Wavelets and Approximation

Ronald DeVore

University of South Carolina

Univariate Approximation

- $f(x)$, $x \in [0,1]$

Univariate Approximation

- $f(x)$, $x \in [0,1]$
- Approximate f by piecewise constants

Univariate Approximation

- $f(x)$, $x \in [0, 1]$
- Approximate f by piecewise constants
- Types of approximation: Linear, Nonlinear

Univariate Approximation

- $f(x)$, $x \in [0,1]$
- Approximate f by piecewise constants
- Types of approximation: Linear, Nonlinear
- Select a metric/norm $\|\cdot\|$ to measure distortion

$$\|f\|_{L_p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}, \quad 0 < p < \infty$$

Univariate Approximation

- $f(x)$, $x \in [0,1]$
- Approximate f by piecewise constants
- Types of approximation: Linear, Nonlinear
- Select a metric/norm $\|\cdot\|$ to measure distortion

$$\|f\|_{L_p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}, \quad 0 < p < \infty$$

- $\|f\|_{L_\infty(\Omega)} := \sup_{x\in\Omega} |f(x)|, \quad p = \infty$

Linear approximation

- Divide $[0, 1]$ into n equal length intervals

Linear approximation

- Divide $[0,1]$ into n equal length intervals
- $\Pi_n := \{[\frac{k}{n}, \frac{k+1}{n}), \quad k = 0, \ldots, n-1\}$

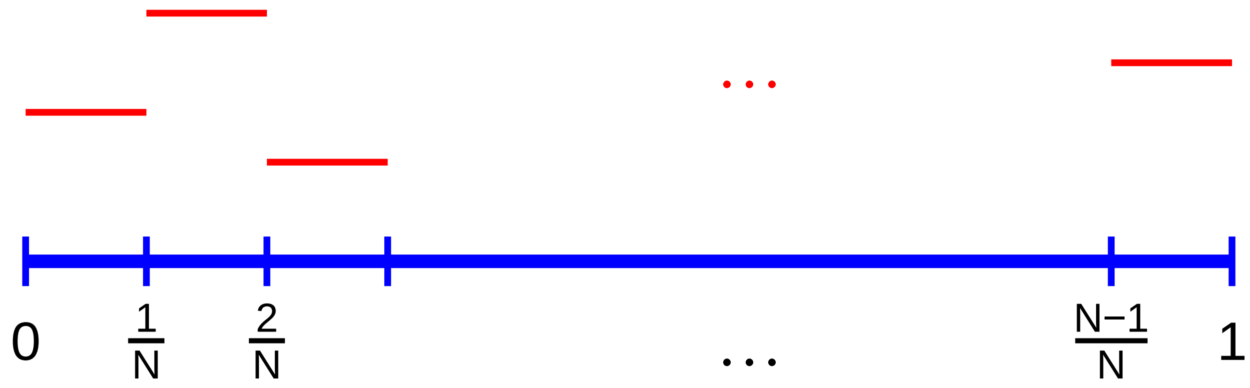
Linear approximation

- Divide $[0,1]$ into n equal length intervals
- $\Pi_n := \{[\frac{k}{n}, \frac{k+1}{n}), \quad k = 0, \ldots, n-1\}$
- $\mathcal{S}_n := \{S : S \; is \; constant \; on \; each \; I \in \Pi_n\}$

Linear approximation

- Divide $[0,1]$ into n equal length intervals
- $\Pi_n := \{[\frac{k}{n}, \frac{k+1}{n}), \quad k = 0, \ldots, n-1\}$
- $\mathcal{S}_n := \{S : S \; is \; constant \; on \; each \; I \in \Pi_n\}$
- $\mathcal{S}_n$ is a linear space

Typical function in $\mathcal{S}_n$



Linear approximation

- Divide $[0,1]$ into n equal length intervals
- $\Pi_n := \{[\frac{k}{n}, \frac{k+1}{n}), \quad k = 0, \dots, n-1\}$
- $\mathcal{S}_n := \{S : S \textit{ is constant on each } I \in \Pi_n\}$
- $\mathcal{S}_n$ is a linear space
- Given $f \in L_p[0,1]$, define error

$$E_n(f)_p := \inf_{S \in \mathcal{S}_n} \|f - S\|_{L_p}$$

Nonlinear: Free Knot Splines

- Divide $[0,1]$ into n intervals $I_i = [\xi_i, \xi_{i+1}]$,
  $0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$

Nonlinear: Free Knot Splines

- Divide $[0,1]$ into n intervals $I_i = [\xi_i, \xi_{i+1}]$,
  $0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$
- $\Sigma_n := \{S : S \textit{ is constant on each } I_i, i = 1, \ldots, n\}$

Nonlinear: Free Knot Splines

- Divide $[0,1]$ into n intervals $I_i = [\xi_i, \xi_{i+1}]$, $0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$
- $\Sigma_n := \{S : S \textit{ is constant on each } I_i, i = 1, \ldots, n\}$
- Σ_n is not a linear space: $S_1, S_2 \in \Sigma_n$ but $S_1 + S_2 \in \Sigma_{2n}$

Nonlinear: Free Knot Splines

- Divide $[0,1]$ into n intervals $I_i = [\xi_i, \xi_{i+1}]$, $0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$
- $\Sigma_n := \{S : S \textit{ is constant on each } I_i, i = 1, \ldots, n\}$
- Σ_n is not a linear space: $S_1, S_2 \in \Sigma_n$ but $S_1 + S_2 \in \Sigma_{2n}$
- Given $f \in L_p[0,1]$, define error

$$\sigma_n(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p}$$

Nonlinear: Adaptive

- For each interval I, $E(I)$ the local L_p error in approximating f by constants

$$E(I) := \inf_c \|f - c\|_{L_p(I)}$$

Nonlinear: Adaptive

- For each interval I, $E(I)$ the local L_p error in approximating f by constants

$$E(I) := \inf_c \|f - c\|_{L_p(I)}$$

- Given error tolerance $\epsilon > 0$ generate partition $\mathcal{P}_\epsilon$ such that $E(I) \leq \epsilon$ for all $I \in \mathcal{P}_\epsilon$

Nonlinear: Adaptive

- For each interval I, $E(I)$ the local L_p error in approximating f by constants

$$E(I) := \inf_c \|f - c\|_{L_p(I)}$$

- Given error tolerance $\epsilon > 0$ generate partition $\mathcal{P}_\epsilon$ such that $E(I) \leq \epsilon$ for all $I \in \mathcal{P}_\epsilon$
- I is good if $E(I) \leq \epsilon$

Nonlinear: Adaptive

- For each interval I, $E(I)$ the local L_p error in approximating f by constants

$$E(I) := \inf_c \|f - c\|_{L_p(I)}$$

- Given error tolerance $\epsilon > 0$ generate partition $\mathcal{P}_\epsilon$ such that $E(I) \leq \epsilon$ for all $I \in \mathcal{P}_\epsilon$
- I is good if $E(I) \leq \epsilon$
- I is bad if $E(I) > \epsilon$

Nonlinear: Adaptive

- Initially if $E([0,1]) \leq \epsilon$ then algorithm terminates and $\mathcal{P}_\epsilon := \{[0,1]\}$

Nonlinear: Adaptive

- Initially if $E([0,1]) \le \epsilon$ then algorithm terminates and $\mathcal{P}_\epsilon := \{[0,1]\}$
- if $E([0,1]) > \epsilon$ then set $\mathcal{B}_\epsilon := \{[0,1]\}$ and $\mathcal{G}_\epsilon := \emptyset$

Nonlinear: Adaptive

- Initially if $E([0,1]) \leq \epsilon$ then algorithm terminates and $\mathcal{P}_\epsilon := \{[0,1]\}$
- if $E([0,1]) > \epsilon$ then set $\mathcal{B}_\epsilon := \{[0,1]\}$ and $\mathcal{G}_\epsilon := \emptyset$
- Recursion: For each $I \in \mathcal{B}_\epsilon$ put child J of I in $\mathcal{G}_\epsilon$ if it is good, put it in $\mathcal{B}_\epsilon$ if it is bad. Remove I from $\mathcal{B}_\epsilon$

Nonlinear: Adaptive

- Initially if $E([0,1]) \leq \epsilon$ then algorithm terminates and $\mathcal{P}_\epsilon := \{[0,1]\}$
- if $E([0,1]) > \epsilon$ then set $\mathcal{B}_\epsilon := \{[0,1]\}$ and $\mathcal{G}_\epsilon := \emptyset$
- Recursion: For each $I \in \mathcal{B}_\epsilon$ put child J of I in $\mathcal{G}_\epsilon$ if it is good, put it in $\mathcal{B}_\epsilon$ if it is bad. Remove I from $\mathcal{B}_\epsilon$
- Stop when $\mathcal{B}_\epsilon = \emptyset$, $\mathcal{P}_\epsilon := \mathcal{G}_\epsilon$, $N_\epsilon := \#(\mathcal{P}_\epsilon)$

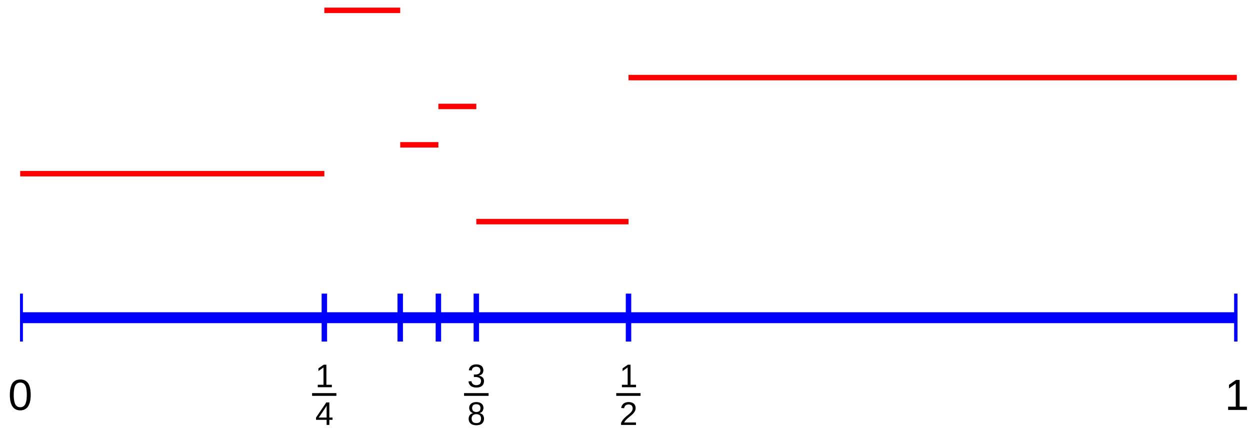
Nonlinear: Adaptive

- $A_{N_\epsilon}(f)$ best approximation to f by piecewise constants on $\mathcal{P}_\epsilon$

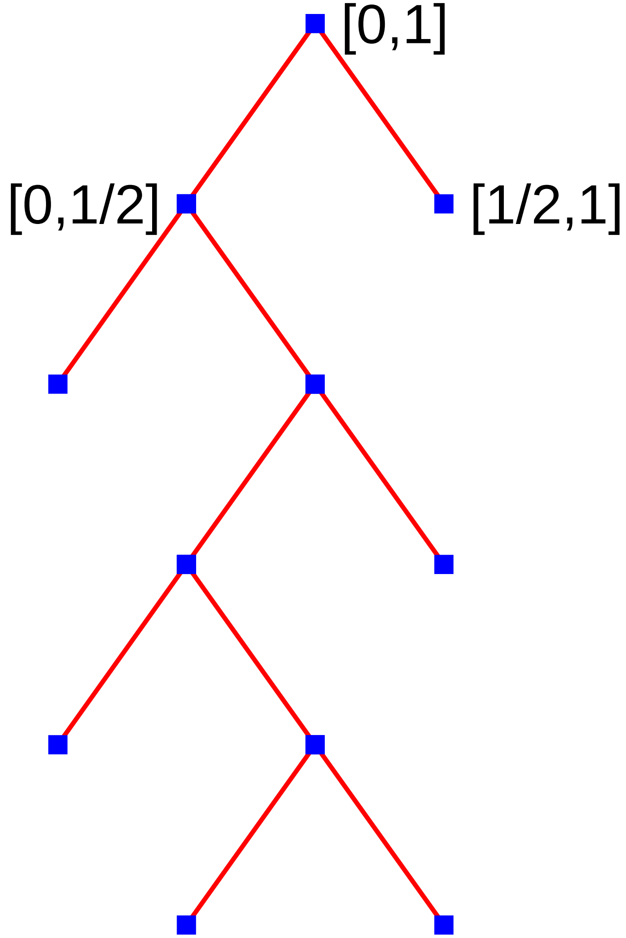
Nonlinear: Adaptive

- $A_{N_\epsilon}(f)$ best approximation to f by piecewise constants on $\mathcal{P}_\epsilon$
- $a_n(f)_p := \inf\{\epsilon : N_\epsilon \leq n\}$

Adaptively generated partition



Tree associated to adaptive partition



Comparison

- Approximation classes: $\alpha > 0$
  define $\mathcal{A}^\alpha(L_p,\ linear\ splines)$ as the set of all $f \in L_p[0,1]$ such that

$$E_n(f)_p \leq Cn^{-\alpha}, \quad n \geq 1$$

Comparison

- Approximation classes: $\alpha > 0$
  define$\mathcal{A}^\alpha(L_p,\ linear\ splines)$ as the set of all $f \in L_p[0,1]$ such that

$$E_n(f)_p \leq Cn^{-\alpha}, \quad n \geq 1$$

- Similarly define $\mathcal{A}^\alpha(L_p)$ for the other forms of approximation

Comparison

- Approximation classes: $\alpha > 0$
  define$\mathcal{A}^\alpha(L_p,\ linear\ splines)$ as the set of all $f \in L_p[0,1]$ such that

$$E_n(f)_p \leq C n^{-\alpha}, \quad n \geq 1$$

- Similarly define $\mathcal{A}^\alpha(L_p)$ for the other forms of approximation
- $\mathcal{A}^\alpha_q(L_p)$ finer scaling: same approximation order α

$$|f|_{\mathcal{A}^\alpha_q(L_p)} := (\sum_{n=1}^{\infty} [n^\alpha E_n(f)_p]^q)^{1/q}$$

Approximation Classes: Linear

- Fix the L_p space to measure error

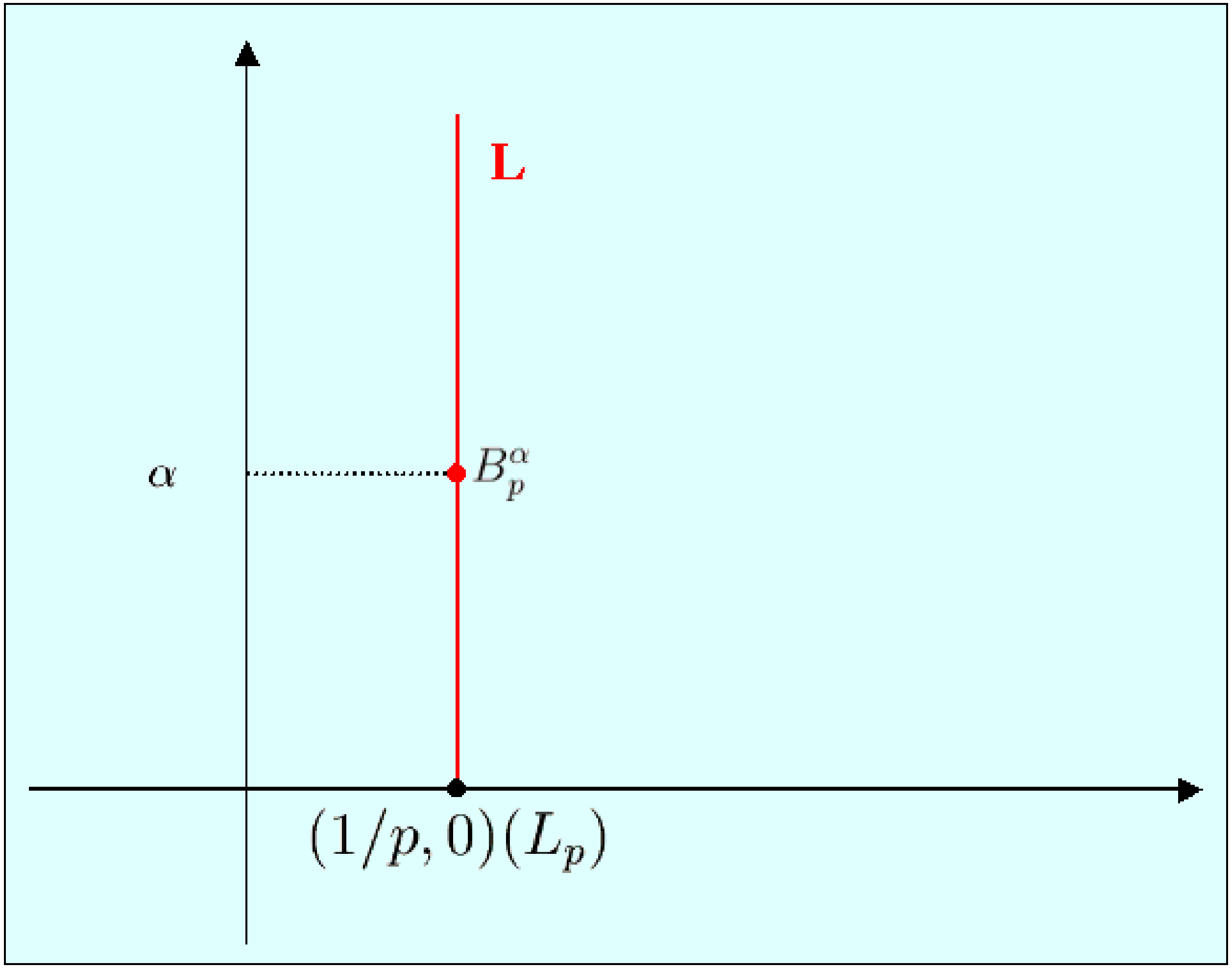
Approximation Classes: Linear

- Fix the L_p space to measure error
- $A^s(L_p,\ linear) = B^s_\infty(L_p)$

Approximation Classes: Linear

- Fix the L_p space to measure error
- $A^s(L_p,\ linear) = B^s_\infty(L_p)$
- Proved by Scherer +

pproximation: $\mathcal{A}^s_\infty(L_p)$ Besov space of smo



Approximation Classes: free knot splines

- Fix the L_p space to measure error

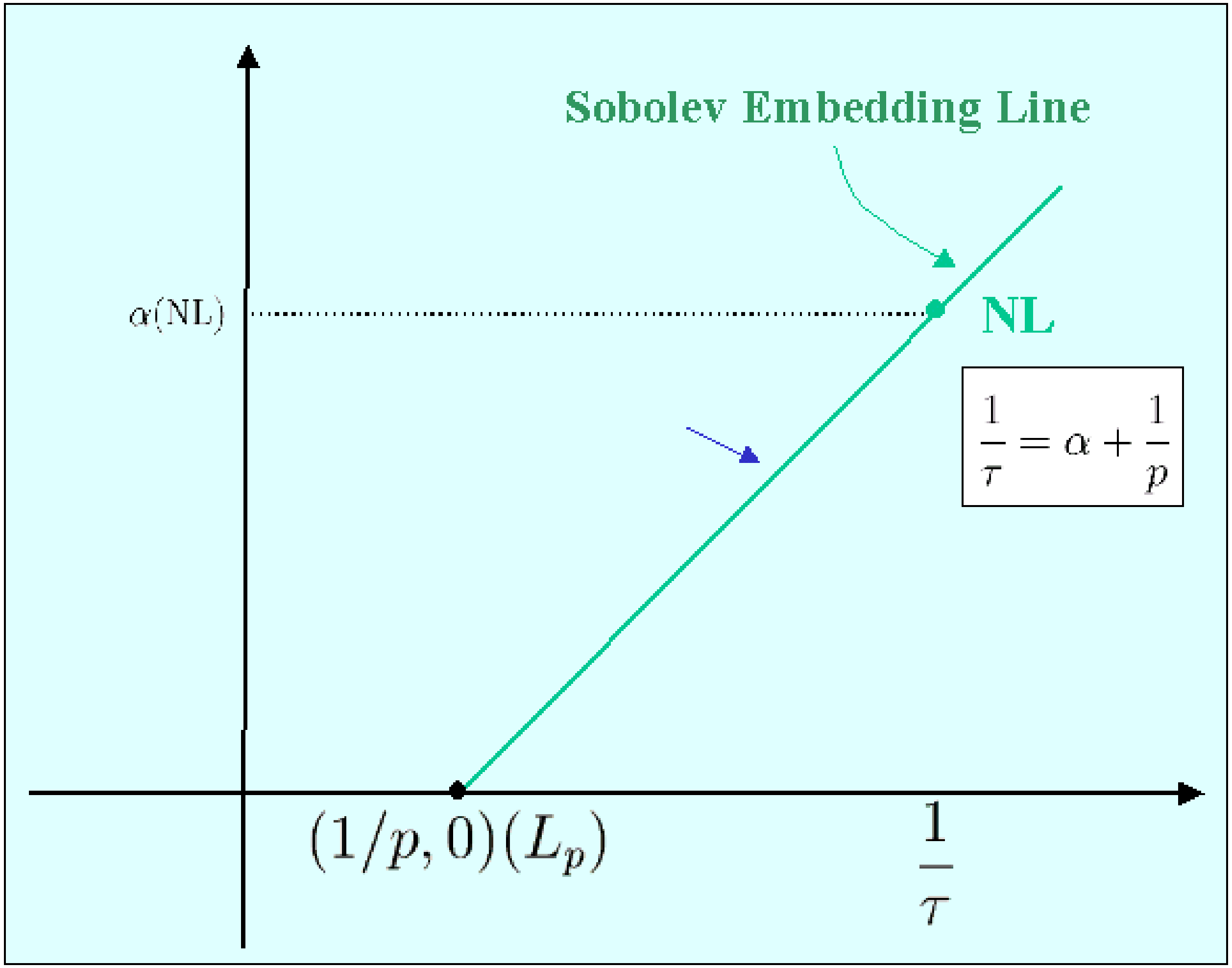
Approximation Classes: free knot splines

- Fix the L_p space to measure error
- $A^s_\tau(L_p,\ nonlinear) = B^s_\tau(L_\tau)$, $\frac{1}{\tau} = s + \frac{1}{p}$

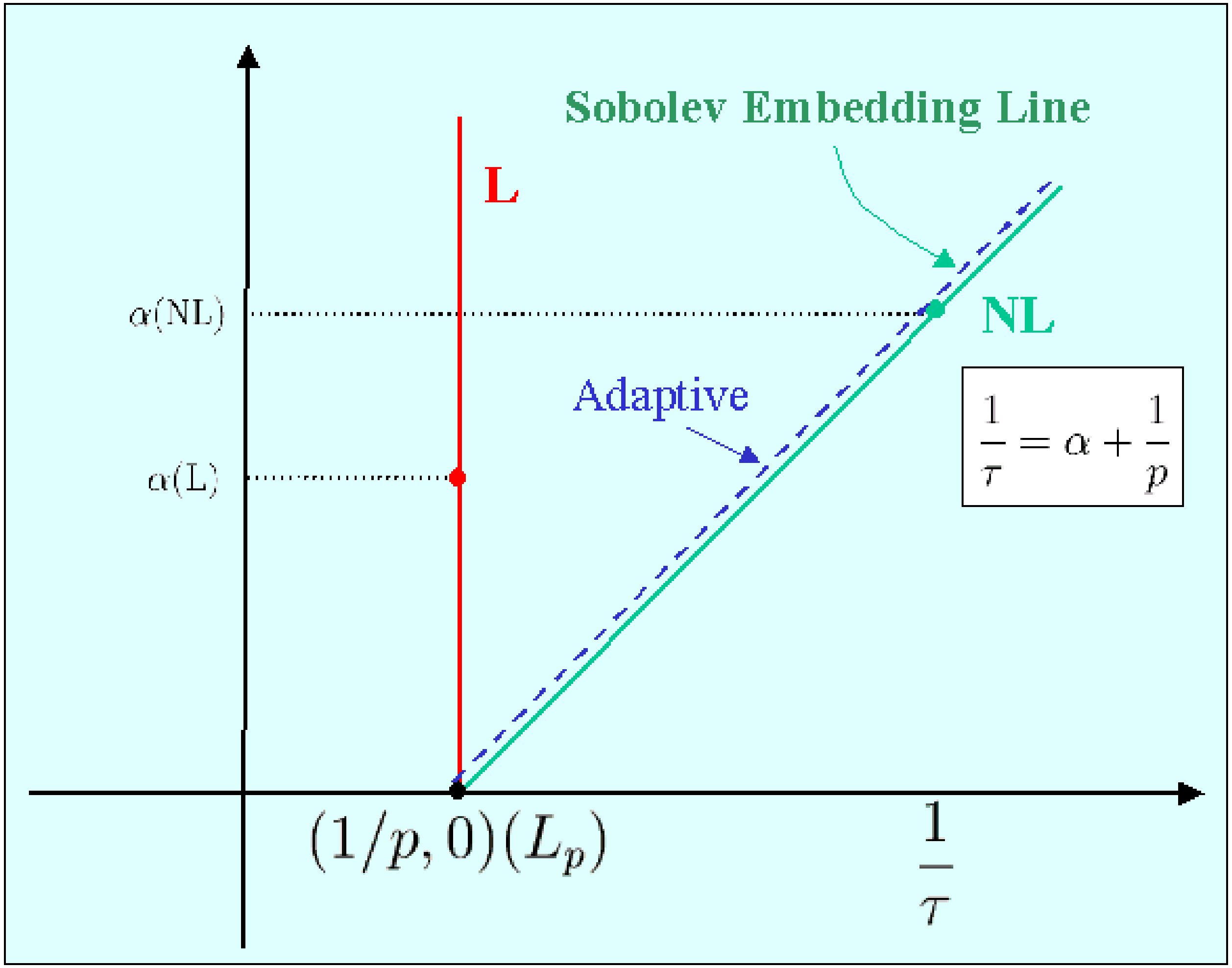
Approximation Classes: free knot splines

- Fix the L_p space to measure error
- $A^s_\tau(L_p,\ nonlinear) = B^s_\tau(L_\tau)$, $\frac{1}{\tau} = s + \frac{1}{p}$
- Petrushev, DeVore-Popov (splines); DeVore-Jawerth-Popov (wavelets)

Approximation class: free knot splines



Adaptive approximation



Example: Approximation in L_∞

- $p = \infty$ approximation order $O(n^{-1})$

Example: Approximation in L_∞

- $p = \infty$ approximation order $O(n^{-1})$
- Linear approximation $f' \in L_\infty$

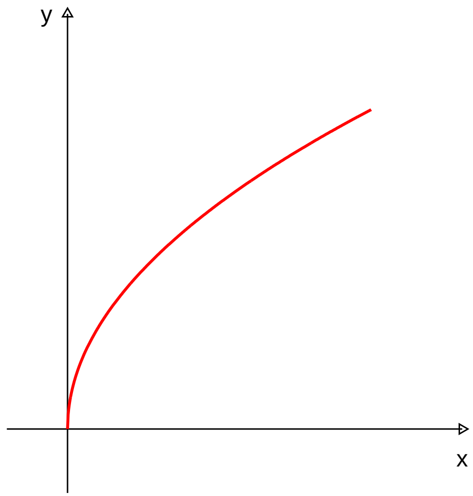
Example: Approximation in L_∞

- $p = \infty$ approximation order $O(n^{-1})$
- Linear approximation $f' \in L_\infty$
- Nonlinear approximation (free knot splines) $f' \in L_1$ or $f \in BV$

Example: Approximation in L_∞

- $p = \infty$ approximation order $O(n^{-1})$
- Linear approximation $f' \in L_\infty$
- Nonlinear approximation (free knot splines) $f' \in L_1$ or $f \in BV$
- Adaptive approximation $f' \in L log L$: for example $f' \in L_p$ for some $p > 1$

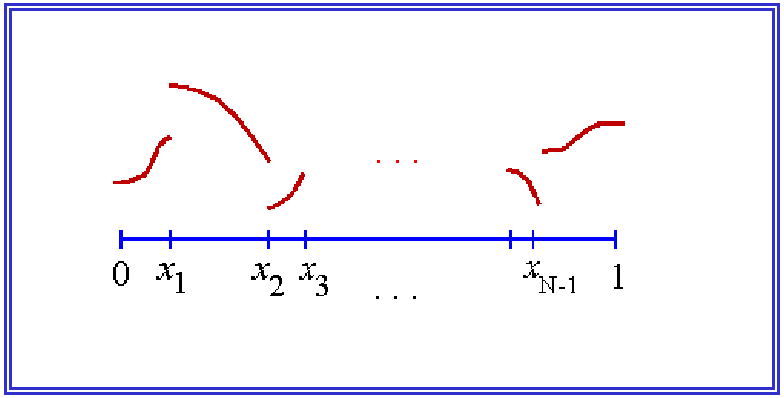
Example: $f(x) = x^\alpha$, $0 < \alpha < 1 - 1/p$



$$E_n(f)_p \approx Cn^{-(\alpha+1/p)} \qquad \sigma_n(f)_p \leq Cn^{-1}$$

Break points/ wavelets concentrate near singularity at 0

Example: piecewise smooth

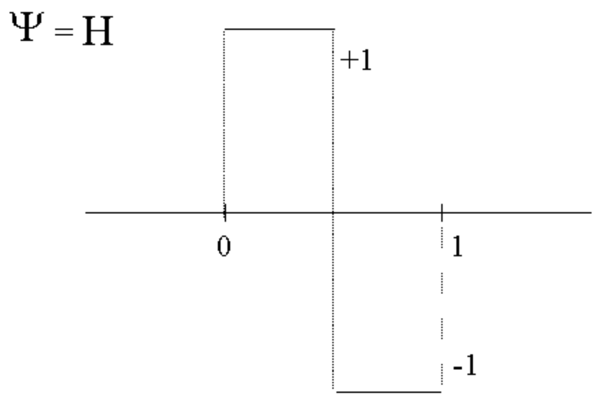


$$E_n(f)_p \geq Cn^{-1/p} \quad \sigma_n(f)_p \leq Cn^{-1}$$

Breakpoints/wavelets concentrate near singularities

Wavelets: Haar Wavelet

$$H(x) := \begin{cases} -1, & x \in [0, 1/2) \\ +1, & x \in [1/2, 1] \end{cases},$$



Wavelets: Haar Basis

- $H_I(x) := 2^{j/2} H(2^j x - k)$, $I = [k2^{-j}, (k+1)2^{-j}]$

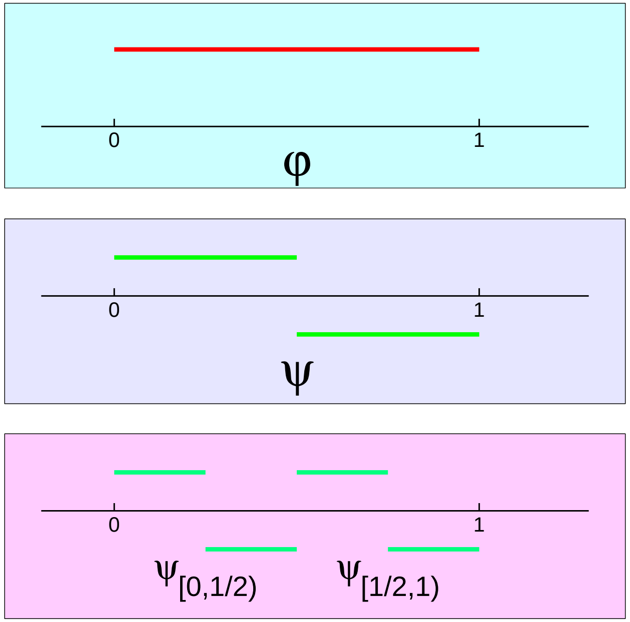
Wavelets: Haar Basis

- $H_I(x) := 2^{j/2} H(2^j x - k)$, $I = [k2^{-j}, (k+1)2^{-j}]$
- $\mathcal{D}_+ := \{I \in \mathcal{D} : |I| \leq 1\}$

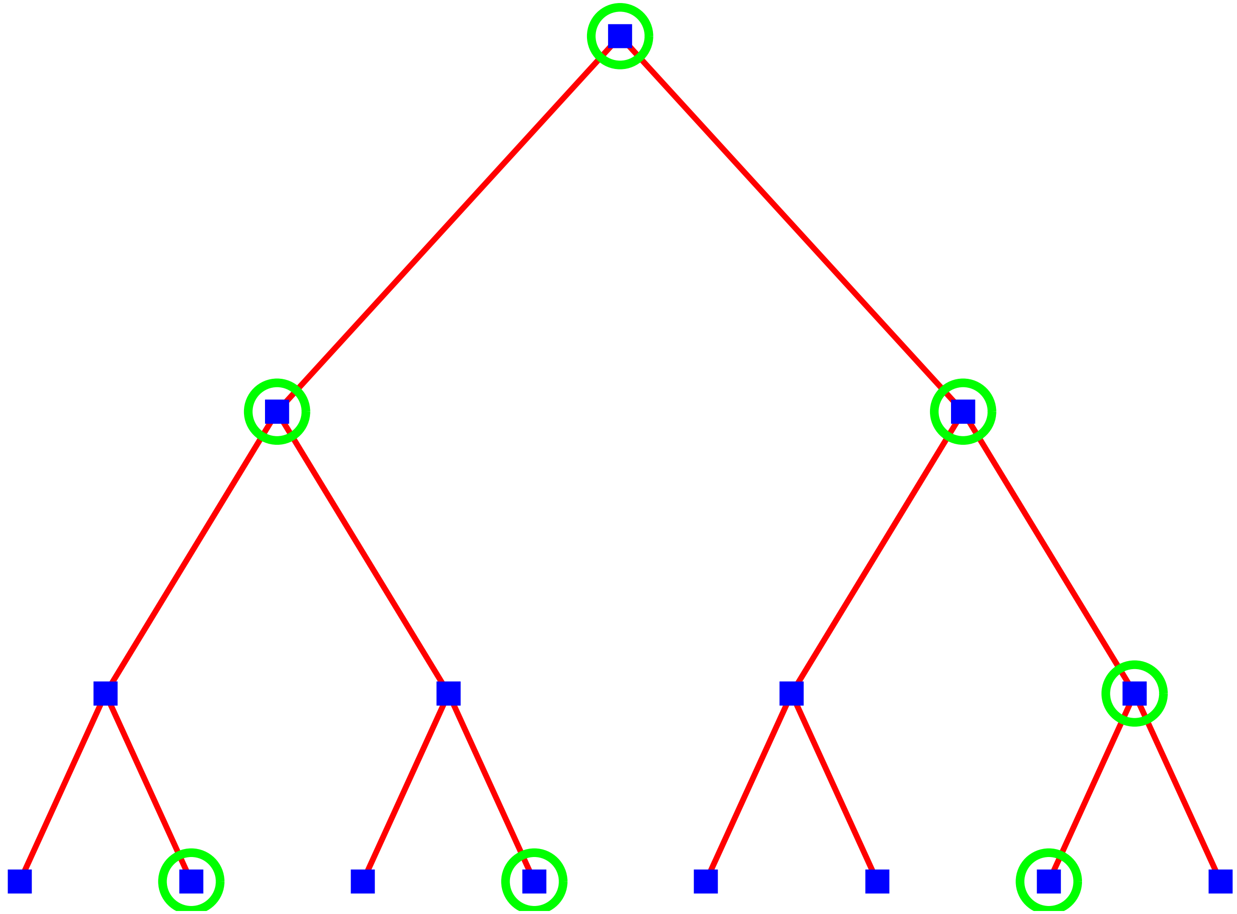
Wavelets: Haar Basis

- $H_I(x) := 2^{j/2} H(2^j x - k)$, $I = [k2^{-j}, (k+1)2^{-j}]$
- $\mathcal{D}_+ := \{I \in \mathcal{D} : |I| \leq 1\}$
- $\{\chi_{[0,1]}\} \cup \{H_I\}_{I \in \mathcal{D}_+}$ is a complete orthonormal system in $L_2[0,1]$

Haar Basis



Wavelet tree



Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals

Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals
- X_n span of first n Haar functions

Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals
- X_n span of first n Haar functions
- X_n is a linear space

Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals
- X_n span of first n Haar functions
- X_n is a linear space
- $E_n^w(f)_p := \inf_{g \in X_n} \|f - g\|_{L_p[0,1]}$

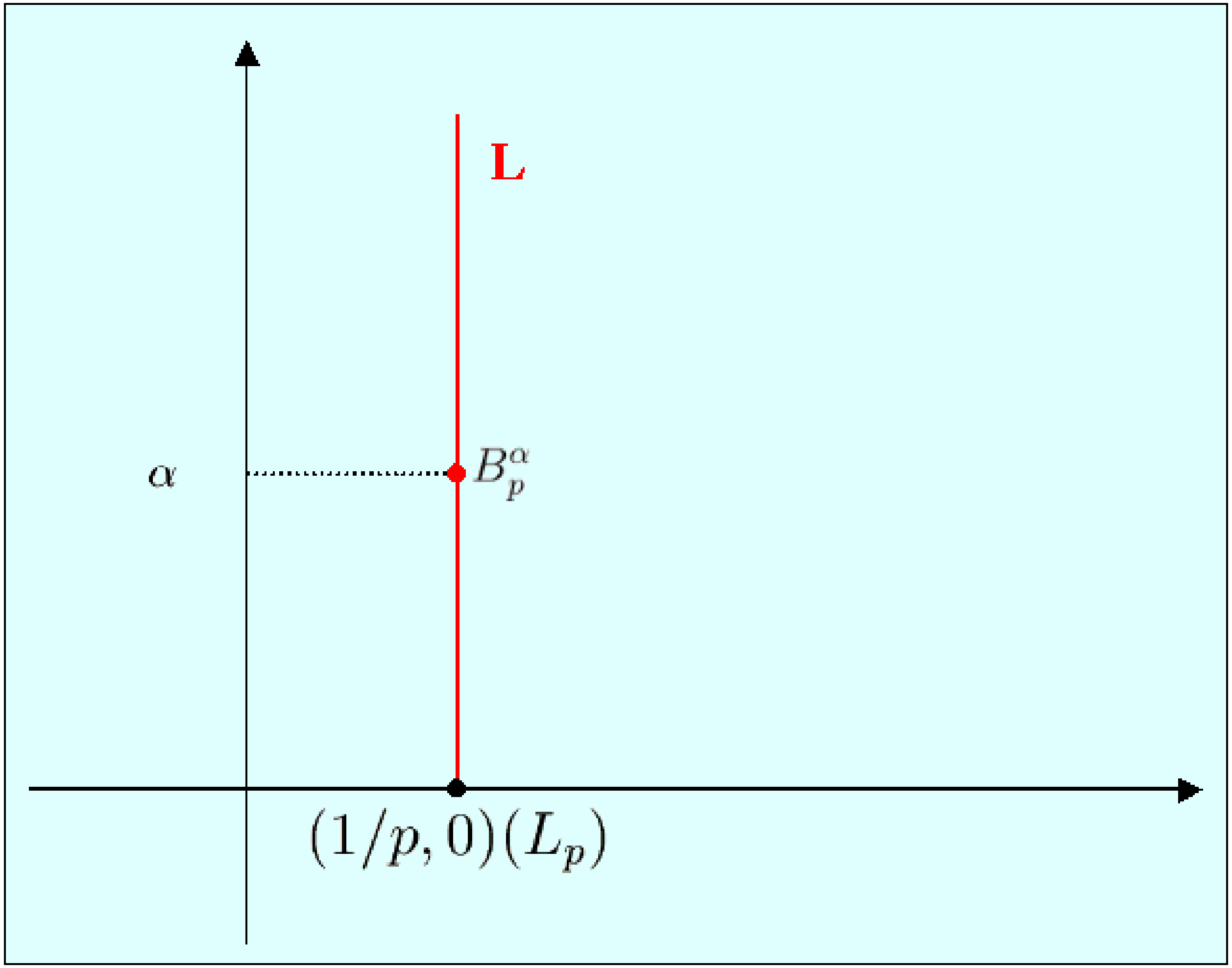
Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals
- X_n span of first n Haar functions
- X_n is a linear space
- $E_n^w(f)_p := \inf_{g \in X_n} \|f - g\|_{L_p[0,1]}$
- This is linear approximation because X_n is a linear space

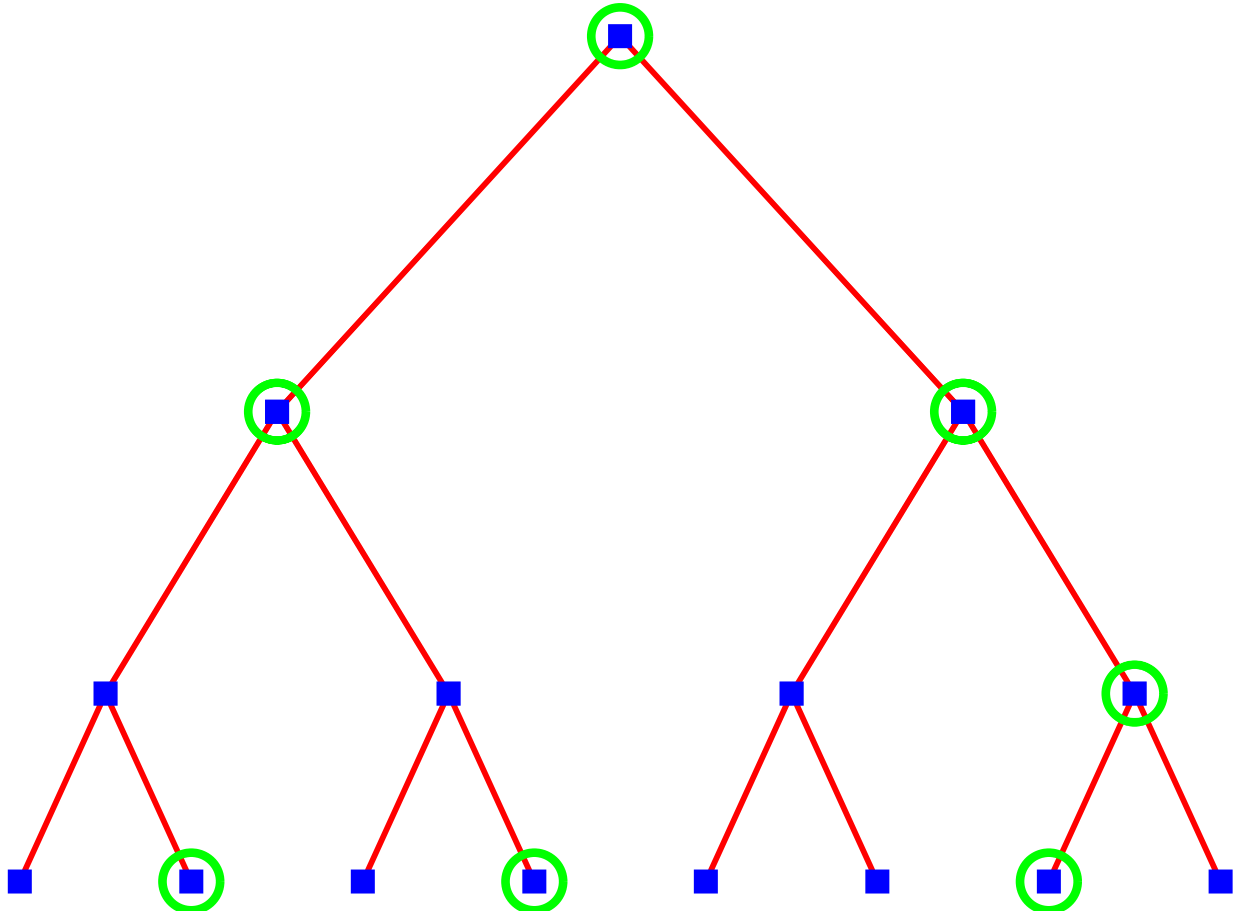
Linear Approximation with Haar Basis

- Natural ordering of dyadic intervals
- X_n span of first n Haar functions
- X_n is a linear space
- $E_n^w(f)_p := \inf_{g \in X_n} \|f - g\|_{L_p[0,1]}$
- This is linear approximation because X_n is a linear space
- The approximation classes for linear approximation with Haar wavelets are identical to those with piecewise constants.

Linear Wavelet: $\mathcal{A}^s_\infty(L_p) = B^s_\infty(L_p)$



n-term approximation



Nonlinear Approximation with Haar Basis

- n-term approximation

Nonlinear Approximation with Haar Basis

- n-term approximation
- $\Sigma_n := \{S = \sum_{I \in \Lambda} c_I H_I : \#(\Lambda) \leq n\}$

Nonlinear Approximation with Haar Basis

- n-term approximation
- $\Sigma_n := \{S = \sum_{I\in\Lambda} c_I H_I : \#(\Lambda) \leq n\}$
- Σ_n is not a linear space

Nonlinear Approximation with Haar Basis

- n-term approximation
- $\Sigma_n := \{S = \sum_{I \in \Lambda} c_I H_I : \#(\Lambda) \leq n\}$
- Σ_n is not a linear space
- $\sigma_n^w(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p[0,1]}$

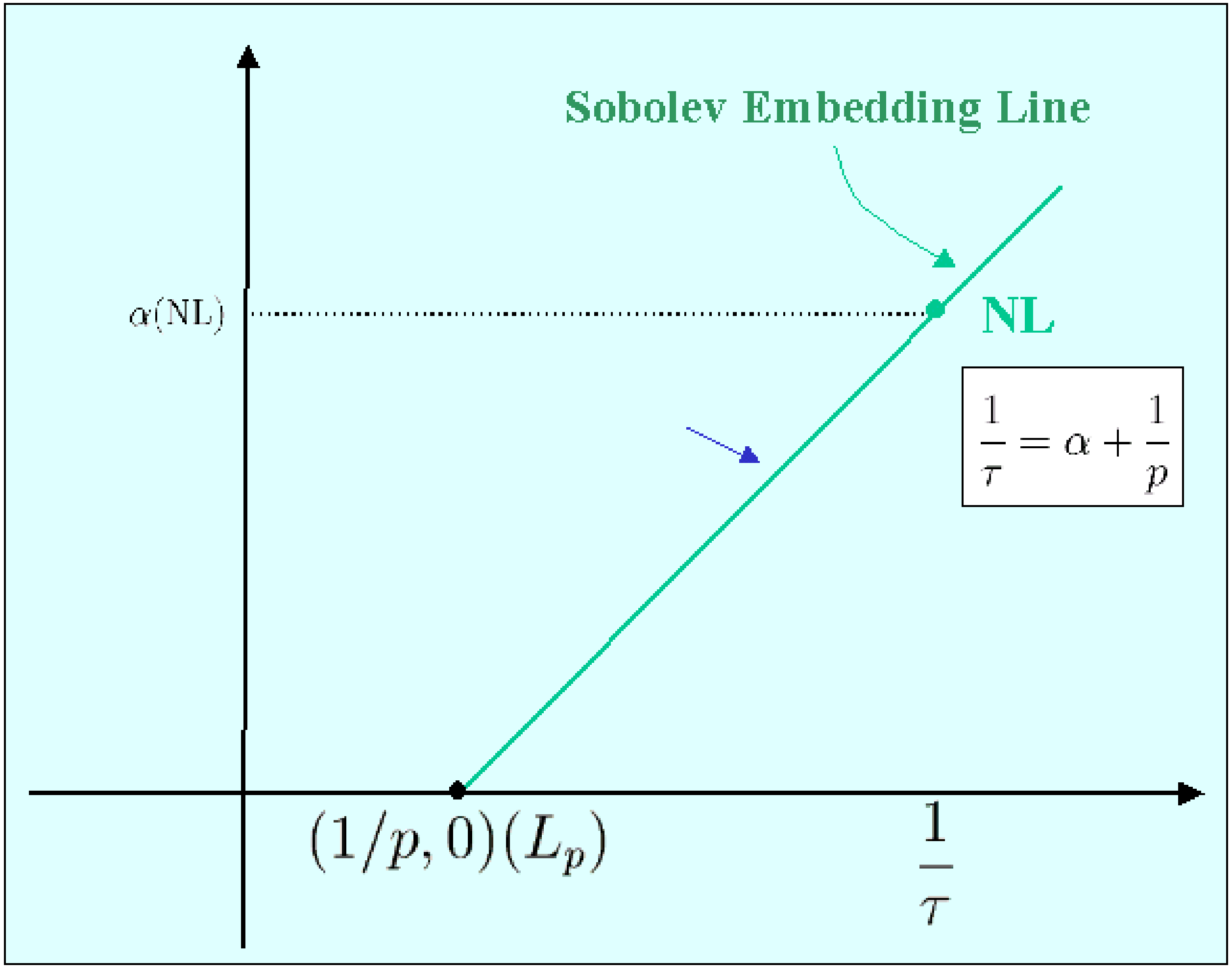
Nonlinear Approximation with Haar Basis

- n-term approximation
- $\Sigma_n := \{S = \sum_{I \in \Lambda} c_I H_I : \#(\Lambda) \leq n\}$
- Σ_n is not a linear space
- $\sigma_n^w(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p[0,1]}$
- This is nonlinear approximation because decisions are made dependent on f

Nonlinear Approximation with Haar Basis

- n-term approximation
- $\Sigma_n := \{S = \sum_{I \in \Lambda} c_I H_I : \#(\Lambda) \leq n\}$
- Σ_n is not a linear space
- $\sigma_n^w(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p[0,1]}$
- This is nonlinear approximation because decisions are made dependent on f
- The approximation classes for nonlinear approximation with Haar wavelets are identical to those with piecewise constants.

roximation class n-term Haar approxima



So Why Wavelets??

- Simplicity of numerical implementation

So Why Wavelets??

- Simplicity of numerical implementation
- How can we find best n-term approximation??

So Why Wavelets??

- Simplicity of numerical implementation
- How can we find best n-term approximation??
- In L_2 take n terms with largest coefficients

So Why Wavelets??

- Simplicity of numerical implementation
- How can we find best n-term approximation??
- In L_2 take n terms with largest coefficients
- DJP: Same strategy works in L_p, $1 < p < \infty$, and other spaces (Sobolev)

Greedy Approximation in $X = L_p$

- Write $f = \sum_{I \in \mathcal{D}} c_I(f)\psi_I$ with $\|\psi_I\|_X = 1$

Greedy Approximation in $X = L_p$

- Write $f = \sum_{I \in \mathcal{D}} c_I(f)\psi_I$ with $\|\psi_I\|_X = 1$
- $\Lambda_n(f)$ indicies of n largest coefficients $|c_I(f)|$

Greedy Approximation in $X = L_p$

- Write $f = \sum_{I \in \mathcal{D}} c_I(f)\psi_I$ with $\|\psi_I\|_X = 1$
- $\Lambda_n(f)$ indicies of n largest coefficients $|c_I(f)|$
- $G_n(f) := \sum_{I \in \Lambda_n(f)} c_I(f)\psi_I$

Greedy Approximation in $X = L_p$

- Write $f = \sum_{I \in \mathcal{D}} c_I(f)\psi_I$ with $\|\psi_I\|_X = 1$
- $\Lambda_n(f)$ indicies of n largest coefficients $|c_I(f)|$
- $G_n(f) := \sum_{I \in \Lambda_n(f)} c_I(f)\psi_I$
- Temlyakov:

$$\|f - G_n(f)\|_X \leq C_X \sigma_n(f)_X$$

Greedy Approximation in $X = L_p$

- Write $f = \sum_{I \in \mathcal{D}} c_I(f)\psi_I$ with $\|\psi_I\|_X = 1$
- $\Lambda_n(f)$ indicies of n largest coefficients $|c_I(f)|$
- $G_n(f) := \sum_{I \in \Lambda_n(f)} c_I(f)\psi_I$
- Temlyakov:

$$\|f - G_n(f)\|_X \leq C_X \sigma_n(f)_X$$

- Greedy strategy is near optimal

Thresholding is near optimal in L_p

- $\Lambda(f,\eta) := \{I : |c_I(f)| > \eta\}$, $N := \#(\Lambda(f,\eta))$

Thresholding is near optimal in L_p

- $\Lambda(f,\eta) := \{I : |c_I(f)| > \eta\}$, $N := \#(\Lambda(f,\eta))$
- $T_\eta(f) := \sum_{I\in\Lambda(f,\eta)} c_I(f)\psi$

Thresholding is near optimal in L_p

- $\Lambda(f,\eta) := \{I : |c_I(f)| > \eta\}$, $N := \#(\Lambda(f,\eta))$
- $T_\eta(f) := \sum_{I\in\Lambda(f,\eta)} c_I(f)\psi$
- $\|f - T_\eta(f)\|_{L_p} \le C_p \sigma_N(f)_p$

Generalizations

- X a Banach space: $\{\psi_j\}$ basis

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$
- $\Lambda_n(f)$ set of indices of n largest $|c_j(f)|$

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$
- $\Lambda_n(f)$ set of indices of n largest $|c_j(f)|$
- $G_n(f) := \sum_{j\in\Lambda_n(f)} c_j(f)\psi_j$

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$
- $\Lambda_n(f)$ set of indices of n largest $|c_j(f)|$
- $G_n(f) := \sum_{j\in\Lambda_n(f)} c_j(f)\psi_j$
- When do we have $\|f - G_n(f)\|_X \le C_X\sigma_n(f)_X$?

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$
- $\Lambda_n(f)$ set of indices of n largest $|c_j(f)|$
- $G_n(f) := \sum_{j \in \Lambda_n(f)} c_j(f)\psi_j$
- When do we have $\|f - G_n(f)\|_X \leq C_X \sigma_n(f)_X$?
- Konjagin-Temlyakov: Near optimal equivalent to the basis is unconditional and democratic

Generalizations

- X a Banach space: $\{\psi_j\}$ basis
- $f = \sum_{j=1}^{\infty} c_j(f)\psi_j$
- $\Lambda_n(f)$ set of indices of n largest $|c_j(f)|$
- $G_n(f) := \sum_{j\in\Lambda_n(f)} c_j(f)\psi_j$
- When do we have $\|f - G_n(f)\|_X \le C_X\sigma_n(f)_X$?
- Konjagin-Temlyakov: Near optimal equivalent to the basis is unconditional and democratic
- Democratic

$$\frac{\|\sum_{I\in\Lambda}\psi_I\|_X}{\|\sum_{I\in\Lambda'}\psi_I\|_X} \le C$$

whenever $\#(\Lambda) = \#(\Lambda')$

Wavelet Bases

- Wavelet bases are democratic in L_p, $1 < p < \infty$, in H_p, $p \leq 1$

Wavelet Bases

- Wavelet bases are democratic in L_p, $1 < p < \infty$, in H_p, $p \leq 1$
- $\|\sum_{I\in\Lambda}\psi_I\|_{L_p} \approx \#(\Lambda)^{1/p}$

Wavelet Bases

- Wavelet bases are democratic in L_p, $1 < p < \infty$, in H_p, $p \leq 1$
- $\|\sum_{I\in\Lambda} \psi_I\|_{L_p} \approx \#(\Lambda)^{1/p}$
- $$C_1 \min_{j\in\Lambda} |c_j(f)|(\#(\Lambda)^{1/p} \leq \|\sum_{j\in\Lambda} c_j(f)\psi_j\|_{L_p}$$

$$\|\sum_{j\in\Lambda} c_j(f)\psi_j\|_{L_p} \leq C_2 \max_{j\in\Lambda} |c_j(f)|(\#(\Lambda)^{1/p}$$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^\infty |c_j|^\tau)^{1/\tau}$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^{\infty} |c_j|^\tau)^{1/\tau}$
- rearrangement of (c_j):
  c_n^* the n-th largest of $|c_j|$, $j \in \{1, 2, \ldots, \}$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^{\infty} |c_j|^\tau)^{1/\tau}$
- rearrangement of (c_j):
  c_n^* the n-th largest of $|c_j|$, $j \in \{1, 2, \ldots, \}$
- weak-ℓ_τ :
  $c_n^* \le M n^{-1/\tau}$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^{\infty} |c_j|^\tau)^{1/\tau}$
- rearrangement of (c_j):
  c_n^* the n-th largest of $|c_j|$, $j \in \{1, 2, \ldots, \}$
- weak-ℓ_τ :
  $c_n^* \le M n^{-1/\tau}$
- smallest M gives $|(c_j)|_{w\ell_\tau}$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^{\infty} |c_j|^\tau)^{1/\tau}$
- rearrangement of (c_j):
  c_n^* the n-th largest of $|c_j|$, $j \in \{1, 2, \dots, \}$
- weak-ℓ_τ :
  $c_n^* \leq M n^{-1/\tau}$
- smallest M gives $|(c_j)|_{w\ell_\tau}$
- $X = L_p$, $1 < p < \infty$, $\{\psi_j\}$ greedy basis.
  $(c_j) \in w\ell_\tau,\ 1/\tau = \alpha + 1/p \leftrightarrow$

$$\sigma_n(f)_p, \|f - G_n(f)\|_{L_p} \leq C M n^{-\alpha}, \quad n = 1, 2, \dots$$

When do we get $\sigma_n(f)_p = O(n^{-\alpha})$

- ℓ_τ: $\|(c_j)\|_{\ell_\tau} := (\sum_{j=1}^\infty |c_j|^\tau)^{1/\tau}$
- rearrangement of (c_j):
  c_n^* the n-th largest of $|c_j|$, $j \in \{1, 2, \dots, \}$
- weak-ℓ_τ :
  $c_n^* \le M n^{-1/\tau}$
- smallest M gives $|(c_j)|_{w\ell_\tau}$
- $X = L_p$, $1 < p < \infty$, $\{\psi_j\}$ greedy basis.
  $(c_j) \in w\ell_\tau,\ 1/\tau = \alpha + 1/p \leftrightarrow$

$$\sigma_n(f)_p, \|f - G_n(f)\|_{L_p} \le C M n^{-\alpha}, \quad n = 1, 2, \dots$$

- $\sum_{n=1}^\infty [n^\alpha \|f - G_n(f)\|_{L_p}]^\tau \frac{1}{n} \approx \|(c_j(f)\|_{\ell_\tau}^\tau$

Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled

Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled
- Require that the wavelet positions chosen in the approximation lie on a tree with n-nodes

Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled
- Require that the wavelet positions chosen in the approximation lie on a tree with n-nodes
- $\Sigma_n^t := \{S = \sum_{I \in \Lambda} c_I H_I : \ \Lambda \ a \ tree \ \#(\Lambda) \le n\}$

Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled
- Require that the wavelet positions chosen in the approximation lie on a tree with n-nodes
- $\Sigma_n^t := \{S = \sum_{I\in\Lambda} c_I H_I : \ \Lambda \ a \ tree \ \#(\Lambda) \le n\}$
- $\sigma_n^t(f)_p := \inf_{S\in\Sigma_n^t} \|f - S\|_{L_p[0,1]}$

Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled
- Require that the wavelet positions chosen in the approximation lie on a tree with n-nodes
- $\Sigma_n^t := \{S = \sum_{I\in\Lambda} c_I H_I : \ \Lambda \ a \ tree \ \#(\Lambda) \le n\}$
- $\sigma_n^t(f)_p := \inf_{S\in\Sigma_n^t} \|f - S\|_{L_p[0,1]}$
- Numerical algorithm $\Lambda_\eta(f)$ as before - complete $\Lambda_\eta(f)$ to a tree Λ_η^t

$$T_\eta^*(f) := \sum_{I\in\Lambda_\eta^t(f)} c_I(f)\psi_I$$

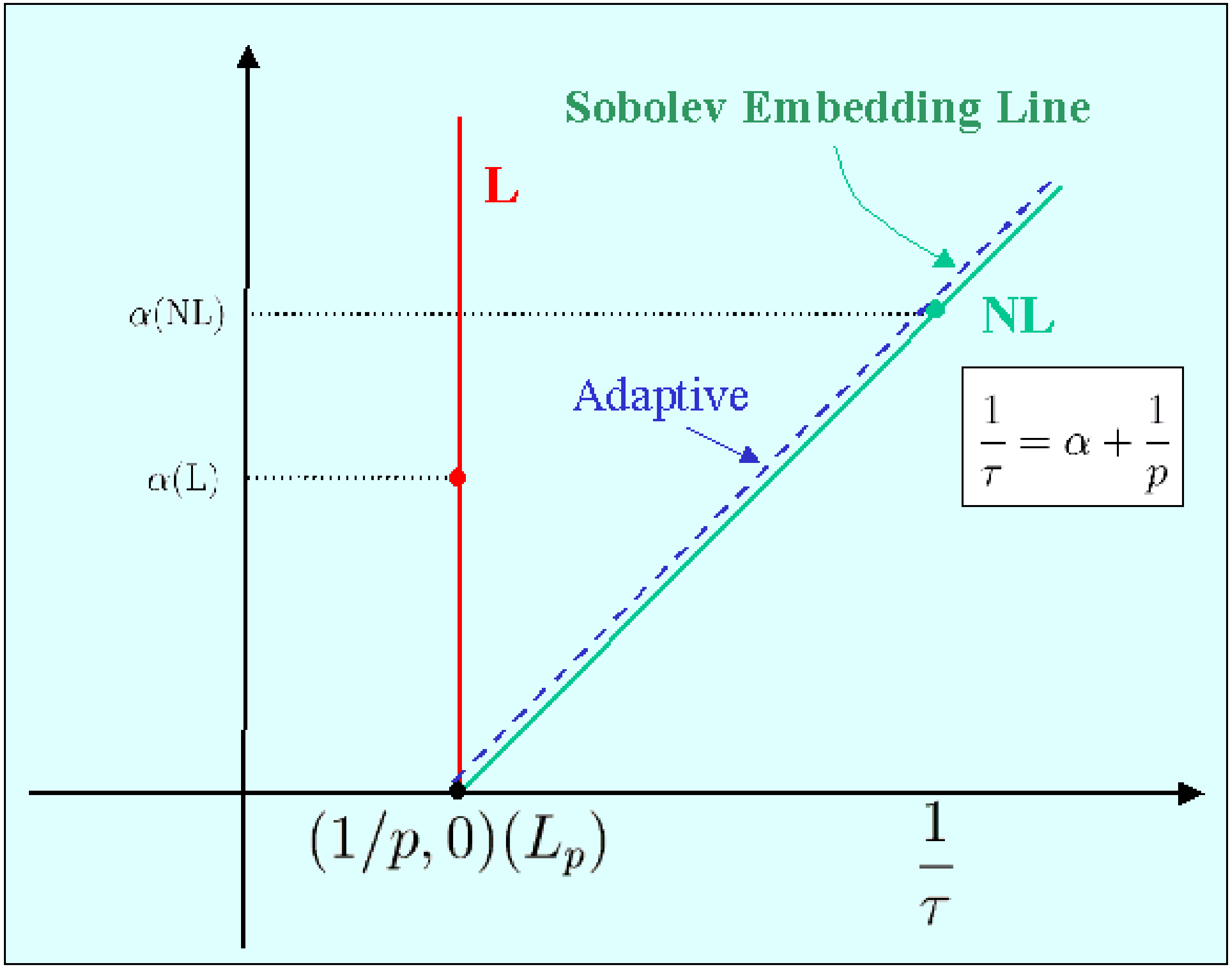
Tree Approximation with Haar Basis

- n-term approximation not numerically implementable: wavelet positions scattered and uncontrolled
- Require that the wavelet positions chosen in the approximation lie on a tree with n-nodes
- $\Sigma_n^t := \{S = \sum_{I \in \Lambda} c_I H_I : \ \Lambda \ a \ tree \ \#(\Lambda) \leq n\}$
- $\sigma_n^t(f)_p := \inf_{S \in \Sigma_n^t} \|f - S\|_{L_p[0,1]}$
- Numerical algorithm $\Lambda_\eta(f)$ as before - complete $\Lambda_\eta(f)$ to a tree Λ_η^t

$$T_\eta^*(f) := \sum_{I \in \Lambda_\eta^t(f)} c_I(f)\psi_I$$

- Approximation properties analogous to adaptive approximation

Tree approximation



Extensions

- Can replace Haar wavelets by biorthogonal wavelets

Extensions

- Can replace Haar wavelets by biorthogonal wavelets
- Approximation results now hold provided $\alpha < r$ where ψ has r vanishing moments and smoothness C^r,

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning
- $\mathcal{A}^{\alpha/d}(L_p) = B^\alpha_\infty(L_p)$

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning
- $\mathcal{A}^{\alpha/d}(L_p) = B^{\alpha}_{\infty}(L_p)$
- Nonlinear Approximation?

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning
- $\mathcal{A}^{\alpha/d}(L_p) = B_\infty^\alpha(L_p)$
- Nonlinear Approximation?
- No analogue of free knot splines: general triangulation

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning
- $\mathcal{A}^{\alpha/d}(L_p) = B^\alpha_\infty(L_p)$
- Nonlinear Approximation?
- No analogue of free knot splines: general triangulation
- n-term approximation using multivariate wavelets the same

Multidimensional results

- Multidimensional: Results hold in $\mathbb{R}^d$ with basis ψ_I^e
- Wavelet Tree Approximation equivalent to adaptive triangulation with isotropic partitioning
- $\mathcal{A}^{\alpha/d}(L_p) = B^{\alpha}_{\infty}(L_p)$
- Nonlinear Approximation?
- No analogue of free knot splines: general triangulation
- n-term approximation using multivariate wavelets the same
- $\mathcal{A}^{\alpha/d}_{\tau}(wavelets, L_p) = B^{\alpha}_{\tau}(L_p)$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$

Compression/Encoding

- How can we evaluate encoders

Compression/Encoding

- How can we evaluate encoders
- Experimental:

  Encoders designed on heuristics

Compression/Encoding

- How can we evaluate encoders
- Experimental:

  Encoders designed on heuristics

- Precise Mathematical Formulation

  Understand rules of game; what it means to be a winner

Compression/Encoding

- How can we evaluate encoders
- Experimental:

  Encoders designed on heuristics

- Precise Mathematical Formulation

  Understand rules of game; what it means to be a winner

- Two essential ingredients

  a. metric ρ to measure distortion

  b. Precise definition of classes K_α to be compressed

Mathematical Formulation

- Distortion: $\rho(S, D_n E_n(S))$

Mathematical Formulation

- Distortion: $\rho(S, D_n E_n(S))$
- Evaluate Performance on a set K of surfaces

$$\delta(K; D_n, E_n) := \sup_{S \in K} \rho(S, D_n E_n(S))$$

Mathematical Formulation

- Distortion: $\rho(S, D_n E_n(S))$
- Evaluate Performance on a set K of surfaces

$$\delta(K; D_n, E_n) := \sup_{S \in K} \rho(S, D_n E_n(S))$$

- Given bit budget n

$$\delta_n(K) := \inf_{E_n, D_n} \delta(K, D_n E_n(S))$$

Mathematical Formulation

- Distortion: $\rho(S, D_n E_n(S))$
- Evaluate Performance on a set K of surfaces

$$\delta(K; D_n, E_n) := \sup_{S \in K} \rho(S, D_n E_n(S))$$

- Given bit budget n

$$\delta_n(K) := \inf_{E_n, D_n} \delta(K, D_n E_n(S))$$

- smallest distortion for the given bit budget

(Near) Optimal Encoding for K

- optimal

$$\delta(K, D_n E_n(S)) = \delta_n(K)$$

(Near) Optimal Encoding for K

- optimal

$$\delta(K, D_n E_n(S)) = \delta_n(K)$$

- near optimal

$$\delta(K, D_n E_n(S)) \le C\delta_n(K)$$

(Near) Optimal Encoding for K

- optimal

$$\delta(K, D_n E_n(S)) = \delta_n(K)$$

- near optimal

$$\delta(K, D_n E_n(S)) \leq C\delta_n(K)$$

- Typically: $\delta_n(K) \approx n^{-s}$ for some $s > 0$

(Near) Optimal Encoding for K

- optimal

$$\delta(K, D_n E_n(S)) = \delta_n(K)$$

- near optimal

$$\delta(K, D_n E_n(S)) \leq C\delta_n(K)$$

- Typically: $\delta_n(K) \approx n^{-s}$ for some $s > 0$
- Game: Find encoder/decoder E/D: for all values of n and all classes K_α, encoder is near optimal

Description of Optimal Encoding: Kolmo

- Given $\epsilon > 0$

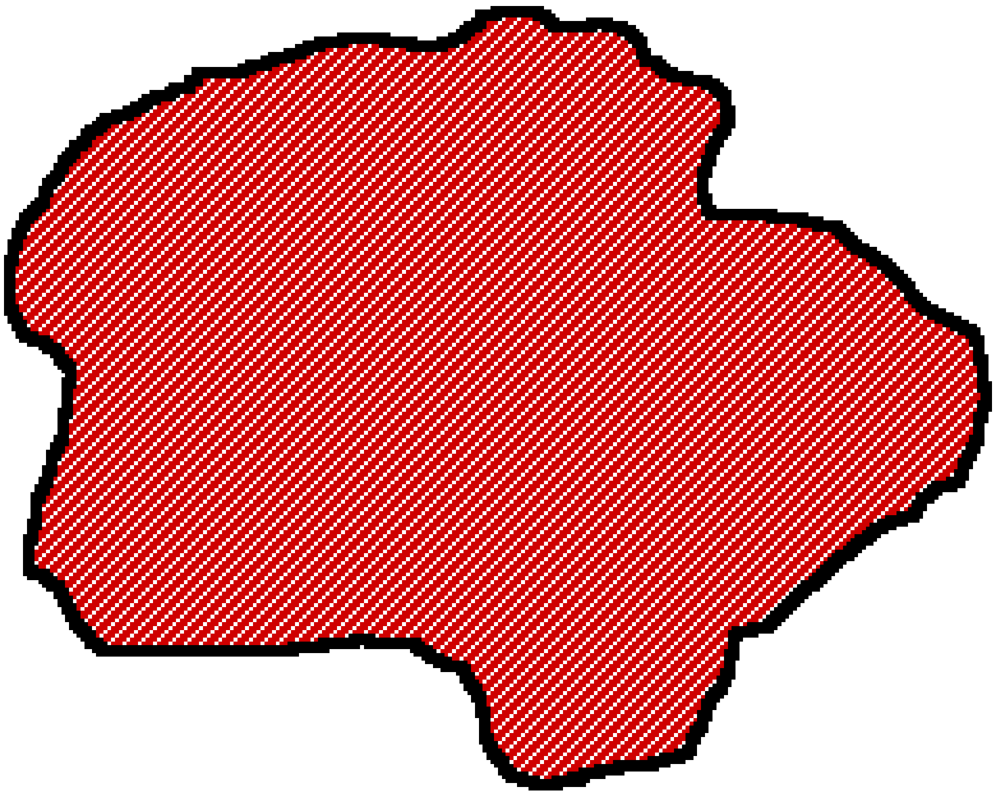
Description of Optimal Encoding: Kolmo

- Given $\epsilon > 0$
- Minimal ϵ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(S_i, \epsilon)$

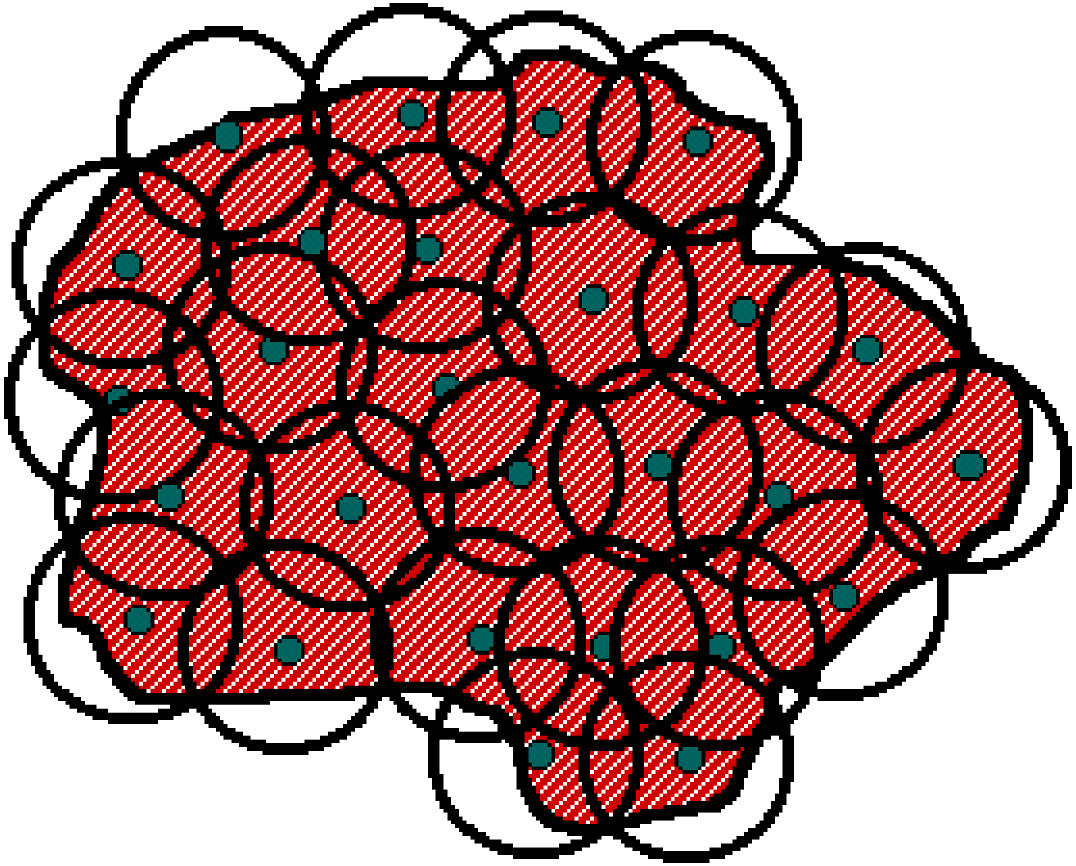
Description of Optimal Encoding: Kolmo

- Given $\epsilon > 0$
- Minimal ϵ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(S_i, \epsilon)$
- Kolmogorov Entropy $H_\epsilon(K) := \log_2 N_\epsilon(K)$

Covering



Covering



Kolmogorov Entropy

- Given $\epsilon > 0$
- Minimal ϵ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(S_i, \epsilon)$
- Kolmogorov Entropy $H_\epsilon(K) := \log_2 N_\epsilon(K)$

- $\delta_n(K) = \inf\{\epsilon : \ H_\epsilon(K) \leq n\}$

Description of Optimal Encoding: Kolmo

- Given $\epsilon > 0$
- Minimal ϵ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(x_i, \epsilon)$
- $\delta_n(K) = \inf\{\epsilon : \ H_\epsilon(K) \leq n$
- Kolmogorov entropy of K gives our benchmark
- Usually not practical encoder

The Issues

1. The metric: least squares

2. The classes

3. Determine Entropy of Classes

4. Build near optimal Encoders/Decoders

Wavelet Encoding

- Tree approximation can be turned into an encoder

Wavelet Encoding

- Tree approximation can be turned into an encoder
- Tree with n nodes can be encoded with n bits

Wavelet Encoding

- Tree approximation can be turned into an encoder
- Tree with n nodes can be encoded with n bits
- Cohen-Dahmen-Daubechies-DeVore (Cohen-Daubechies-Gulleryuz-Orchard) This encoder is optimal on all Besov classes compactly embedded into L_2

Wavelet Encoding

- Tree approximation can be turned into an encoder
- Tree with n nodes can be encoded with n bits
- Cohen-Dahmen-Daubechies-DeVore (Cohen-Daubechies-Gulleryuz-Orchard) This encoder is optimal on all Besov classes compactly embedded into L_2
- EZW, Said-Pearlman,