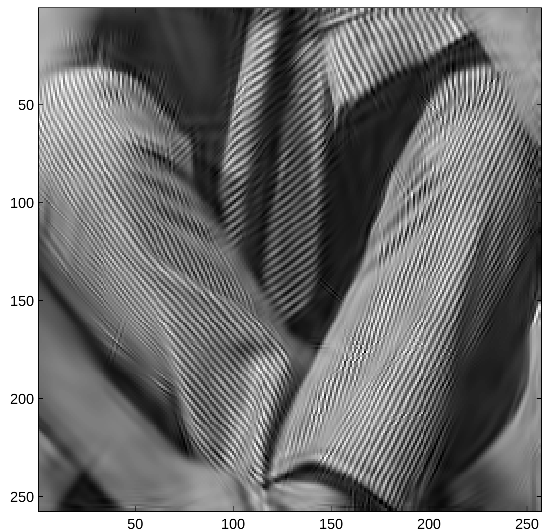
(s) Curvelets and TV



(t) Noisy Detail



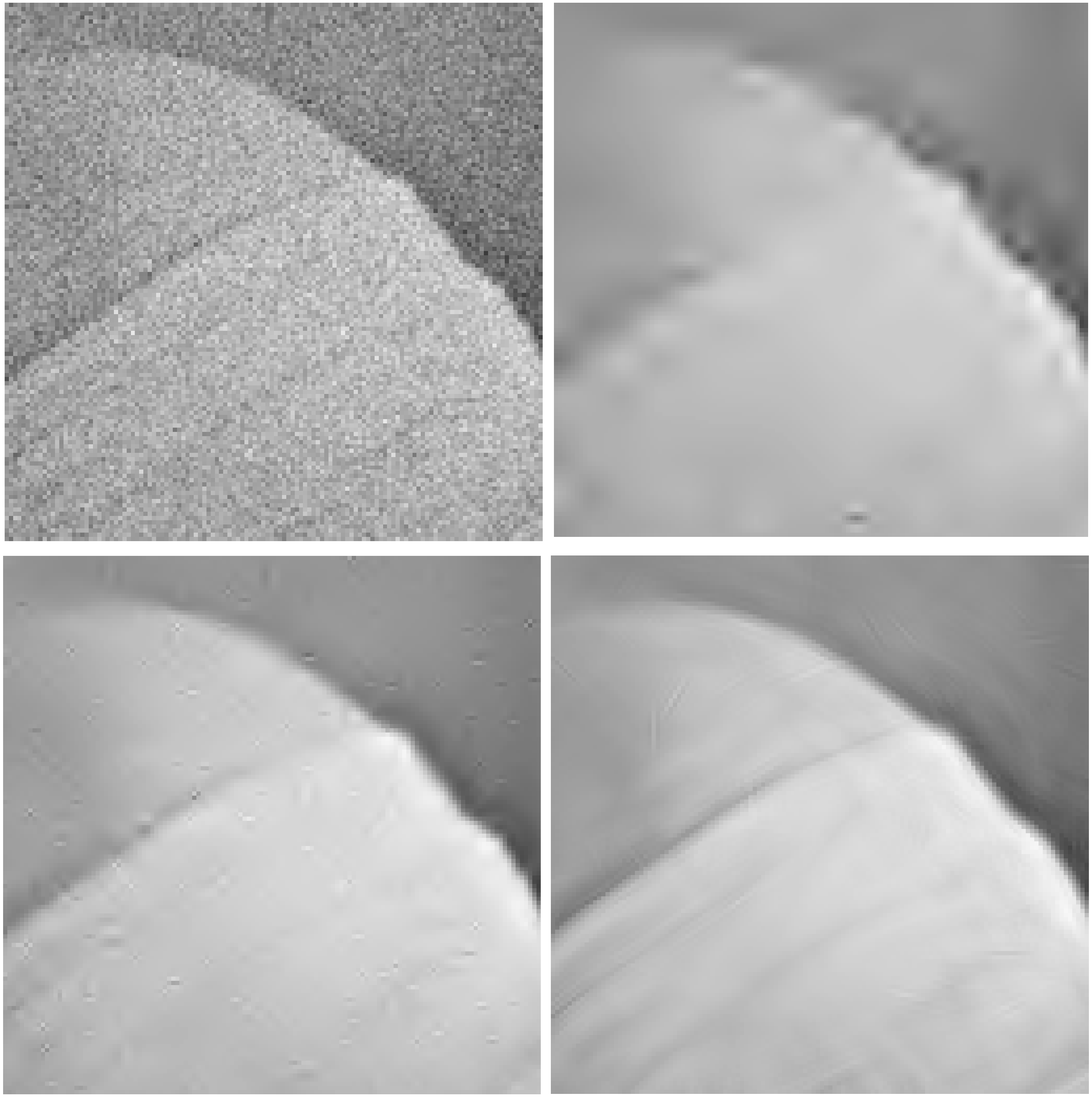
(u) Curvelets

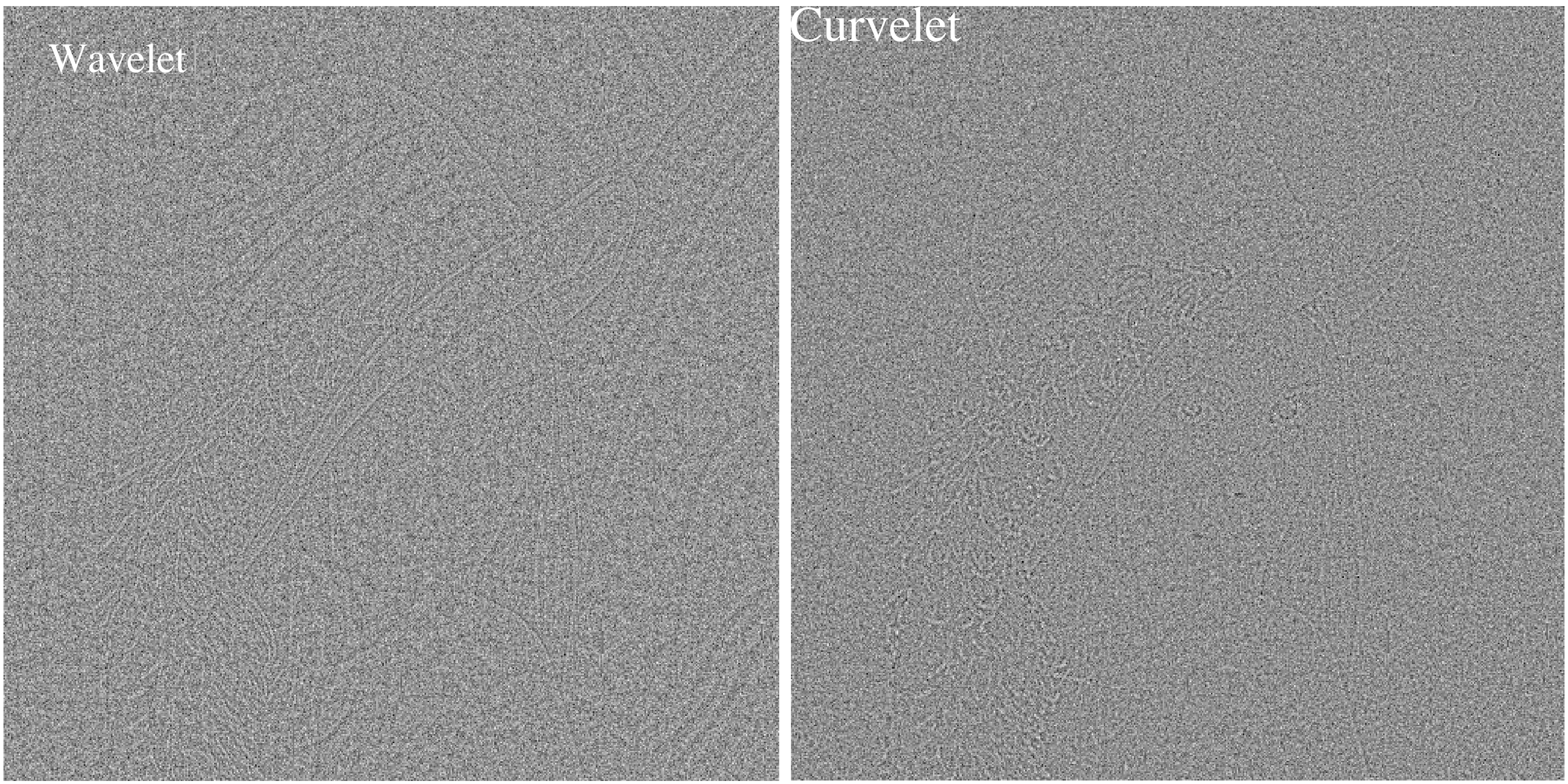


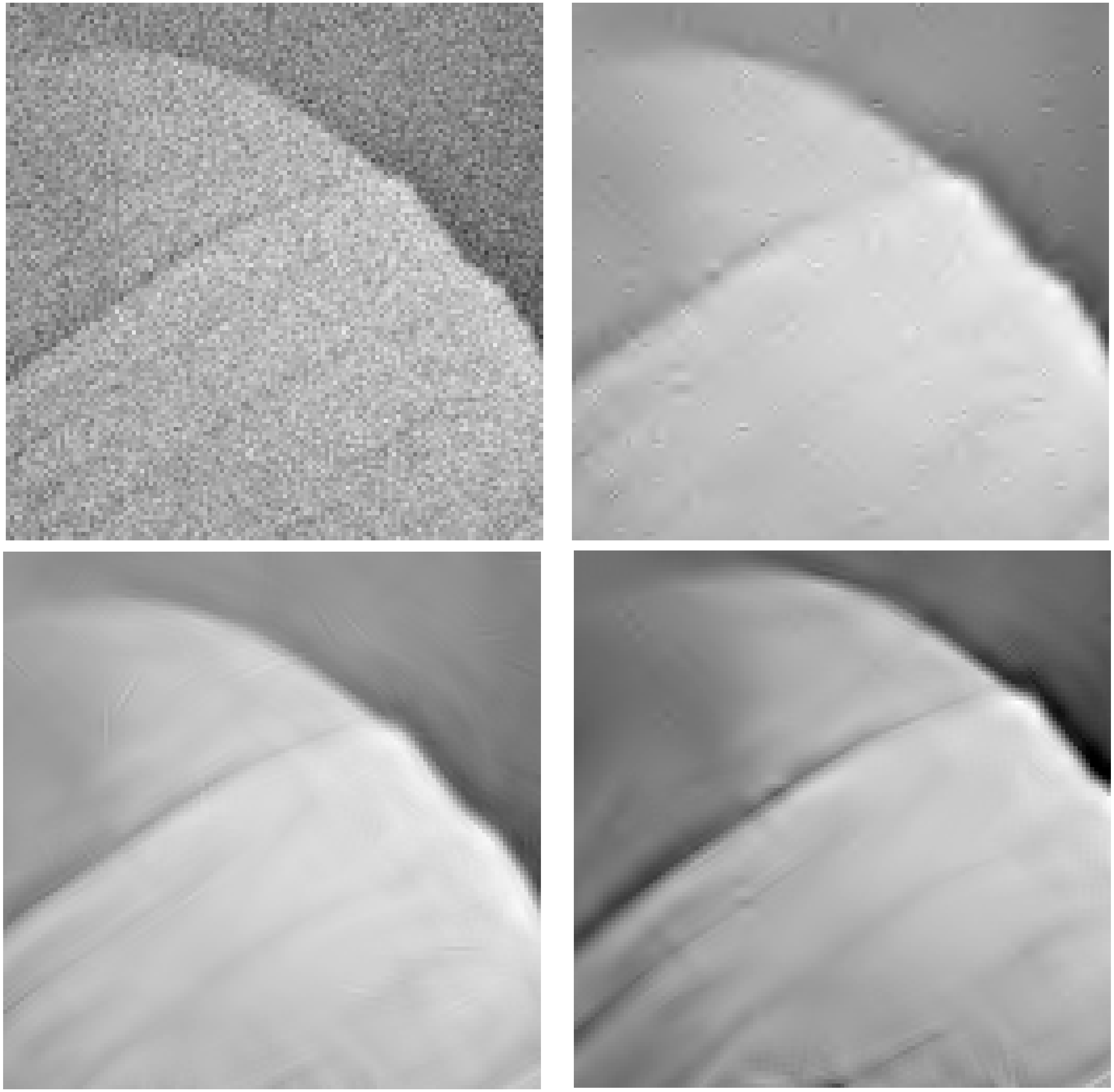
(v) Curvelets and TV

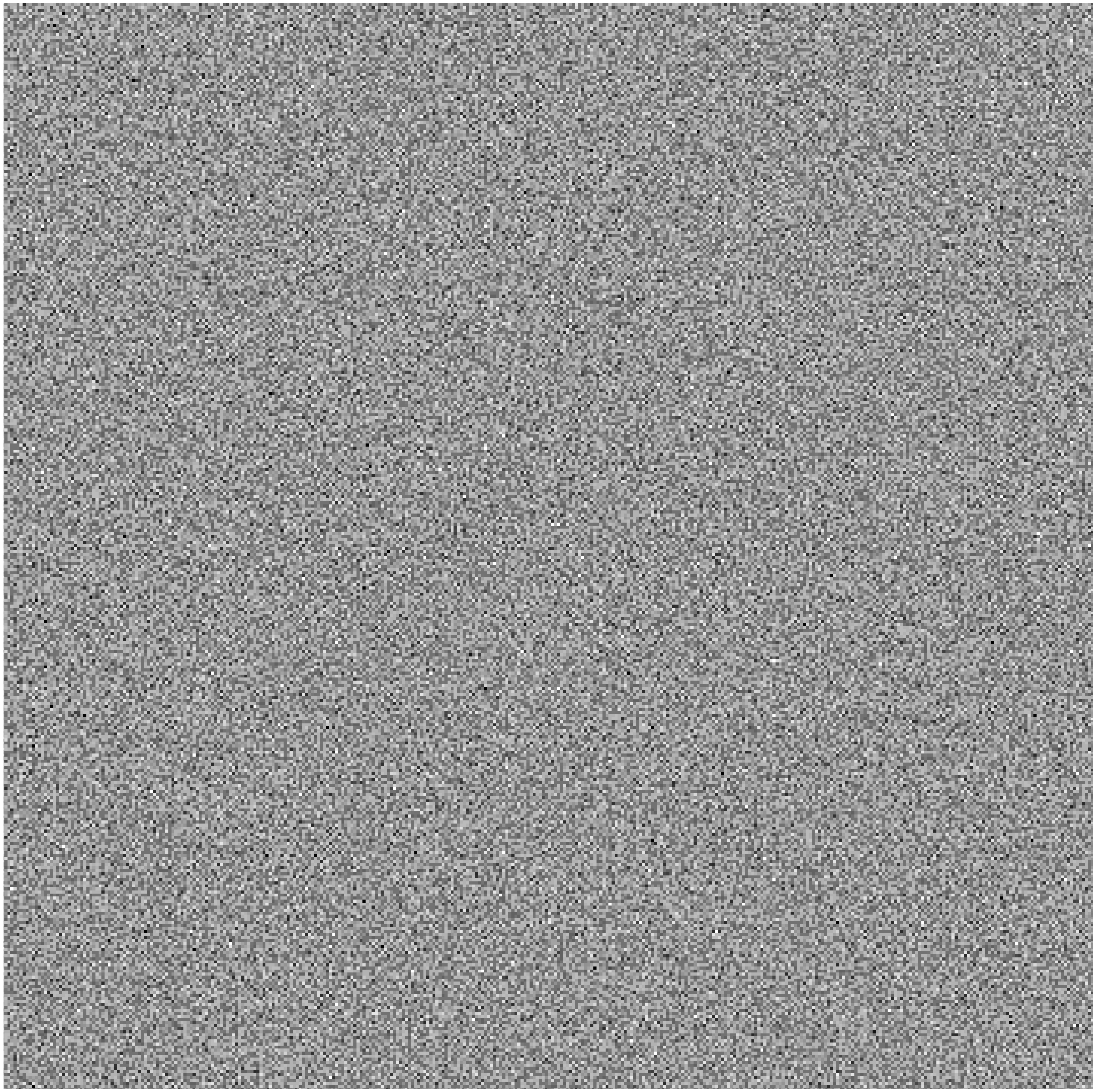
Numerical Experiments, II: Work of Jean-Luc Starck (CEA)

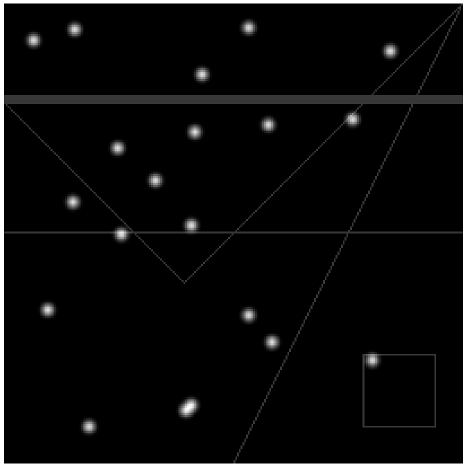
 



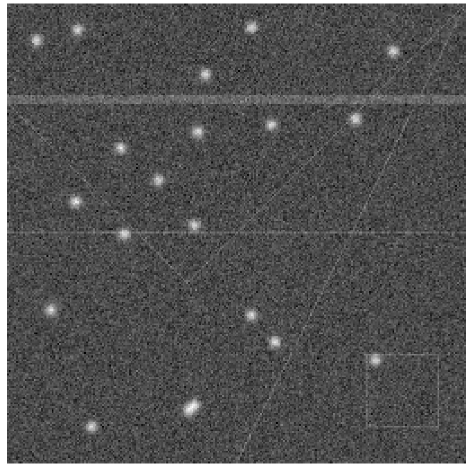

Wavelet
Curvelet



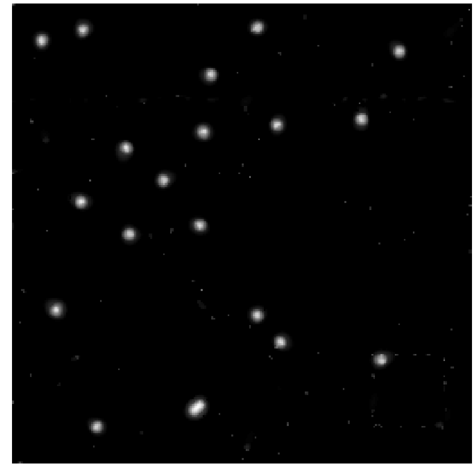




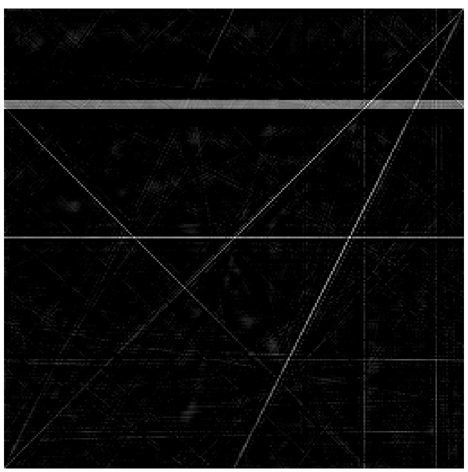
a) Simulated image (Gaussians+lines)



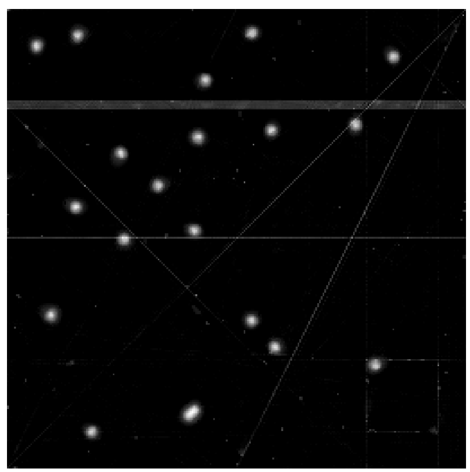
b) Simulated image + noise



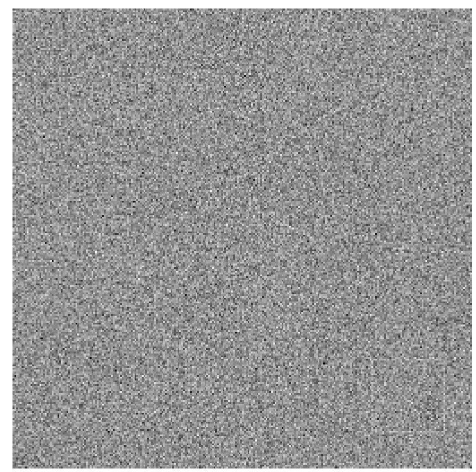
c) A trous algorithm



d) Curvelet transform



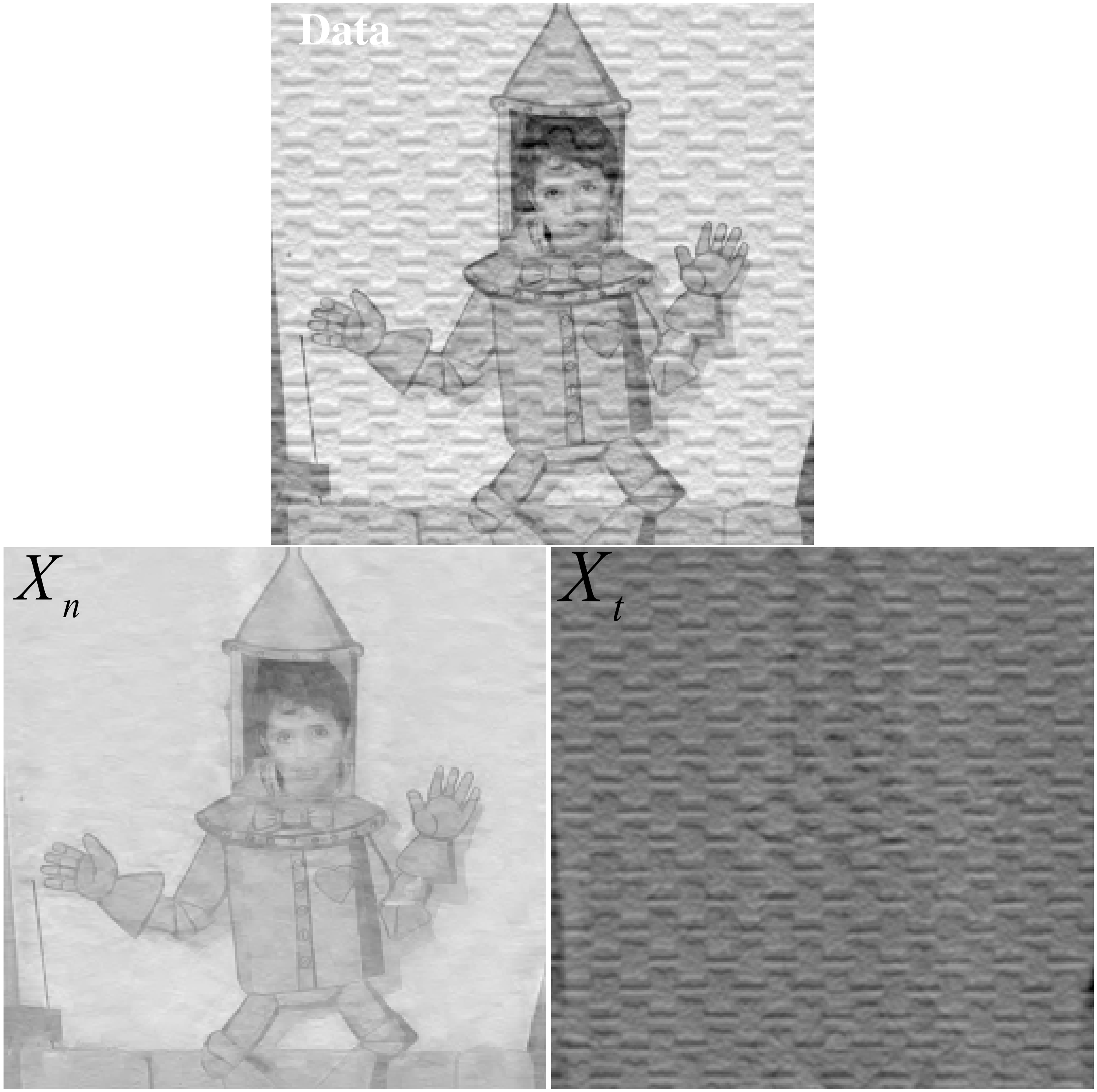
e) coaddition c+d



f) residual = e-b

Separation of Texture from Piecewise Smooth Content

<u>The separation task</u>: decomposition of an image into a texture and a natural (piecewise smooth) scene part.


Data
X_n
X_t