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Wavelet Techniques in Multifractal Analysis

Stéphane Jaffard

This paper is dedicated to Benoît Mandelbrot on the occasion of his jubilee.

Abstract. We show how estimates on the Hausdorff dimensions of the Hölder singularities of a function can be derived from the properties of its expansion on an orthonormal wavelet basis. The validity of the multifractal formalism is examined and results concerning the genericity of multifractality in a given function space are deduced.

Multifractal analysis is a recent field, introduced in the context of turbulence in the mid 80s; its purpose is to analyze the pointwise Hölder regularity of functions, and to understand how this regularity fluctuates from point to point. One tries to determine the *spectrum of singularities* of the function f studied: It is a novel function which associates to each Hölder exponent H the dimension of the sets of points where f has this given pointwise regularity.

We will give a detailed account of the interaction between wavelet analysis and multifractal analysis. This interaction was made possible because of two remarkable properties of wavelet expansions:

- It is possible to characterize the pointwise Hölder regularity of a function by simple estimates on the decay rate of its wavelet coefficients, corresponding to the wavelets localized near the point considered.
- Wavelets are unconditional bases of "most" function spaces.

The second point is relevant because a central problem in multifractal analysis is to relate the spectrum of singularities of a function f to the function spaces which contain f. Formulas which perform this bridge are called *multifractal formalisms.*

This paper is split into five parts:

Section 1 is introductory; we define the pointwise Hölder regularity and the different notions of dimension that we will use. These notions are illustrated by the study of two functions: The Weierstrass "'nowhere" differentiable functions, and the devil's staircase. These toy examples allow us to introduce, in a very simple

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setting, some of the tools that will be explored and used in the following: The wavelet criterion for irregularity and the mass distribution principle.

Section 2 is devoted to the wavelet characterization of the pointwise Hölder exponent and the relationship between Hölder regularity and local oscillation. We give a new formulation of this criterion in terms of local suprema of the wavelet coefficients, the *wavelet leaders*, which paves the way for the new multifractal formalism introduced in the following section.

Section 3 is devoted to the study of this new multifractal formalism. First, a heuristic justification is given; though it follows the lines of the initial argument given by Parisi and Frisch, this formulation is given in terms of wavelet leaders and not, as usual, in terms of either increments, or wavelet coefficients. This has the advantage of eliminating some causes of failure of the multifractal formalism. We prove that this multifractal formalism yields, for all functions that have a minimal uniform regularity, an upper bound for their spectrum of singularities; we give an alternative derivation of the multifractal formalism in terms of local oscillations, and we prove that both methods are equivalent.

Section 4 is a complement to Section 3: We derive properties satisfied by the *scaling function* which appear in the formulation of the multifractal formalism. We show that it indicates which oscillation spaces contain the function considered and we give some properties of these spaces.

Section 5 gives "generic" results of multifractality. First, we discuss what genericity can mean in infinite dimensional function spaces. We focus on one particularly important notion: Prevalence. Finally, we perform the multifractal analysis of "almost every" function (in the sense of prevalence) belonging to a given Besov or oscillation space.

1. What is multifractal analysis?

In this first section we introduce the main concepts that are relevant in multifractal analysis, and we perform the "multifractal analysis" of two very simple examples: the Weierstrass functions and the devil's staircase.

1.1. Pointwise regularity.

Multifractal functions are used as models for signals whose regularity may change abruptly from one point to the next. Our first task is therefore to define what is meant by pointwise regularity. It is a way to quantify, using a positive parameter α, the fact that the graph of a function may be more or less "ruguous" at a point x_0.

Definition 1. Let α be a nonnegative real number, and $x_0 \in \mathbb{R}^d$; a function $f : \mathbb{R}^d \to \mathbb{R}$ is $C^\alpha(x_0)$ if there exists $C > 0$, $\delta > 0$ and a polynomial P of degree at most $[\alpha]$ such that

$$\text{if} \quad |x - x_0| \leq \delta, \qquad |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1.1}$$

Remarks: We use the (unusual) convention $[\alpha] = \alpha - 1$ if α is an integer. This convention implies that the polynomial P is unique even when α is an integer; the constant term of P is necessarily $f(x_0)$; P is called the Taylor expansion of f at x_0 of order α.

We will also use a slightly weaker notion; f is $C^\alpha_{log}(x_0)$ if there exists $C>0$, $\delta>0$ and a polynomial P of degree at most $[\alpha]$ such that

$$\text{if } |x-x_0|\le\delta, \qquad |f(x)-P(x-x_0)|\le C|x-x_0|^\alpha \log\left(1/|x-x_0|\right). \tag{1.2}$$

Definition 2. The Hölder exponent of f at x_0 is

$$h_f(x_0)=\sup\{\alpha: \quad f \quad \text{is} \quad C^\alpha(x_0)\}.$$

Note that (1.1) implies that f is bounded in a neighbourhood of x_0; therefore, the Hölder exponent is defined only for locally bounded functions. The Hölder exponent is defined point by point and describes the local regularity variations of f. Some functions have a constant Hölder exponent; they are called *monohölder functions*. We will see that it is the case of the Weierstrass functions; it is also the case of the sample paths of the Brownian motion which satisfy almost surely: $\forall x$, $h_B(x)=1/2$. Such functions display a "very regular irregularity". As a first example, we now study the pointwise regularity of the Weierstrass functions; indeed, this will allow us to see, in a very simple setting, how wavelet techniques can be used to prove irregularity.

1.2. The Weierstrass functions. The Weierstrass functions are defined by

$$W_{A,B}(x)=\sum_{n=1}^{\infty}A^n\cos(B^nx),$$

where B is assumed to be larger than 1, so that the series is lacunary, and A is assumed to be smaller than 1, so that the series converges normally. Derivating term by term, it is clear that, if $AB<1$, $W_{A,B}$ est differentiable. We assume that $AB>1$ which will actually imply that $W_{A,B}$ is nowhere differentiable.

Proposition 1. If $A<1<AB$, the Hölder exponent of $W_{A,B}$ is a constant function which is equal to $\alpha=-\log A/\log B$.

Proof of Proposition 1: Let us first check that $W_{A,B}$ is $C^\alpha(x_0)$ for any $x_0\in\mathbb{R}$. In the difference

$$W_{A,B}(x)-W_{A,B}(x_0)=\sum_{n=1}^{\infty}A^n\left(\cos(B^nx)-\cos(B^nx_0)\right),$$

we can either bound the difference of cosines simply by 2 or, using the mean value theorem, by $B^n|x-x_0|$. Let

$$N=\left[\frac{-\log(|x-x_0|)}{\log B}\right].$$

Using the first bound for $n\ge N$ and the second one for $n<N$, we get

$$|W_{A,B}(x)-W_{A,B}(x_0)|\le\sum_{n=1}^{N}A^nB^n|x-x_0|+2\sum_{n=N}^{\infty}A^n.$$

We have to sum up two geometric series. Because of the value taken for N, the first sum is bounded by $C(AB)^N\le C|x-x_0|^\alpha$, and the second one is bounded by $CA^N\le C|x-x_0|^\alpha$, hence the regularity result holds.

In order to prove irregularity of $W_{A,B}$ at every point, we will use a technique which predates the wavelet methods developed in Section 2.

DEFINITION 3. Suppose that $\psi(x)$ satisfies the following conditions

$$|\psi(x)| \leq \frac{C}{1+|x|^2} \quad \text{et} \quad \int_{\mathbb{R}} \psi(x)dx = 0.$$

Let $f \in L^\infty(\mathbb{R})$, $a > 0$ and $b \in \mathbb{R}$; the continuous wavelet transform of f is defined by

$$C(a,b) = \int_{\mathbb{R}} f(x)\psi\left(\frac{x-b}{a}\right)\frac{dx}{a}. \tag{1.3}$$

LEMMA 1. If $f \in C^\alpha(x_0)$ for $\alpha \in [0,1]$, then the continuous wavelet transform of f satisfies

$$|C(a,b)| \leq C(a^\alpha + |b-x_0|^\alpha).$$

Proof of Lemma 1: Since ψ has a vanishing integral,

$$C(a,b) = \int_{\mathbb{R}} (f(x)-f(x_0))\psi\left(\frac{x-b}{a}\right)\frac{dx}{a},$$

so that

$$\begin{aligned} |C(a,b)| &\leq C\int_{\mathbb{R}} |x-x_0|^\alpha \left|\psi\left(\frac{x-b}{a}\right)\right|\frac{dx}{a} \\ &\leq C\int_{\mathbb{R}} \frac{|x-b|^\alpha + |b-x_0|^\alpha}{1+\left|\frac{x-b}{a}\right|^2}\frac{dx}{a} \\ &\leq C(a^\alpha + |b-x_0|^\alpha). \end{aligned}$$

Let us come back to Proposition 1. The idea of its proof is to pick ψ so that the computation of $C(a,b)$ may be as simple as possible. It is the case if, in (1.3), only one frequency of the Weierstrass function is selected. A possible choice is therefore to pick for ψ a function whose Fourier transform $\widehat{\psi}$ is C^2 and satisfies

$$supp(\widehat{\psi}) \subset \left[\frac{1}{B}, B\right] \quad \text{et} \quad \widehat{\psi}(1) = 2.$$

The hypotheses of Lemma 1 are clearly satisfied and, if $a = B^{-N}$,

$$\begin{aligned} C(B^{-N},b) &= \sum_{n=1}^{\infty} A^n \int_{\mathbb{R}} \cos(B^n x)\psi\left(\frac{x-b}{B^{-N}}\right)\frac{dx}{B^{-N}} \\ &= \sum_{n=1}^{\infty} A^n \int_{\mathbb{R}} \cos(B^{n-N}u + B^n b)\psi(u)du. \end{aligned}$$

Because of our choice for $\widehat{\psi}$, the only nonvanishig integral corresponds to $n = N$ so that

$$C(B^{-N},b) = A^N e^{iB^N b};$$

thus,

$$\text{if} \quad a = B^{-N}, \quad \text{then} \quad |C(a,b)| = a^\alpha.$$

Lemma 1 implies that the Hölder exponent cannot be larger than α at any point and Proposition 1 follows. In particular, the Weierstrass functions are nowhere differentiable.

1.3. Measure and Hausdorff dimension. Multifractal analysis studies functions whose Hölder exponent can jump from one point to the next. If such is the case, the numerical computation of the Hölder exponent of a signal is completely instable, and also quite meaningless. One rather wishes to obtain a more global information such as: Does the Hölder exponent take a given value H? And, if such is the case, what is the size of the set of points where h_f takes this value? This question rises the following problem: Which kind of 'size' should we use? The answer can be justified by the study of many mathematical functions: Usually, there is a 'most probable' exponent, which is taken almost everywhere; therefore, 'size' cannot mean 'Lebesgue measure', which would not allow to distinguish between the sizes of all but one sets of points where a given Hölder exponent appears. The *Hausdorff dimension* is more fitted for this purpose. Let us first recall the notion of δ-dimensional Hausdorff measure, see [**33, 34**].

Definition 4. Let $A \subset \mathbb{R}^d$. If $\varepsilon > 0$ and $\delta \in [0, d]$, we denote

$$M_\varepsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where R is an ϵ-covering of A, i.e. a covering of A by bounded sets $\{A_i\}_{i \in \mathbb{N}}$ of diameters $|A_i| \le \varepsilon$. The infimum is therefore taken on all ϵ-coverings.

For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is

$$mes_\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta.$$

(Note that the limit exists (it can take the value $+\infty$) since M_ϵ^δ is a decreasing function of ϵ.)

Definition 5. Let $A \subset \mathbb{R}^d$; then, there exists $\delta_0 \in [0, d]$ such that

$$\begin{aligned} &\forall \delta < \delta_0, \quad mes_\delta(A) = +\infty \\ &\forall \delta > \delta_0, \quad mes_\delta(A) = 0. \end{aligned}$$

This critical δ_0 is called the Hausdorff dimension of A.

The existence of δ_0 is a consequence of two straightforward assertions: If $\delta < \delta'$,

$$mes_\delta(A) < +\infty \Rightarrow mes_{\delta'}(A) = 0 \quad \text{and} \quad mes_{\delta'}(A) > 0 \Rightarrow mes_\delta(A) = +\infty.$$

In order to bound the Hausdorff dimension of a set, it is sufficient to consider particular coverings. By contrast, it is harder to obtain a lower bound directly from the definition, since it requires to consider all possible coverings. The following "mass distribution principle" replaces the study of all ϵ-coverings by the construction of one measure (which is often naturally supplied by the fractal set considered).

Proposition 2. Let m be a probability measure supported by $A \subset \mathbb{R}^d$. Assume that there exist $\delta \in [0, d]$, $C > 0$ and $\epsilon > 0$ such that, for any ball B of diameter less than ϵ,

$$m(B) \le C|B|^\delta.$$

Then $mes_\delta(A) \ge 1/C$.

Proof: Let $\{B_i\}$ be an arbitrary ϵ-covering of A. Then

$$1 = m(A) = m\left(\bigcup B_i\right) \le \sum m(B_i) \le C \sum |B_i|^\delta.$$

Other definitions of fractal dimension have been introduced and are used in multifractal analysis. The simplest notion is supplied by the *box dimensions.*

DEFINITION 6. Let $A \subset \mathbb{R}^d$; if $\varepsilon > 0$, let $N_\epsilon(A)$ be the smallest number of sets of radius ϵ required to cover A.

The upper box dimension of A is

$$\overline{dim_B}(A) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

The lower box dimension of A is

$$\underline{dim_B}(A) = \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

One important drawback of the box dimensions is that, if A is dense, then the box dimensions take invariably the value d. Since most multifractal functions of interest have dense sets of Hölder singularities, box dimensions are unable to draw any distinction between the sizes of these sets. This explains why box dimensions are not used in the definition of the spectrum of singularities.

Another notion of dimension which retains some flavour of the box dimension, but has better mathematical properties, is supplied by the *packing dimension.* Invented by C. Tricot, see [**99**], it can be defined by

$$dim_p(A) = \inf \left\{ \sup_i \left(\underline{dim_B} A_i : A \subset \bigcup_{i=1}^{\infty} A_i \right) \right\}$$

(the infimum is taken over all possible partitions of A into a countable collection A_i). The packing dimension has been extensively used in the context of multifractal measures; however its use is much more scarce for functions; see, however, Proposition 8 where an upper bound for packing dimensions of Hölder singularities is obtained; see also [**50**] where other bounds of packing dimensions are obtained for a different notion of Hölder singularity.

1.4. The spectrum of singularities.

The Hölder exponent of multifractal functions may take a given value on a fractal set. If such is the case, one wishes to determine the Hausdorff dimension of this set.

DEFINITION 7. Let $f : \mathbb{R}^d \to \mathbb{R}$, and $H \geq 0$. If H is a value taken by the function $h_f(x)$, let

$$E_H = \{x_0 \in \mathbb{R}^d : \ h_f(x_0) = H\}. \tag{1.4}$$

The spectrum of singularities (or Hölder spectrum) of f is

$$d_f(H) = dim(E_H)$$

(we use the convention $d_f(H) = -\infty$ if H is not a Hölder exponent of f). The support of the spectrum is the set of values of H for which $E_H \neq \emptyset$.

Remarks:

- We will also consider the set

$$G_H = \{x_0 \in \mathbb{R}^d : \ f \notin C^H_{log}(x_0)\}. \tag{1.5}$$

- These global notions can also be defined locally; if $\Omega \subset \mathbb{R}^d$ is a nonempty open set, we define
$$E_H^\Omega = E_H \cap \Omega \quad \text{and} \quad d_f^\Omega(H) = dim(E_H^\Omega);$$
clearly, $d_f(H) = \sup_\Omega d_f^\Omega(H)$.
- The spectrum of singularities is defined on $\mathbb{R}^+ \bigcup \{+\infty\}$.

If h_f takes at least two finite values, f is said to be *mutifractal.* If h_f takes only one finite value, f is said to be *monofractal*; one example is the devil's staircase, considered in Section 1.5. In the examples we will consider, $d_f(H)$ will usually take nonnegative values on a whole interval $[H_{min}, H_{max}]$. If $H_{min} \neq H_{max}$, its computation requires the study of an infinite number of fractal sets E_H; in such cases, the term 'multifractal' is completely justified.

To perform the multifractal analysis of a function means to determine its spectrum of singularities.

One can meet two types of multifractal functions. A first one is supplied by inhomogeneous functions, which are smoother in some regions than in others. This case is often met in image analysis. Indeed, an image is a patchwork of textures with different characteristics. Their spectrum of singularities reflects the mono- (or multi-) fractal nature of each component, and also of the boundaries (which may also be fractal) where discontinuities appear. In such situations, the determination of the local spectrum of singularities $d_f^\Omega(H)$ for the different 'components' Ω is more relevant. By contrast, *homogeneous* multifractal functions present the same characteristics everywhere. The following definition makes this notion precise.

DEFINITION 8. Let f be a multifractal function (therefore, its Hölder exponent takes at least two values); f is an homogeneous multifractal function if the spectrum of singularities of its restriction to any nonempty open set Ω is independent of Ω (and therefore coincides with the whole spectrum of singularities of f).

1.5. The devil's staircase. The *devil's staircase* is probably the simplest fucntion whose multifractal analysis can be completed in an elementary way. Its Hölder singularities are located on a fractal set, the *triadic Cantor set*, which is defined as follows.

Let $x \in [0, 1]$; x can be written (in base 3) as
$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \quad \text{with} \quad a_j \in \{0, 1, 2\}.$$
The triadic Cantor set K is the set of x such that $a_j \in \{0, 2\}$ for all j.

The devil's staircase is defined as follows: If $x \notin K$, at least one of the a_j is equal to 1. Let $l = \inf\{j : \; a_j = 1\}$; then
$$\mathcal{D}(x) = \sum_{j=1}^{l-1} \frac{a_j}{2^{j+1}} + \frac{a_l}{2^l}.$$
The function $\mathcal{D}(x)$ is thus defined almost everywhere on $[0, 1]$. It is then extended by continuity on $[0, 1]$.

One easily checks that $\mathcal{D}$ is increasing and that, on K, its Hölder exponent takes the value $\log 2/\log 3$; it is equal to $+\infty$ at the other points.

In order to compute the spectrum of singularities of $\mathcal{D}$, we have to determine the Hausdorff dimension of K. Let $\epsilon > 0$ and n be such that $3^{-n} \leq \epsilon < 3^{-n+1}$. Since K can be embedded in the union of 2^n intervals of length 3^{-n}, using these intervals as ϵ-covering, we obtain that the Hausdorff dimension of K is at most δ.

A lower bound for the Hausdorff dimension of K is obtained through an application of the mass distribution principle. Since $\mathcal{D}$ is increasing from $[0,1]$ to $[0,1]$, its derivative μ is a probability measure. Furthermore, since $\mathcal{D}$ is locally constant on the complement of K, then $Supp(\mu) = K$. Let $I = [x_0, x] \subset [0,1]$. The Höder regularity of $\mathcal{D}$ implies that

$$\mu(I) = \mathcal{D}(x) - \mathcal{D}(x_0) \leq 2|I|^{\log 2/\log 3}.$$

Therefore, $m_{\delta,0}(K) \geq \frac{1}{2}$, and the mass distribution principle yields that

$$dim(K) \geq \frac{\log 2}{\log 3}.$$

Hence, we have proved the following statement.

Proposition 3. The devil's staircase is monofractal; its spectrum of singularities is supported on $\{\frac{\log 2}{\log 3}\} \cup \{+\infty\}$ where

$$d_{\mathcal{D}}\left(\frac{\log 2}{\log 3}\right) = \frac{\log 2}{\log 3} \quad \text{and} \quad d(+\infty) = 1.$$

1.6. Notes. The notion of pointwise Hölder regularity has been generalized in several ways. A first one was introduced by A. P. Calderon and A. Zygmund in 1961, and leads to a weaker condition, see [**23**]: f belongs to $T^p_\alpha(x_0)$ if there exists a polynomial $P(x-x_0)$ of degree at most $[\alpha]$ such that

$$\left(\frac{1}{r^d}\int_{B(x_0,r)} |f(x) - P(x-x_0)|^p \, dx\right)^{1/p} \leq Cr^\alpha. \tag{1.6}$$

Clearly, if $f \in C^\alpha(x_0)$, then $\forall p \geq 1$, $f \in T^p_\alpha(x_0)$. This definition is particularly useful when one deals with functions which are not locally bounded. Indeed, (1.6) makes sense as soon as $f \in L^p_{loc}$; therefore, it is a natural substitute for pointwise Hölder regularity when functions in L^p_{loc} are considered. This is particularly relevant when dealing with applications where the signal studied is not locally bounded; for instance, it is the case in fully developed turbulence where singularities of Hölder exponent -1 corresponding to thin vorticity filaments can be observed, and in mammography images, where microcalcifications also appear as singularities of Hölder exponent -1, see [**5**]. If $f \in L^p_{loc}$, one defines the *p-exponent at* x_0 by

$$h_{f,p}(x_0) = \sup\{\alpha : \ f \in T^p_\alpha(x_0)\},$$

and the p-spectrum of f by

$$d_{f,p}(H) = dim\left(\{x_0 : \ h_{f,p}(x_0) = H\}\right).$$

These notions were studied by C. Melot in [**85**].

Let us mention another situation where the p-exponent is more relevant than the Hölder exponent. Suppose that Ω is a domain with a fractal boundary; up to now, only one parameter was available to classify these domains: The dimension of their boundary. At each point x_0 of the boundary, the Hölder exponent of the characteristic function $f = 1_\Omega$ takes the value 0; this is in sharp contrast with the 1-exponent which can clearly take any value in $[0,\infty]$, depending on the behavior of the boundary near x_0; this remark allows one to perform a *multifractal analysis of fractal boundaries* and leads to a whole function (the 1-spectrum) as discriminating parameter between these domains; this yields a a much richer classification tool for fractal domains, see [**59**].

On the other hand, Hölder regularity can be strengthened as follows, see [**50**]. We denote by $meas(A)$ the Lebesgue measure of a set A. Let $\alpha > -d$; a point x_0 is a *strong α-singularity* of f if there exist $C, C' > 0$ such that $\forall$ P polynomial of degree at most α, $\forall j, \exists A_j, B_j$

$$\begin{cases} meas(A_j) \geq C2^{-dj}, \; meas(B_j) \geq C2^{-dj} \\ \forall x \in A_j \cup B_j, \; |x - x_0| \leq 2^{-j} \\ \forall x \in A_j, \forall y \in B_j, \; (f(x) - P(x - x_0)) - (f(y) - P(y - x_0)) \geq C'2^{-\alpha j}. \end{cases}$$

Note that, if $\alpha < 1$, the last condition reduces to $f(x) - f(y) \geq C'2^{-\alpha j}$. Bounds on the packing dimension of the strong α-singularities have been obtained in [**50**]; see also [**59**] where estimates on the the packing dimension of the boundary of a domain in $\mathbb{R}^d$ are obtained in terms of the wavelet coefficients of the characteristic function of the domain.

A natural question is to determine which nonnegative functions $h(x)$ are Hölder exponents. An exact characterization of the Hölder exponents of continuous functions was obtained by P. Anderson, see [**2**] (following results of S. Jaffard [**49**] and, independently of K. Daoudi, J. Lévy-Véhel and Y. Meyer [**30**]): *$h(x)$ is the Hölder exponent of a continuous function f if and only if it is the lim inf of a sequence of continuous functions.* (Note that the problem is still open if f is not assumed to be continuous). The first constructive proofs used deterministic constructions for f. In practice, for simulation purposes, one needs to construct random processes. This led to the introduction of *multifractional Brownian motions*, which are Gaussian processes generalizing the Brownian motion. This study was initiated by A. Benassi, S. Jaffard et D. Roux in [**18**], and independently by J. Lévy-Véhel et R. F. Peltier in [**76**]. The most general constructions were obtained by A. Ayache and J. Lévy-Véhel, see [**10**]. One can also consult the survey paper [**16**] by A. Benassi, S. Cohen and J. Istas, which covers a wider range of related problems.

The notion of Hölder exponent is adapted to functions whose regularity changes abruptly from point to point. When it is not the case, the more stable notion of *local Hölder exponent* can be used:

$$H_f(x_0) = \inf\{\alpha : \; \exists \delta > 0, \; f \in C^\alpha([x_0 - \delta, x_0 + \delta])\}.$$

J. Lévy-Véhel has studied the properties of this exponent. In particular, in collaboration with S. Seuret, he proved the following characterization: The local Hölder exponents of continuous functions are exactly the nonnegative, lower semicontinuous functions, see [**77**]. However, up to now, this notion has had no impact on multifractal analysis: Indeed, all multifractal functions we will meet have a constant local Hölder exponent.

The idea of associating fractals to measures or functions can be traced back to the books by Mandelbrot in the 70s and 80s [**82, 83**]; see also [**84**] where the main contributions of B. Mandelbrot to multifractal analysis are collected, and [**32**] where C. Evertsz and B. Mandelbrot give an extremely pedagogical introduction to multifractal measures. Multifractals were introduced by G. Parisi and U. Frisch in [**92**] for modeling fully developed turbulence, following the pioneering work of B.

Mandelbrot; indeed, B. Mandelbrot had introduced multiplicative cascades models in [**81**], which were studied by J.-P. Kahane et J. Peyriere in [**66**] (see also [**14, 15, 89**] and references therein); it is remarkable that these cascades, initially introduced as models for the dissipation of energy in turbulent flows *before the notion of multifractal was introduced* actually supplied the first mathematical models of multifractal measures.

Here is a very simple example of a multiplicative cascade which is a multifractal measure. It is constructed recursively on $[0,1]$ as follows. Let $a \in (0,1)$, $b = 1-a$, and $p \in (0,1)$. We pick at random $\mu([0,1/2]) = a$ or b, with probabilities respectively p and $1-p$, and $\mu((1/2,1]) = 1-\mu([0,1/2])$; Once the measure of a dyadic interval λ has been determined, we split this interval into two "sons" of the same length; the measure of its left son λ' is also picked at random and takes the value $\mu(\lambda') = a\mu(\lambda)$ or $(1-a)\mu(\lambda)$ with probabilities respectively p and $1-p$, and the measure of its right son is $\mu(\lambda)-\mu(\lambda')$. One thus constructs iteratively a random probability measure on $[0,1]$. This is the simplest example of a multiplicative random cascade; there exists a huge literature on multifractal cascades, see [**14, 35, 89**] and references therein for a mathematical analysis of these cascades, and [**101, 12, 15**] for important recent generalizations.

We won't consider multifractal measures in these notes; let us just mention that the notion of Hölder exponent is replaced by the *local dimension* of the measure μ defined as

$$h_\mu(x_0) = \liminf_{\delta\to 0} \frac{\log\left(\mu([x_0-\delta, x_0+\delta])\right)}{\log(\delta)},$$

see [**19, 21, 32, 44, 91, 34**]. However, a relationship between Hölder regularity for measures and functions can be explicitely established in the one-dimensional setting: If μ is a probability measure supported on $\mathbb{R}$, let $f(x) = \mu((-\infty, x])$; one immediately checks that

$$\text{if} \quad h_\mu(x_0) \in [0,1), \qquad \text{then} \quad h_\mu(x_0) = h_f(x_0).$$

The problem we mentioned concerning the Hölder exponent can also be raised for the spectrum of singularities: Which functions $d(H)$ are spectra of singularities of continuous functions? We only have partial answers: limsup of nonnegative piecewise constant functions are spectra, see [**50**], but we do not know if this class covers all possible spectra. Furthermore, one would rather wish to characterize the spectra of homogeneous multifractal functions; in this case, only very partial results are known, see [**55**].

There exists many generalizations of Cantor's triadic set and of the devil's staircase; see for instance [**65**] where more general perfect sets are constructed and their relationships with trigonometric series are studied. Another generalization, fractal strings, was introduced by M. Lapidus and C. Pomerance in [**71, 72**]; it was then used in a number of papers, including those by M. Lapidus and H. Maier [**69, 70**], C. He and M. Lapidus, see [**42, 43**] and M. Lapidus and C. Pomerance, see [**73**]; see also the reference book [**74**] by M. Lapidus and M. van Frankenhuysen.

The box dimensions yield an important information on the fractal geometry of the graph of functions, which is independent of the spectrum of singularities. General formulas which allow to derive the box dimensions of the graphs of arbitrary functions from their wavelet expansions are available, see [**53**] and [**67**]. The problem of determining the Hausdorff dimensions of graphs is often very hard, especially in deterministic settings; see however [**17**], and [**11, 95, 96**] where estimates are derived from wavelet expansions.

The notion of homogeneous multifractal functions is important for modelling; indeed, with regards to fully developed turbulence, Parisi and Frisch conjectured that its behavior is *universal*, i.e. that the spectrum of singularities does not depend on the particular fluid considered, on the limit conditions,... In particular, if this hypothesis is verified, the spectrum of singularities of the velocity of a turbulent fluid should be independent of the region where the fluid is being inspected, and therefore, the velocity of a turbulent fluid is expected to be an homogeneous multifractal function. Usually, the mathematical functions which are known to be multifractal are actually homogeneous multifractals. We will see several examples backing this assertion.

2. Wavelets and Hölder regularity

Orthonormal wavelet bases are a privileged tool to study multifractal functions for several reasons. A first one, exposed in this section, is that the Hölder exponent can be characterized by simple local decay conditions on the wavelet coefficients. Another reason, discussed in Section 3, concerns the study of the *multifractal formalism*. It is a formula which is used to derive the spectrum of singularities of a function f from the knowledge of some function spaces which contain f. Here again, wavelets play a key role since these spaces are defined by simple conditions on the wavelet coefficients.

We will just recall some properties of orthonormal wavelet bases that will be useful in the following. We refer the reader to [**27, 31, 64, 75, 80, 87**] for detailed expositions of this subject.

2.1. Orthonormal wavelet bases. Orthonormal wavelet bases are constructed through the help of a *multiresolution analysis*. It is a sequence of closed subspaces of $L^2(\mathbb{R}^d)$ denoted by V_j ($j \in \mathbb{Z}$) and satisfying:

(1) $\forall j \in \mathbb{Z}$, $V_j \subset V_{j+1}$,
(2) $\forall j \in \mathbb{Z}$, $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$,
(3) $\exists \varphi(x) \in V_0$ such that the functions $\varphi(x-k)$ ($k \in \mathbb{Z}^d$) form an orthonormal basis of V_0.
(4) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

The multiresolution analysis is r-smooth if φ is C^r and if the $\partial^\alpha\varphi$, for $|\alpha| \le r$, have fast decay.

The subspace W_j is defined as the orthogonal complement of V_j in V_{j+1}. Under these assumptions, there exist $2^d - 1$ functions $\psi^{(i)}$ satisfying the same regularity and decay properties as φ and such that the $\psi^{(i)}(x-k)$ ($i = 1, ..., 2^d - 1$, $k \in \mathbb{Z}^d$) form an orthonormal basis of W_0. Using Point 2, it follows that $\forall j \in \mathbb{Z}$, the $2^{dj/2}\psi^{(i)}(2^j x - k)$ ($k \in \mathbb{Z}^d$) form an orthonormal basis of W_j.

By construction, the W_j are orthogonal, and we can use two possible decompositions of $L^2(\mathbb{R}^d)$ as a direct orthogonal sum

$$L^2 = \bigoplus_{j=-\infty}^{+\infty} W_j \quad \text{or} \quad L^2 = V_0 + \bigoplus_{j=0}^{+\infty} W_j,$$

which leads to two possible ways to write a function of L^2

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_i c^i_{j,k}\psi^{(i)}(2^jx-k), \tag{2.1}$$

or

$$f(x)=\sum_{k\in\mathbb{Z}^d}C_k\varphi(x-k)+\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\sum_i c^i_{j,k}\psi^{(i)}(2^jx-k); \tag{2.2}$$

the $c^i_{j,k}$ are the wavelet coefficients of f

$$c^i_{j,k}=2^{dj}\int_{\mathbb{R}^d}f(x)\psi^{(i)}(2^jx-k)dx, \tag{2.3}$$

and

$$C_k=\int_{\mathbb{R}^d}f(x)\varphi(x-k)dx. \tag{2.4}$$

A consequence of Point 2 is that the functions $2^{dj/2}\varphi(2^jx-k)$ form an orthonormal basis of V_j. Thus, if $P_j(f)$ denotes the orthogonal projection of f on V_j,

$$P_j(f)(x)=\sum_k C_{j,k}\varphi(2^jx-k)\quad\text{where}\quad C_{j,k}=2^{-dj}\int f(x)\varphi(2^jx-k)dx.$$

Remarks: In (2.1) or (2.2), we do not choose the L^2 normalisation for the wavelets, but rather an L^∞ normalisation which is better fitted to the study of Hölder regularity.

Note that (2.3) and (2.4) make sense even if f does not belong to L^2; indeed, if one uses smooth enough wavelets, these formulas can be interpreted as a duality product betweeen smooth functions (the wavelets) and distributions (f). We will see the cases of Sobolev and Besov spaces in Section 4.1.

A simple example of multiresolution analysis is obtained, in dimension 1, by taking $\Phi=1_{[0,1]}$; in this case, V_j is the space of square integrable functions which are constant on each interval $[\frac{k}{2^j},\frac{k+1}{2^j}]$, and a possible choice for Ψ is $1_{[0,1/2]}-1_{[1/2,1]}$. The corresponding wavelet basis is called the *Haar basis*.

Among the families of wavelet bases that exist, two will be particularly useful for us:

- Lemarié-Meyer wavelets, such that φ and $\psi^{(i)}$ both belong to the Schwartz class;
- Daubechies wavelets, such that the functions φ and $\psi^{(i)}$ can be chosen arbitrarily smooth and with compact support.

We will often assume that the multiresolution analysis is of *tensor product type* i.e., if $x=(x_1,\dots x_d)$, then

$$\varphi(x)=\Phi(x_1)\Phi(x_2)\dots\Phi(x_d),$$

where $\Phi(x)$ is associated to a 1-dimensional multiresolution analysis. If $\Psi(x)$ denotes the corresponding 1-dimensional wavelet, we can take for d-dimensional wavelets the functions

$$\psi^{(i)}(x)=\Psi_1(x_1)\Psi_2(x_2)\dots\Psi_d(x_d),$$

where Ψ_l denotes either Φ or Ψ, and where the choice $\Phi(x_1)\Phi(x_2)\dots\Phi(x_d)$ is the only one excluded (thus there are indeed 2^d-1 wavelets).

If the multiresolution analysis is r-smooth, the wavelets have a corresponding number of vanishing moments, see [**87**]:

$$\text{If } |\alpha| < r, \quad \text{then} \quad \int_{\mathbb{R}^d} \psi^{(i)}(x) x^\alpha dx = 0.$$

Therefore, if the wavelets are in the Schwartz class, all their moments vanish.

2.2. Wavelets and uniform regularity. We start by characterizing uniform regularity properties in terms of the wavelet coefficients. Let us recall the definition of the Hölder C^α spaces.

Definition 9. Let $\alpha \in (0,1)$. A function f belongs to $C^\alpha(\mathbb{R}^d)$ if $f \in L^\infty(\mathbb{R}^d)$ and if $\exists C > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad |f(x) - f(y)| \leq C|x-y|^\alpha. \tag{2.5}$$

Let $\alpha > 1$ and $\alpha \notin \mathbb{N}$. A function f belongs to $C^\alpha(\mathbb{R}^d)$ if $f \in L^\infty(\mathbb{R}^d)$ and has partial derivatives of any order $|\beta| \leq [\alpha]$ and if, when $|\beta| = [\alpha]$, then

$$\forall x, y \in \mathbb{R}^d \qquad |\partial^\beta f(x) - \partial^\beta f(y)| \leq C|x-y|^{\alpha - [\alpha]}. \tag{2.6}$$

The C^α norm of f is the sum of $\| f \|_\infty$ and of the infimum of the Cs such that (2.5) or (2.6) holds. The following proposition shows that the space C^α is characterized by a very simple decay condition on the wavelet coefficients.

Proposition 4. Let $\alpha > 0$ be such that $\alpha \notin \mathbb{N}$ and assume that the multiresolution analysis is of tensor product type and r-smooth with $r \geq [\alpha] + 1$. Then

$$f \in C^\alpha(\mathbb{R}^d) \iff \exists C > 0 : \begin{cases} \qquad\qquad \forall k \in \mathbb{Z}^d \quad |C_k| \leq C \\ \forall j \geq 0, \forall i, \ \ \forall k \in \mathbb{Z}^d \quad |c^i_{j,k}| \leq C 2^{-\alpha j}. \end{cases} \tag{2.7}$$

Remarks: In the following, we will usually not mention the regularity needed for the wavelets, which will be assumed to be "smooth enough"; the minimal regularity required following easily from the computations.

Proposition 4 holds for any wavelet basis which is r-smooth with $r > \alpha$ because the matrix of a change of wavelet basis is continuous on the space of coefficients satisfying (2.7), see [**87**].

Proof: Suppose that $f \in C^\alpha(\mathbb{R}^d)$; then,

$$|C_k| \leq \int_{\mathbb{R}^d} |f(x)|\, |\varphi(x-k)| dx \leq C \| f \|_\infty .$$

The wavelet $\psi^{(i)}(x)$ can be written $\Psi_1(x_1)\Psi_2(x_2)\dots\Psi_d(x_d)$, where the Ψ_l are equal to Φ or Ψ, and at least one of them is equal to Ψ. Suppose that it is the case for Ψ_1. If $k = (k_1, \dots, k_d)$,

$$c^i_{j,k} = 2^{dj} \int_{\mathbb{R}^d} f(x_1, \dots, x_d) \Psi(2^j x_1 - k_1) \dots \Psi_d(2^j x_d - k_d) dx_1 \dots dx_d.$$

The function $\Psi(x)$ has its first $[\alpha]+1$ moments vanishing; therefore it can be written

$$\Psi(x) = \frac{d^{[\alpha]}\tilde{\Psi}}{dx^{[\alpha]}}(x)$$

where $\tilde{\Psi}$ still has fast decay and a vanishing integral. Let

$$\tilde{\psi}(x_1, \dots x_d) = \tilde{\Psi}(x_1)\Psi_2(x_2)\dots\Psi_d(x_d).$$

Integrating by parts $[\alpha]$ times in the direction x_1, we get

$$\begin{aligned} c^i_{j,k} &= 2^{dj} \int f(x)\psi^{(i)}(2^j x-k)dx \\ &= (-1)^{[\alpha]}2^{dj}2^{-[\alpha]j}\int \frac{d^{[\alpha]}f}{dx_1^{[\alpha]}}(x)\tilde{\psi}(2^jx-k)dx \\ &= (-1)^{[\alpha]}2^{dj}2^{-[\alpha]j}\int \left(\frac{d^{[\alpha]}f}{dx_1^{[\alpha]}}(x)-\frac{d^{[\alpha]}f}{dx_1^{[\alpha]}}\left(\frac{k}{2^j}\right)\right)\tilde{\psi}(2^jx-k)dx \end{aligned}$$

(because $\tilde{\psi}$ has a vanishing integral); therefore, using the decay of $\tilde{\psi}$,

$$| c^i_{j,k} |\leq C2^{dj}2^{-[\alpha]j}\int \left|x-\frac{k}{2^j}\right|^{\alpha-[\alpha]}\frac{C}{(1+\mid 2^jx-k\mid)^{d+2}}dx \leq C2^{-\alpha j}.$$

Let us now prove the converse result. Assume that

$$|C_k|\leq C \quad \text{and} \quad |c^i_{j,k}|\leq C2^{-\alpha j}. \tag{2.8}$$

Let

$$f_{-1}(x)=\sum_k C_k\varphi(x-k)$$

and, if $j\geq 0$,

$$f_j(x)=\sum_i\sum_k c^i_{j,k}\psi^{(i)}(2^jx-k); \tag{2.9}$$

let us consider the series

$$\sum_{j=-1}^{+\infty} f_j(x). \tag{2.10}$$

Using (2.8) and the decay of φ and $\psi^{(i)}$, one obtains that (2.9) converges uniformly on any compact to a limit f_j which has the same regularity as the wavelets. Furthermore,

$$\forall j\geq -1, \quad | f_j(x) |\leq C\sum_k \frac{2^{-\alpha j}}{(1+\mid 2^jx-k\mid)^{d+1}}\leq C2^{-\alpha j};$$

therefore, the series (2.10) converges normally to a function which we denote momentarily by g and which belongs to L^∞. Similarly, using the decay of the $\partial^\beta\varphi$ and of the $\partial^\beta\psi^{(i)}$, one obtains

$$\text{if} \quad \beta|\leq[\alpha]+1, \qquad | \partial^\beta f_j(x) |\leq C2^{(|\beta|-\alpha)j}. \tag{2.11}$$

We can differentiate term by term the series $\sum f_j$ up to the order $[\alpha]$, and still obtain a normally convergent series; thus g belongs to $C^{[\alpha]}$ and

$$\forall|\beta|\leq[\alpha], \qquad |\partial^\beta g(x)|\leq C.$$

Let x and y in $\mathbb{R}^d$ and j_0 be defined by $2^{-j_0-1}\leq| x-x_0 |< 2^{-j_0}$; if $|\beta|=[\alpha]$, then $| \partial^\beta g(x)-\partial^\beta g(y) |$ is bounded by

$$\sum_{j\leq j_0} | \partial^\beta f_j(x)-\partial^\beta f_j(y) | + \sum_{j>j_0} | \partial^\beta f_j(x) | + \sum_{j>j_0} | \partial^\beta f_j(y) | .$$

Using (2.11) and the mean value theorem, the first term is bounded by

$$\sum_{j\le j_0} \mid x-y\mid \sup_{|\beta|=[\alpha]+1}\sup_{[x,x_0]} \mid \partial^\beta f_j(t)\mid \quad \le C\mid x-y\mid \sum_{j\le j_0} 2^{([\alpha]+1-\alpha)j}$$

$$\le C\mid x-x_0\mid 2^{([\alpha]+1-\alpha)j_0}.$$

Coming back to the definition of j_0, it follows that the first term is bounded by $C\mid x-x_0\mid^\alpha$. The second and third terms are bounded by

$$\sum_{j>j_0} 2^{([\alpha]-\alpha)j} \le C2^{([\alpha]-\alpha)j_0} \le C\mid x-x_0\mid^{\alpha-[\alpha]}.$$

Let us now prove that $f=g$. We consider the sequence $F_j = f-\sum_{j=-1}^{J} f_j$; its wavelet coefficients vanish for $j\le J$ and it shares the same wavelet coefficients as f if $j\ge J$. Therefore, the same computation as above shows that $\| F_J\|_\infty \le C2^{-J}$, so that $f=g$; hence the converse result holds.

2.3. Characterization of the Hölder exponent. We start by introducing some definitions and notations. A dyadic cube of scale j is a cube of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times\cdots\times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right),$$

where $k=(k_1,\dots k_d)\in\mathbb{Z}^d$. Instead of indexing the wavelets and wavelet coefficients with the three indices (i,j,k), we will use dyadic cubes. Since i takes 2^d-1 values, we can assume that it takes values in $\{0,1\}^d-(0,\dots,0)$; we will use the following notations:

- $\lambda\ \ (=\lambda(i,j,k)) \ \ = \dfrac{k}{2^j}+\dfrac{i}{2^{j+1}}+\left[0,\dfrac{1}{2^{j+1}}\right)^d.$
- $c_\lambda = c^i_{j,k}$
- $\psi_\lambda(x)=\psi^{(i)}(2^jx-k)$,
- $\mu_\lambda = \dfrac{k}{2^j}$.

The wavelet ψ_λ is essentially localized near the cube λ; more precisely, when the wavelets are compactly supported

$$\exists C>0 \ \text{ such that } \ \forall i,j,k, \qquad supp\,(\psi_\lambda)\subset C\cdot\lambda$$

(where $C\cdot\lambda$ denotes the cube of same center as λ and C times wider). Finally, Λ_j will denote the set of dyadic cubes λ which index a wavelet of scale j, i.e. wavelets of the form $\psi_\lambda(x)=\psi^{(i)}(2^jx-k)$ (note that Λ_j is a subset of the dyadic cubes of side 2^{j+1}). We take for norm on $\mathbb{R}^d$

$$\text{if } x=(x_1,\dots,x_d), \quad |x| = \sup_{i=1,\dots,d}|x_i|;$$

so that the diameter of a dyadic cube of side 2^{-j} is 2^{-j}.

Definition 10. The wavelet leaders are defined by

$$d_\lambda = \sup_{\lambda'\subset\lambda}|c_{\lambda'}|. \tag{2.12}$$

Note that if $f \in L^\infty$, then

$$|c_\lambda| \leq 2^{dj} \int |f(x)||\psi_\lambda(x)|dx \leq C \sup |f(x)|;$$

so that the wavelet leaders are finite.

Definition 11. Two dyadic cubes λ_1 et λ_2 are called adjacent if they are at the same scale and if $dist(\lambda_1, \lambda_2) = 0$ (note that a dyadic cube is adjacent to itself). We denote by $\lambda_j(x_0)$ the dyadic cube of side 2^{-j} containing x_0 and by $Adj(\lambda)$ the set of 3^d dyadic cubes adjacent to λ. Then

$$d_j(x_0) = \sup_{\lambda' \in Adj(\lambda_j(x_0))} d_{\lambda'}.$$

Definition 12. The influence cone at x_0 consists of the set of dyadic cubes of the form $\lambda_j(x_0)$ and their adjacent cubes.

The wavelet characterization of the Hölder exponent requires the following regularity hypothesis, which is slightly stronger than continuity.

Definition 13. A function f is uniformly Hölder if there exists $\epsilon > 0$ such that $f \in C^\epsilon(\mathbb{R}^d)$.

The following theorem allows to characterize the pointwise regularity by a decay condition of the $d_j(x_0)$ when $j \to +\infty$.

Theorem 1. Let $\alpha > 0$. If f is $C^\alpha(x_0)$, then there exists $C > 0$ such that

$$\forall j \geq 0, \qquad d_j(x_0) \leq C 2^{-\alpha j}. \tag{2.13}$$

Conversely, if (2.13) holds and if f is uniformly Hölder, then f belongs to $C^\alpha_{log}(x_0)$.

Remark: If f is uniformly Hölder, its regularity at x_0 is therefore determined by the behavior of the d_λ in the influence cone at x_0. More precisely, one can notice that the characterization supplied by Proposition 4 can be rewritten

$$\exists C \ \ \forall \lambda \qquad d_\lambda \leq C 2^{-\alpha j}.$$

Therefore pointwise and uniform Hölder regularity are characterized by the same decay condition, either written locally (in the influence cone) or uniformly.

Proof of Theorem 1: If f is $C^\alpha(x_0)$, then there exists a polynomial P of degree at most $[\alpha]$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

Let $j \geq 0$; if $\lambda' \subset 3\lambda_j(x_0)$, then, using the vanishing moments of the wavelets,

$$c_{\lambda'} = 2^{dj'} \int f(x) \psi^{(i)}(2^{j'}x - k')dx = 2^{dj'} \int (f(x) - P(x - x_0)) \psi^{(i)}(2^{j'}x - k')dx;$$

thus $|c_{\lambda'}|$ is bounded by

$$C 2^{dj'} \int \frac{| \ x - x_0 \ |^\alpha}{(1 + | \ 2^{j'}x - k' \ |)^{d+\alpha+1}} dx \leq C 2^{dj'} \int \frac{| \ x - \mu_{\lambda'} \ |^\alpha + | \ \mu_{\lambda'} - x_0 \ |^\alpha}{(1 + | \ 2^{j'}x - k' \ |)^{d+\alpha+1}} dx$$

(because, $\forall \alpha, a, b > 0$, $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$). The change of variable $t = 2^{j'}x - k'$ yields

$$|c_{\lambda'}| \leq C(2^{-\alpha j'} + |\mu_{\lambda'} - x_0|^\alpha). \tag{2.14}$$

Since $j' \geq j-1$ and $|\mu_{\lambda'} - x_0| \leq 4d2^{-j}$, we have $|c_{\lambda'}| \leq C2^{-\alpha j}$, so that $d_j(x_0) \leq C2^{-\alpha j}$.

Let us now prove the converse result. First note that, since f is uniformly Hölder, Proposition 4 implies that the wavelet series of f converges uniformly to f on any compact. Let j' be given. We will first estimate the size of $f_{j'}$ and of its partial derivatives. If λ' is a cube of side $2^{-j'}$, denote by λ $(= \lambda(\lambda'))$ the dyadic cube defined by

- If $\lambda' \subset 3\lambda_{j'}(x_0)$, then $\lambda = \lambda_{j'}(x_0)$,
- else, if $j = \sup\{l : \ \lambda' \subset 3\lambda_l(x_0)\}$, then $\lambda = \lambda_j(x_0)$ (and it follows that $2^{-j-1} \leq |\mu_{\lambda'} - x_0| \leq 4d2^{-j}$).

In the first case, by hypothesis, $|c_{\lambda'}| \leq d_{j'}(x_0) \leq C.2^{-\alpha j'}$, and the sum on the corresponding λ' satisfies (as in the proof of Proposition 4)

$$|\sum_{\lambda'} c_{\lambda'}\psi_{\lambda'}(x)| \leq C2^{-\alpha j'}.$$

In the second case, $|c_{\lambda'}| \leq d_j(x_0) \leq C2^{-\alpha j} \leq C|x_0 - \mu_{\lambda'}|^\alpha$ and the sum on the corresponding values of λ' of scale j' satisfies

$$\begin{aligned} \left|\sum_{\lambda'} c_{\lambda'}\psi_{\lambda'}(x)\right| &\leq C\sum_{k'} \frac{|x_0 - \mu_{\lambda'}|^\alpha}{(1+|2^{j'}x - k'|)^{d+\alpha+1}} \\ &\leq C\sum_{k'} \frac{|x - x_0|^\alpha + |x - \mu_{\lambda'}|^\alpha}{(1+|2^{j'}x - k'|)^{d+\alpha+1}} \\ &\leq C(|x-x_0|^\alpha + 2^{-\alpha j'}). \end{aligned}$$

Thus

$$|f_{j'}(x)| \leq C\left(2^{-\alpha j'} + |x - x_0|^\alpha\right); \tag{2.15}$$

Similarly, one obtains, for any β such that $|\beta|$ is smaller than the regularity of the wavelets,

$$|\partial^\beta f_{j'}(x)| \leq C2^{|\beta| j'}\left(2^{-\alpha j'} + |x - x_0|^\alpha\right). \tag{2.16}$$

In particular, if $|\beta| \leq [\alpha]$, the series $\sum_{j=-1}^{\infty} \partial^\beta f_j(x_0)$ converges absolutely; (recall that, by convention, if α is an integer, $[\alpha] = \alpha - 1$) and we can define the polynomial

$$P(x - x_0) = \sum_{|\beta| \leq [\alpha]} \frac{(x - x_0)^\beta}{\beta!} \sum_{j=-1}^{\infty} \partial^\beta f_j(x_0);$$

P is the sum of the polynomials

$$P_j(x - x_0) = \sum_{|\beta| \leq [\alpha]} \frac{(x - x_0)^\beta}{\beta!} \partial^\beta f_j(x_0);$$

Since P_j is the Taylor expansion of f_j of degree $[\alpha]$ at x_0,

$$|f_j(x) - P_j(x - x_0)| \leq C|x - x_0|^{[\alpha]+1} \sup_{[x, x_0]} \sum_{|\beta| = [\alpha]+1} |\partial^\beta f_j|$$

$$\leq C|x-x_0|^{[\alpha]+1}2^{([\alpha]+1)j}\left(2^{-\alpha j}+|x-x_0|^{\alpha}\right). \tag{2.17}$$

We will now bound

$$|f(x)-P(x-x_0)|=\left|\sum_{j\geq -1}(f_j(x)-P_j(x-x_0))\right|. \tag{2.18}$$

By hypothesis, there exists $\epsilon>0$ such that $f\in C^{\epsilon}(\mathbb{R}^d)$. Let j_0 and j_1 be defined by

$$2^{-j_0-1}\leq\mid x-x_0\mid<2^{-j_0}\quad\text{and}\quad j_1=\left[\frac{\alpha j_0}{\epsilon}\right].$$

We have $j_0\leq C|\log|x-x_0||$. If $j\leq j_0$, using (2.17), the sum of the corresponding terms of (2.18) is bounded by

$$\sum_{j\leq j_0}C|x-x_0|^{[\alpha]+1}2^{([\alpha]+1)j}\left(2^{-\alpha j}+|x-x_0|^{\alpha}\right)\leq Cj_0|x-x_0|^{\alpha}.$$

If $j_0<j\leq j_1$, (2.15) implies that $|f_j(x)|\leq C|x-x_0|^{\alpha}$; so that

$$\sum_{j=j_0}^{j_1}|f_j(x)|\leq Cj_1|x-x_0|^{\alpha}\leq\frac{C}{\epsilon}|x-x_0|^{\alpha}\left|\log(|x-x_0|)\right|.$$

If $j>j_1$, since f is uniformly Hölder, Proposition 4 implies that $|f_j(x)|\leq C2^{-\epsilon j}$, and

$$\sum_{j=j_1}^{\infty}|f_j(x)|\leq C2^{-\epsilon j_1}\leq C2^{-\alpha j_0}\leq C|x-x_0|^{\alpha}.$$

Finally, using (2.16),

$$\begin{aligned}\sum_{j\geq j_0}|P_j(x-x_0)|\leq\;& C\sum_{j\geq j_0}\sum_{|\beta|\leq[\alpha]}\frac{|x-x_0|^{\beta}}{\beta!}|\partial^{\beta}f_j(x_0)|\\ &\leq C\sum_{j\geq j_0}\sum_{|\beta|\leq[\alpha]}|x-x_0|^{\beta}2^{|\beta|j}2^{-\alpha j}\\ &\leq C|x-x_0|^{\alpha}.\end{aligned}$$

The converse part of Theorem 1 is thus proved.

2.4. Quasisure Hölder regularity in $C^{\alpha}(\mathbb{R}^d)$. We will now prove a quasi-sure result which is a direct consequence of Theorem 1. Let us first recall Baire's category theorem:

If E is a complete metric space, every countable intersection of open dense sets is dense.

If a property $\mathcal{P}(x)$ holds (at least) on a countable intersection of open dense sets, it is said to hold *quasi-surely*. Note that a quasi-sure property $\mathcal{P}$ does not necessarily hold on a large set; indeed, in $\mathbb{R}^d$ a quasi-sure property may hold only on a set of measure 0 (and even of dimension 0). It rather means that the property holds on a "very dense" set. Indeed, countable intersections of such sets are still dense.

PROPOSITION 5. Let $\alpha > 0$ and such that $\alpha \notin \mathbb{N}$. The functions of $C^\alpha(\mathbb{R}^d)$ are quasi-surely mono-Hölder with Hölder exponent $h(x) = \alpha$. (The set of mono-Hölder functions of exponent α even contains a dense open set of $C^\alpha(\mathbb{R}^d)$).

Proof of Proposition 5: We assume that a smooth enough wavelet basis has been chosen. Any function $f \in C^\alpha$ can be written in the form

$$f(x) = \sum_k c_k \varphi(x-k) + \sum_i \sum_{j \geq 0} \sum_k c^i_{j,k} \psi^{(i)}(2^j x - k),$$

where

$$\sup_{j,k}(|c_k|, 2^{\alpha j}|c^i_{j,k}|) \leq C.$$

We can choose for equivalent norm in $C^\alpha(\mathbb{R}^d)$ the infimum of the Cs such that this inequality holds. If N is a given integer, let E_N be the subset of $C^\alpha(\mathbb{R}^d)$ defined by: $F \in E_N$ if all $|c_k|$ and $2^{\alpha j}|c^i_{j,k}|$ are *nonvanishing* integer multiples of 2^{-N}. Clearly

$$\forall f \in C^\alpha(\mathbb{R}^d) \quad dist(f, E_N) \leq 2^{-N},$$

so that $\bigcup_{N=1}^\infty E_N$ is dense in $C^\alpha(\mathbb{R}^d)$. In a metric space E, we denote by $B(x,R)$ the open ball of center x and radius R. Let $F_N = E_N + B(0, 2^{-N-1})$. F_N is an open set. Any function f of F_N satisfies

$$\forall j,k \quad 2^{\alpha j}|c^i_{j,k}| \geq 2^{-N-1}$$

and the direct part of Theorem 1 implies that the Hölder exponent of f is everywhere at most α. Since, on the other hand, any function f of F_N belongs to $C^\alpha(\mathbb{R}^d)$, f is clearly a mono-Hölder function of exponent α. The union of the F_N is the expected open dense set.

2.5. Hölder exponent and local oscillations. We will now see a consequence of Theorem 1 which supplies an alternative definition for Hölder regularity. In this new definition, the polynomial P, which may be difficult to exhibit, does not appear explicitly.

Let $f : \mathbb{R}^d \to \mathbb{R}$ and $h \in \mathbb{R}^d$. The first order difference of f is

$$(\Delta^1_h f)(x) = f(x+h) - f(x).$$

If $n > 1$, the differences of order n are defined recursively by

$$(\Delta^n_h f)(x) = (\Delta^{n-1}_h f)(x+h) - (\Delta^{n-1}_h f)(x).$$

DEFINITION 14. If A is a convex subset of $\mathbb{R}^d$, the oscillation of order n of f on A is

$$\mathbf{OS}^n_f(A) = \sup_{[x,x+nh]\subset A} |(\Delta^n_h f)(x)|.$$

Note that

$$\mathbf{OS}^1_f(A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x).$$

PROPOSITION 6. If $f \in C^\alpha(x_0)$, then

$$\forall n \geq [\alpha]+1 \quad \forall \epsilon > 0 \quad |\mathbf{OS}^n_f(B(x_0,\epsilon))| \leq C\epsilon^\alpha. \tag{2.19}$$

Conversely, if f is uniformly Hölder and if (2.19) holds, then $f \in C^\alpha_{log}(x_0)$.

Note that, if $0 < \alpha < 1$, the triangular inequality implies that

$$f \in C^\alpha(x_0) \iff \forall \epsilon > 0 \ \ |\mathbf{OS}^1_f(B(x_0, \epsilon))| \le C\epsilon^\alpha.$$

Proof of Proposition 6: Assume that $f \in C^\alpha(x_0)$. Since

$$|f(x) - P(x - x_0)| \le C|x - x_0|^\alpha,$$

it follows that

$$\text{if} \quad |x - x_0| \le \epsilon, \qquad \text{then} \quad |\Delta^n_h(f(x) - P(x - x_0))| \le 2^n C\epsilon^\alpha.$$

But, if $n > deg(P)$, $\Delta^n_h P(x - x_0) = 0$. It follows that $|\Delta^n_h f(x)| \le C\epsilon^\alpha$.

In order to prove the converse result, we need the following property of one-dimensional wavelets.

Lemma 2. Assuming that the multiresolution analysis is n-smooth, for any $n \ge 1$, there exists a function θ_n with fast decay and such that $\psi = \Delta^n_{1/2}\theta_n$. Furthermore, if ψ is compactly supported, we can pick θ_n also compactly supported.

Proof of Lemma 2: Recall that, in the construction of one-dimensional wavelets by multiresolution analysis, the embedding $V_{-1} \subset V_0$ implies the existence of coefficients l_k such that

$$\varphi\left(\frac{x}{2}\right) = \sum_k l_k \varphi(x - k);$$

which can be rewritten in the Fourier domain as

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi),$$

where m_0 is C^∞ and 2π-periodic. Furthermore, m_0 vanishes at π to the order at least r, see [**87**]. One can pick for wavelet ψ the function

$$\widehat{\psi}(2\xi) = e^{-i\xi}\overline{m_0}(\xi + \pi)\widehat{\varphi}(\xi).$$

It follows that $\widehat{\psi}(2\xi)$ vanishes at the multiples of 2π to the order at least r. Furthermore, if ϕ is compactly supported, m_0 is a trigonometric polynomial, so that (after perhaps translating ψ), $\bar{m}_0(\xi + \pi)$ can be written

$$\bar{m}_0(\xi + \pi) = (e^{i\xi} - 1)^r P(\xi), \tag{2.20}$$

where P is a trigonometric polynomial, and therefore

$$P(\xi) = \sum_{k=-n}^{n} a_k e^{ik\xi}.$$

In dimension 1, to apply Δ^n_h to f amounts to multiply $\widehat{f}(\xi)$ by $e^{ih\xi} - 1$. Therefore, we look for a function θ_n such that

$$\widehat{\psi}(\xi) = (e^{i\xi/2} - 1)^n \widehat{\theta_n}(\xi).$$

We can define θ_n by

$$\widehat{\theta}_n(\xi) = e^{-i\xi/2}\frac{\overline{m_0}(\xi + \pi)}{(e^{i\xi/2} - 1)^n}\widehat{\varphi}\left(\frac{\xi}{2}\right).$$

Since the function $n(\xi) = \dfrac{\overline{m_0}(\xi + \pi)}{(e^{i\xi/2} - 1)^n}$ is C^∞ and 4π-periodic, θ_n has fast decay. Furthermore, if φ is compactly supported, (2.20) implies that $n(\xi)$ is a trigonometric polynomial, and therefore θ_n is compactly supported. Hence Lemma 2 holds.

Let us now prove the converse part in Proposition 6. Assume that, in dimension d, we use the tensor-product wavelet basis. Then

$$\psi^{(i)}(x) = \Psi^1(x_1)\dots\Psi^d(x_d),$$

where the Ψ^i are either ψ or φ, but at least one of them is equal to ψ. Suppose for instance, that $\Psi^1(x_1) = \psi(x_1)$. Then

$$\begin{aligned} c^i_{j,k} &= 2^{dj}\int f(x)\Psi^1(2^j x_1 - k_1)\dots\Psi^d(2^j x_d - k_d)dx \\ &= \int f\left(\frac{x+k}{2^j}\right)\Psi^1(x_1)\dots\Psi^d(x_d)dx. \end{aligned}$$

Clearly, in dimension 1,

$$\int f(x)(\Delta^n_h g)(x)dx = (-1)^n \int (\Delta^n_{-h} f)(x)g(x)dx; \tag{2.21}$$

so that

$$c^i_{j,k} = (-1)^n \int \left(\Delta^n_{-e_1/2} f\right)\left(\frac{x+k}{2^j}\right)\theta_n(x_1)\Psi^2(x_2)\dots\Psi^d(x_d)dx, \tag{2.22}$$

with $e_1 = (1,0,\dots,0)$.

Let $K_0 = B(0,1)$ and if $l > 0$, $K_l = B(0,2^l) - B(0,2^{l-1})$. If $x \in K_l$, $\dfrac{x+k}{2^j}$ and $\dfrac{x+k}{2^j} - \dfrac{ne_1}{2.2^j}$ belong to $B(x_0, |x_0-\mu_\lambda| + n2^{l-j})$. Therefore

$$\begin{aligned} &\left|\int_{K_l}\left(\Delta^n_{-e_1/2} f\right)\left(\frac{x_1+k}{2^j}\right)\theta_n(x_1)\Psi^2(x_2)\dots\Psi^d(x_d)dx\right| \\ &\le C\int_{K_l}\frac{\left(|x_0-\mu_\lambda| + n2^{l-j}\right)^\alpha}{(1+|x|)^{\alpha+d+1}}dx \le \frac{\left(|x_0-\mu_\lambda| + n2^{l-j}\right)^\alpha}{2^{l(\alpha+1)}}, \end{aligned}$$

which is bounded by $C\dfrac{|x_0-\mu_\lambda|^\alpha}{2^{l(\alpha+1)}}$ if $n2^{l-j} \le |x_0-\mu_\lambda|$, and else by $C\dfrac{2^{-\alpha j}}{2^l}$. The sum over l is therefore clearly bounded by $|x_0-\mu_\lambda|^\alpha + 2^{-\alpha j}$; hence (2.14) holds and the converse part in Proposition 6 is now a consequence of the converse part of Theorem 1.

The following corollary follows from Theorem 1 and Proposition 6. It characterizes the Hölder exponent by local decay conditions of the oscillation, or, equivalently, by local decay conditions of local suprema of the wavelet coefficients.

Corollary 1. If f is uniformly Hölder, the Hölder exponent of f can be computed using either

$$h_f(x_0) = \liminf_{j\to\infty}\left(\frac{\log(d_j(x_0))}{\log(2^{-j})}\right), \tag{2.23}$$

or

$$h_f(x_0) = \sup_{n\in\mathbb{N}}\left[\liminf_{\epsilon\to 0}\left(\frac{\log(|\mathbf{OS}^n_f(B(x_0,\epsilon))|)}{\log\epsilon}\right)\right]. \tag{2.24}$$

Remark: The supremum over n in formula (2.24) should be understood as follows: The term between brackets is equal to n as long as $n < [h_f(x_0)]$ and becomes independent of n if $n \ge [h_f(x_0)]$.

2.6. Notes. The first construction of an orthonormal wavelet basis was performed by Haar in 1909; we saw that the corresponding wavelet is discontinuous. The first arbitrarily smooth orthonormal wavelet bases were constructed by Strömberg in 1981. Wavelets in the Schwartz class were introduced by P.-G. Lemarié and Y. Meyer in 1986. The construction by multiresolution analysis was performed by S. Mallat and Y. Meyer in 1989, and compactly supported wavelets were introduced by I. Daubechies shortly afterwards.

The wavelet characterization of the space $C^\alpha(\mathbb{R}^d)$ is remarkable; it is in sharp contrast with Fourier expansions: Indeed, the space of periodic C^α functions cannot be characterized by a condition bearing on the moduli of their Fourier coefficients, see [**102**].

Hardy and Littlewood are the forerunners of the wavelet method used to prove irregularity at a point. They had noticed that, if f is smooth at x_0, the convolution of f with a smooth, well localized function ψ of vanishing integral (a "wavelet") has to be small near x_0, see [**40, 41**]. Irregularity results are obtained by contraposition. (One can interpret wavelet coefficients as a convolution product of $f(x)$ with $\psi(2^j x)$, this convolution being then sampled at the points $k2^{-j}$.) Hardy and Littlewood applied this method to the Weierstrass functions, but also to the "non-differentiable Riemann function" $\sum \frac{\sin(2\pi n^2 x)}{n^2}$, which is much more difficult to handle since its Hölder exponent is everywhere discontinuous; they showed in 1914 that this function is not differentiable, except perhaps at the rational points $\frac{2p+1}{2q+1}$, and J. Gerver showed in 1970 that, indeed, it is differentiable at these points, see [**37**]. This was the first function, with unbounded variation, which was proved to be multifractal (in 1996, see [**51**]). More properties of this remarkable function are exposed in [**62**].

If irregularity results using the wavelet technique were at least implicit in Hardy and Littlewood's paper in the 1910s, the first converse result only appeared in 1989, in [**47**]; nonetheless, one should mention a remarkable result of J.-P. Kahane who proved in 1976 the existence of slow points of the Brownian motion using its decomposition on the Schauder basis, which predates these wavelet techniques, see [**63, 64**].

We mentioned in Section 1.6 several generalizations of pointwise Hölder regularity. Another generalization, the *scaling exponent,* is based on the Littlewood-Paley decomposition of f; it has been introduced by Y. Meyer in [**88**]. Moduli of continuity more general than $\theta(x) = x^\alpha$ and their wavelet characterization are studied in [**60**].

The idea of studying the properties of a function using the wavelet leaders d_λ rather than the wavelet coefficients first appeared in the resolution of the following problem: Find a formula which yields the upper box dimension of the graph of a function; the answer is expressed in terms of the d_λ, see [**53**].

Condition (2.13) used to be written in the form

$$|c_\lambda| \leq C 2^{-\alpha j}(1 + |2^j x - k|)^\alpha$$

(which is clearly equivalent to (2.13)). This condition is a particular case of the two-microlocal conditions $C^{s,s'}(x_0)$ defined by

$$|c_\lambda| \leq C 2^{-sj}(1 + |2^j x - k|)^{-s'}. \tag{2.25}$$

Two-microlocal conditions were introduced by J.-M. Bony as a tool in the study of the propagation of singularities of solutions of PDEs. The set of couples (s, s') such that $f \in C^{s,s'}(x_0)$ is called the *two-microlocal domain* of f at x_0. It yields a very accurate information on the local oscillations of f near x_0 and the regularity of its fractional derivatives and primitives at x_0. The properties of the two-microlocal domain were investigated by J. Lévy-Véhel, S. Seuret and their collaborators, see [**78**], and the paper in the present volume [**79**].

The converse part in Theorem 1 requires a uniformly Hölder assumption. Even if this assumption can actually be slightly weakened, a uniform regularity assumption, strictly stronger than continuity, is necessary, as shown by Yves Meyer, see [**48**]. Therefore, wavelet methods cannot be used to study functions which have a dense set of discontinuities. Such functions are not just a mathematical curiosity; for instance, most Lévy processe (processes with independent, stationary increments), which play a central role in probability, share this property, see [**54**]. It is also the case for Davenport series which are of the form

$$\sum_{n=1}^{\infty} a_n\{nx\} \quad \text{where} \quad \{x\} = x - [x] - \frac{1}{2},$$

see [**58**]. Note however that, in [**61**], pointwise Hölder estimates are obtained with only a Besov regularity assumption which implies no uniform C^ϵ regularity (but is strictly stronger than continuity). Similarly, the logarithmic correction in the converse part of Theorem 1 is best possible, see [**48**].

The generalizations of the pointwise regularity conditions mentioned in Section 1.6 also have wavelet characterizations; for instance, C. Melot has obtained a wavelet characterization of the $T^p_\alpha(x_0)$ condition, see [**85**].

The first quasi-sure result concerning Hölder regularity is a famous theorem by Banach, in 1931, which states that a continuous function is almost surely nowhere differentiable, see [**13**]. Z. Buczolich and J. Nagy proved in [**22**] that quasi-all monotone continuous functions are multifractal and their spectrum is

$$\begin{aligned} d(H) &= H && \text{if} \quad H \in [0,1], \\ &= -\infty && \text{if} \quad H > 1. \end{aligned}$$

3. The multifractal formalism

The spectrum of singularities of many mathematical functions can be determined directly from its definition. On the opposite, for many real-life signals, whose Hölder exponent is expected to be everywhere discontinuous, the numerical determination of their Hölder regularity is not feasible, and therefore, one cannot expect to have direct access to their spectrum of singularities. In such cases, one has to find an indirect way to compute $d(H)$; the multifractal formalism is a formula which is expected to yield the spectrum of singularities of f from "global" quantities which are numerically computable. Mathematically, these quantities are interpreted as indicating that f belongs to a certain subset of a family of function spaces. There exist slight variants on these formulas, which depend on the family of function spaces used. Their validity never holds in complete generality. However, three types of verification can be performed:

- The multifractal formalism is proved under additional assumptions on f (a self-similarity assumption, or for a class of particular random processes, or even for very specific functions f, see [**3, 50, 51**] for instance).

- It is proved for a "large" subset of the function space considered; the quasi-sure result of Proposition 5 in Section 2.4 supplies such an example. We will also see prevalence results in Section 5.
- The multifractal formalism is shown to yield an upper bound of the spectrum of singularities of any (uniformly Hölder) function, see Section 3.2.

In this section, we will first describe the heuristic arguments used in the derivation of the multifractal formalism. They will be presented in two different contexts: First, in Section 3.1 using wavelet expansion, and in Section 3.3 directly on increments of the function. This will lead us naturally to introduce some new function spaces, the *oscillation spaces*, which will be studied afterwards. In Section 3.2, we prove that, indeed, the multifractal formalism yields a general upper bound of the spectrum.

3.1. Derivation of the multifractal formalism: The wavelet method. A *multifractal formalism* is a formula which is expected to yield the spectrum of singularities of a function from the estimation of 'global' quantities which are numerically computable. The first formula, based on L^p norms of the increments of the function, was proposed by G. Parisi and U. Frisch and in 1985, see [**92**]. Soon afterwards, A. Arneodo, E. Bacry and J.-F. Muzy proposed an alternative formula based on the continuous wavelet transform, see [**4, 7**]. In this section, we derive a slightly different formula based on the wavelet leaders d_λ. The advantage will be that one does not have to assume that the Hölder singularities are "cusp-singularities"; a cusp singularity at x_0 displays a behavior similar to $|x-x_0|^\alpha$, as opposed to *chirps* which display strong oscillations like $|x-x_0|^\alpha \sin(|x-x_0|^{-\beta})$, see [**6**].

Using heuristic arguments, we now derive this multifractal formalism. We will form 'global quantities' based on the wavelet leaders d_λ. Indeed, if f is uniformly Hölder, which we assume from now on, Theorem 1 shows that, if $h_f(x_0)=H$, there exists an infinite number of dyadic cubes λ which contain x_0 (or are adjacent to such a cube) such that

$$\lim_{j\to+\infty}\left(\frac{\log d_\lambda}{\log(2^{-j})}\right)=H;$$

which we will write $d_\lambda \sim 2^{-Hj}$. Global quantities which are natural to consider are the l^p-averages of the d_λ at all scales. In order to keep as much information as possible on the d_λ, we don't restrict the computation to $p\geq 1$ or even $p>0$; therefore, if Ω is a bounded domain of $\mathbb{R}^d$ and $p\in\mathbb{R}^*$, let

$$S^\Omega_{p,j}=2^{-dj}\sum_{\lambda\in\Lambda_j\cap\Omega} d^p_\lambda,$$

where the sum is taken on all wavelet leaders of scale j corresponding to dyadic cubes localized in Ω. The order of magnitude of $S^\Omega_{p,j}$ when $j\to+\infty$ is estimated with the help of the *scaling function* of f on Ω defined by

$$\omega^\Omega_f(p)=\liminf_{j\to+\infty}\left(\frac{\log\left(S^\Omega_{p,j}\right)}{\log(2^{-j})}\right), \tag{3.1}$$

which means that $S^{\Omega}_{p,j}$ is of the order of magnitude of $2^{-\omega^{\Omega}_f(p)j}$ when $j \to +\infty$. More precisely, by definition of $\omega^{\Omega}_f(p)$,

$$\forall \delta > 0, \exists C > 0 \qquad \sum_{\lambda \in \Lambda_j \cap \Omega} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d)j} \le C 2^{\delta j}$$

and there exist $j_n \to +\infty$ such that

$$\forall \delta > 0, \exists C > 0 \qquad \sum_{\lambda \in \Lambda_{j_n} \cap \Omega} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d)j_n} \ge C 2^{-\delta j_n}.$$

These two conditions can be rewritten as follows: For any $\delta > 0$,

$$\sum_{j=0}^{J} \sum_{\lambda \in \Lambda_j \cap \Omega} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d+\delta)j} \longrightarrow +\infty \quad \text{when} \quad J \longrightarrow +\infty \tag{3.2}$$

and

$$\sum_{j=0}^{\infty} \sum_{\lambda \in \Lambda_j \cap \Omega} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d-\delta)j} \le C. \tag{3.3}$$

Let $x_0 \in \mathbb{R}^d$ and $\epsilon > 0$. If $h_f(x_0) = H$, there exists a sequence of dyadic cubes λ in the influence cone at x_0 such that

$$|\lambda| \le \epsilon \quad \text{and} \quad 2^{(-H-\delta)j} \le d_\lambda \le 2^{(-H+\delta)j}. \tag{3.4}$$

For each $x_0 \in \Omega$ at which the Hölder exponent takes the value H, let us pick such a cube λ, and denote by $\Lambda(H)$ the set of these cubes and their adjacent neighbours. The cubes which belong to $\Lambda(H)$ therefore constitue an ϵ-covering of E_H. For this particular subset of dyadic cubes, (3.3) implies that

$$\sum_{\lambda \in \Lambda(H)} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d-\delta)j} \le C,$$

and therefore, taking (3.4) into account,

$$\forall \delta > 0, \qquad \sum_{\lambda \in \Lambda(H)} |\lambda|^{(-\omega^{\Omega}_f(p)+d+Hp+\delta p+\delta)} \le C;$$

thus $d^{\Omega}_f(H)$ (which denotes the spectrum of singularities of f restricted to Ω) satisfies

$$d^{\Omega}_f(H) \le \omega^{\Omega}_f(p) - d - Hp.$$

Furthermore, (3.2) implies that, for at least one value of H, which we will denote by H_0,

$$\sum_{\lambda \in \Lambda(H_0)} d^p_\lambda 2^{(\omega^{\Omega}_f(p)-d)j} \ge C 2^{-\delta j}$$

(because otherwise, when one 'sums over all values of H', the corresponding quantity would decrease exponentially). Thus there exists H_0 such that

$$d^{\Omega}_f(H_0) = -\omega^{\omega}_f(p) + d + H_0 p.$$

Summing up, we see that the spectrum of f restricted to Ω satifies, for any $p \ne 0$,

$$d^{\Omega}_f(H) \le -\omega^{\Omega}_f(p) + d + Hp \quad \text{and} \quad \exists H_0 : \quad d^{\Omega}_f(H_0) = -\omega^{\Omega}_f(p) + d + H_0 p;$$

which can be rewritten as follows:

$$\omega^{\Omega}_f(p) = \inf_H (d - d^{\Omega}_f(H) + pH). \tag{3.5}$$

In order to determine the spectrum on all of $\mathbb{R}^d$, writing $\mathbb{R}^d$ as a countable union of bounded domains, we have

$$d_f(H) = \sup_{\Omega \subset \mathbb{R}^d} d_f^{\Omega}(H);$$

and therefore, we have to replace in (3.5) the function $\omega_f^{\Omega}(p)$ by

$$\omega_f(p) = \inf_{\Omega} \omega_f^{\Omega}(p). \tag{3.6}$$

We thus obtain

$$\omega_f(p) = \inf_H (d - d_f(H) + pH). \tag{3.7}$$

We will show in Section 4.3 that $\omega_f(p)$ (which is defined on $\mathbb{R} \setminus \{0\}$) extends to a concave function on $\mathbb{R}$. Denote by $\tilde{d}_f(H)$ the concave hull of $d_f(H)$ (i.e. the smallest concave function larger than $d_f(H)$). Formula (3.7) can be interpreted as stating that $-\tilde{d}_f(H)$ and $-\omega_f(p) + d$ are convex conjugate functions, and each can therefore be deduced from the other by a Legendre transform. It follows that

$$\tilde{d}_f(H) = \inf_p \ (Hp - \omega_f(p) + d)\,. \tag{3.8}$$

Formula (3.8) tells us that we only expect to recover the concave hull of the spectrum of singularities from the scaling function. However, it often happens that $d_f(H)$ actually is a concave function, in which case (3.8) allows us to recover $d_f(H)$ completely. **When such is the case,**

$$d_f(H) = \inf_p \ (Hp - \omega_f(p) + d)\,, \tag{3.9}$$

and we will say that the multifractal formalism is satisfied by f.

3.2. Upper bound of the spectrum of singularities. When p is positive, the scaling function can be given a functional interpretation. For that, we introduce the *oscillation spaces* which are defined by the following condition on the wavelet expansion.

Definition 15. Let $s \in \mathbb{R}$ and $p > 0$; a distribution f belongs to $\mathcal{O}_p^s(\mathbb{R}^d)$ if its wavelet coefficients satisfy

$$\sum_k |C_k|^p \leq C$$

and

$$\exists C > 0 \quad \forall j \geq 0 \quad 2^{(sp-d)j} \sum_k {d_\lambda}^p \leq C. \tag{3.10}$$

Remarks: These function spaces are a particular case of the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$ previously considered in [**53, 57**].

It follows immediately from Definition 15 that, if $p > 0$, then

$$\omega_f(p) = \sup\{s : \ f \in \mathcal{O}_{p,loc}^{s/p}\},$$

therefore the function $\omega_f(p)$ indicates which $\mathcal{O}_p^s$ spaces contain the function f.

One easily checks that the oscillation spaces are Banach spaces if $p \geq 1$; they are complete metric spaces if $0 < p < 1$ for the metric defined as follows: Let f

and g be two functions in $\mathcal{O}^\omega$. The coefficients of f are denoted c_λ and C_k as in Definition 15, and those of g are denoted $\tilde{c}_\lambda$ and $\tilde{C}_k$; then
(3.11)

$$d(f,g) = \| f-g \|_{B^{0,\infty}_\infty} \sum_k |C_k - \tilde{C}_k|^p + \sup_{j\geq 0} \left(2^{(sp-d)j} \sum_k \left(\sup_{\lambda'\subset\lambda} |\tilde{c}_{\lambda'} - c_{\lambda'}| \right)^p \right).$$

The following result states that the multifractal formalism yields an upper bound for the spectrum of any function.

THEOREM 2. Let f be a uniformly Hölder function. Then

$$d_f(H) \leq \inf_{p\in\mathbb{R}^*} (Hp - \omega_f(p) + d). \tag{3.12}$$

First, we consider the case where p is positive. Recall that the sets G_H were defined in (1.5).

PROPOSITION 7. Let $p > 0$; if f is a uniformly Hölder function in $\mathcal{O}^s_p$,

$$\forall H \geq s - \frac{d}{p}, \quad dim(G_H) \leq d + Hp - sp. \tag{3.13}$$

Furthermore, if $s - \frac{d}{p} > 0$, then $G_H = \emptyset$ for any $H < s - \frac{d}{p}$.

Proof: With regards to the case $H < s - \frac{d}{p}$, if $f \in \mathcal{O}^s_p$,

$$|c_\lambda| \leq d_\lambda \leq C 2^{-(s-\frac{d}{p})j}$$

so that $\forall x_0$, $h_f(x_0) \geq s - \frac{d}{p}$.

Let us now prove the first assertion. Let

$$G_{j,H} = \left\{ \lambda : \ |d_\lambda| \geq 2^{-Hj} \right\},$$

and denote by $N_{j,H}$ the cardinality of $G_{j,H}$. By hypothesis, $f \in \mathcal{O}^s_p$ so that $2^{(sp-d)j} \sum |d_\lambda|^p \leq C$, and

$$2^{(sp-d)j} N_{j,H} 2^{-Hpj} \leq C,$$

so that $N_{j,H} \leq C 2^{(d-sp+Hp)j}$.

Denote by $F_{j,H}$ the set of cubes λ of scale j such that either $\lambda \in G_{j,H}$, or λ is adjacent to a cube of $G_{j,H}$. Clearly,

$$Card(F_{j,H}) \leq 3^d Card(G_{j,H}) \leq 3^d C 2^{(d-sp+Hp)j}.$$

Denote by $F_h = \limsup_{j\to+\infty} F_{j,H}$ the set of points that belong to an infinite number of $F_{j,H}$. If $x_0 \notin F_H$, then there exists j_0 $(= j_0(x))$ such that, for any $j \geq j_0$, we have $d_j(x_0) \leq 2^{-Hj}$; thus we can choose C $(= C(x))$ large enough so that

$$\forall j \geq 0, \quad d_j(x_0) \leq C 2^{-Hj}.$$

Theorem 1 implies that $f \in C^H_{log}(x_0)$; so that $G_H \subset F_H$.

It remains to bound the dimension of F_H. Let $\epsilon > 0$, and

$$j_0 = \inf\{ j : \sqrt{d} 2^{-j} \leq \epsilon \}.$$

We choose for ϵ-covering of F_H all the cubes λ such that $j \geq j_0$ and $\lambda \in F_{j,H}$. Clearly,

$$\begin{aligned} \sum Diam(B_\lambda)^\delta \leq & \; C \sum_{j=j_0}^{\infty} Card(F_{j,H}) \left(\sqrt{d} 2^{-j}\right)^\delta \\ & \leq C \sum_{j=j_0}^{\infty} 2^{(d-sp+Hp-\delta)j}, \end{aligned}$$

which is finite if $\delta > d + Hp - sp$; hence the first part of the proposition holds.

Let us now check that the last point of the proposition follows. If $x_0 \in E_H$, $h_f(x_0) = H$, and $\forall H' > H$, $x_0 \in G_{H'}$; so that $E_H \subset G_{H'}$. Let $p > 0$; by definition of $\omega_f(p)$ we have $\forall \epsilon > 0$, $f \in \mathcal{O}_{p,loc}^{(\omega_f(p)-\epsilon)/p}$, so that

$$\forall H' > H, \; \forall \epsilon > 0 \;\; d(H) = dim(E_H) \leq dim(G_{H'}) \leq d + H'p - \omega_f(p) + \epsilon,$$

and thus the spectrum of f restricted on any bounded set Ω satisfies

$$d^\Omega(H) \leq d + Hp - \omega_f(p).$$

Since this upper bound is valid for all $p > 0$, (3.12) follows (with the infimum taken only on $\mathbb{R}^+$).

We consider now the case where p is negative. In this case, we will obtain a result which is stronger than Theorem 2 since it yields a bound for the packing dimension of the Hölder singularities.

Proposition 8. Let $p < 0$; if $f \in \mathcal{O}_p^s$, the packing dimension of B_H is bounded by $d - sp + Hp$.

Note that, in contradistinction with the case $p > 0$, we do not have to make any uniform regularity assumption here.

Proof: Let $\delta > 0$ and J such that $2^{-J} \leq \delta$. If $f \in C^H(x_0)$, there exists $A > 0$ such that

$$\forall j \geq J, \qquad \sup_{\lambda' \subset \lambda_j(x_0)} |c_{\lambda'}| \leq A 2^{-Hj};$$

so that, since $p < 0$,

$$\left(\sup_{\lambda' \subset \lambda_j(x_0)} |c_{\lambda'}| \right)^p \geq A^p 2^{-Hpj}. \tag{3.14}$$

Denote by Ω_A the set of points x where (3.14) holds for any $j \geq J$. Clearly,

$$B_H \subset \bigcup_{A>0} \Omega_A,$$

where the union can be written as a countable union. Since $f \in \mathcal{O}_p^s$, there are at most $CA^p 2^{(d-sp+Hp+\epsilon)j}$ cubes λ satisfying (3.14), so that the upper box dimension of each set Ω_A is bounded by $d - sp + Hp$. The proposition follows by countable union, and the upper bound (3.12) also follows (with the infimum taken on $\mathbb{R}^-$); therefore (3.12) is completely proved.

3.3. Derivation of the multifractal formalism: The oscillation method. In order to derive a multifractal formalism, we started from (2.23); however, one can alternatively start from the characterization of pointwise regularity based on local oscillations (2.24): If $h_f(x_0) = H$, there exists an infinite number of balls $B(x_0, \epsilon_j)$ such that, for n large enough,

$$\lim_{\epsilon_j \to 0} \frac{\log\left(|\mathbf{OS}_f^n(B(x_0, \epsilon_j))|\right)}{\log(\epsilon_j)} = H.$$

In this formulation, the "global" quantities which are natural to consider are

$$T^{\Omega}_{p,\epsilon} = \epsilon^{dj} \sum_{l \in \epsilon\mathbb{Z}^d \cap \Omega} |\mathbf{OS}_f^n(B(l, 3\sqrt{d}\epsilon))|^p \tag{3.15}$$

(we use balls of radius $3\sqrt{d}\epsilon$ so that there is enough overlapping, but the precise value 3 is not important). Let

$$\nu_f^{\Omega}(p) = \liminf_{\epsilon \to 0} \left(\frac{\log\left(T^{\Omega}_{p,\epsilon}\right)}{\log(\epsilon)} \right) = d + \liminf_{\epsilon \to 0} \left(\frac{\log\left(\sum_{l \in \epsilon\mathbb{Z}^d \cap \Omega} |\mathbf{OS}_f^n(B(l, 3\sqrt{d}\epsilon))|^p \right)}{\log(\epsilon)} \right).$$

The argument we developed using the $T^{\Omega}_{p,j}$ can be reproduced. In the present setting, it leads to the following formulation of the multifractal formalism:

$$\tilde{d}(H) = \inf_p \; (Hp - \nu_f(p) + d)\,, \tag{3.16}$$

where

$$\nu_f(p) = \inf_{\Omega} \nu_f^{\Omega}(p).$$

When p is positive, the function $\nu_f(p)$ also has a function space interpretation.

Definition 16. Let $p > 0$ and $s \in \mathbb{R}$. A function f belongs to $\mathcal{V}_p^s(\mathbb{R}^d)$ if $f \in L^\infty(\mathbb{R}^d)$ and if, for $n > [s]$,

$$\exists C > 0 \quad \forall \epsilon \geq 0 \quad \epsilon^{d-sp} \sum_{l \in \epsilon\mathbb{Z}^d} |\mathbf{OS}_f^n(B(l, 3\sqrt{d}\epsilon))|^p \leq C. \tag{3.17}$$

Note that

$$\nu_f(p) = \sup\{s : f \in \mathcal{V}^{s/p}_{p,loc}(\mathbb{R}^d)\}. \tag{3.18}$$

The function $\nu_f(p)$ indicates to which $\mathcal{V}_p^s$ spaces the function f belongs locally.

Before considering the problem of the validity of the multifractal formalism, one should first check that its two formulations using either wavelets or oscillations lead to the same formula. There exists only a partial result in this direction which concerns positive values of p. We will show that, if f is uniformly Hölder, for any $p > 0$,

$$\omega_f(p) = \nu_f(p).$$

This result is a consequence of the following theorem.

THEOREM 3. The following imbeddings hold

$$\forall \epsilon > 0, \ \ \forall \eta > 0 \quad C^{\epsilon} \cap V^{s+\eta,p} \hookrightarrow \mathcal{O}_p^s \hookrightarrow V^{s,p}; \tag{3.19}$$

thus, if $f : \ \mathbb{R}^d \to \mathbb{R}$ is uniformly Hölder, then the p-oscillation exponent of f is given by

$$\begin{aligned} \omega_f(p) \quad &= \sup\{s : \ f \in \mathcal{O}_p^{s/p,0}\} \\ &= \limsup_{j\to+\infty} \frac{\log\left(\displaystyle\sum_{\lambda\in\Lambda_j} \sup_{\lambda'\subset\lambda} |c_{\lambda'}|^p\right)}{j\log 2}. \end{aligned} \tag{3.20}$$

Let us stress the fact that this result deals only with the case $p > 0$. We do not know if the scaling function $\omega_f(p)$ can be characterized by local oscillations when p is negative.

3.4. Proof of Theorem 3. We assume that we use compactly supported wavelets. (The general case will follow afterwards using Theorem 4 which states that the scaling function is independent of the wavelet basis). Coming back to the comparison of wavelet coefficients and oscillations performed in Section 2.5, one sees that the function $\theta_n(x_1)\Psi^2(x_2)\dots\Psi^d(x_d)$ which appears in (2.22) is compactly supported, say in $[-A, A]^d$. Let $B_\lambda = B(k2^{-j}, A\sqrt{d}2^{-j})$. It follows from (2.22) that $|c_\lambda| \le C\cdot \mathbf{OS}_f^n(B_\lambda)$; thus, if $\lambda' \subset \lambda$,

$$|c_{\lambda'}| \le C\cdot \mathbf{OS}_f^n(B_{\lambda'}) \le C\cdot \mathbf{OS}_f^n(B_\lambda);$$

so that $d_\lambda \le C\cdot \mathbf{OS}_f^n(B_\lambda)$. Therefore,

$$2^{(sp-d)j} \sum_{\lambda\in\Lambda_j} d_\lambda^p \le C 2^{(sp-d)j} \sum_{\lambda\in\Lambda_j} \left[\mathbf{OS}_f^n(B_\lambda)\right]^p.$$

In order to obtain a converse estimate, we first assume that f is compactly supported. We will bound the oscillations of each component $f_{j'}$ on the ball B_λ by $\displaystyle\sum_{\lambda\in\Lambda_j} d_\lambda^p$.

If $j' \ge j$, then

$$f_{j'} = \sum_{\lambda'\in\Lambda_{j'}} c_{\lambda'}\psi_{\lambda'}.$$

Using the localization of the wavelets, it follows that

$$\mathbf{OS}_{f_{j'}}^n(B_\lambda) \le C \parallel f \parallel_{L^\infty(B_\lambda)} \le C \sup_{C\lambda'\cap B_\lambda\neq\emptyset} |c_{\lambda'}|.$$

In particular, since $f \in C^\epsilon(\mathbb{R}^d)$, it follows that

$$\mathbf{OS}_{f_{j'}}^n(B_\lambda) \le C 2^{-\epsilon j'}. \tag{3.21}$$

Let $\tilde{d}_\lambda = \displaystyle\sup_{C\lambda\cap\mu\neq\emptyset} \ \sup_{\mu'\subset\mu} |c_{\mu'}|$. We have

$$\mathbf{OS}_{f_{j'}}^n(B_\lambda) \le C\tilde{d}_\lambda. \tag{3.22}$$

Since $\tilde{d}_\lambda$ is a supremum of d_λ on at most $(C+2)^d$ cubes, it follows that

$$\sum_{\lambda\in\Lambda_j} \tilde{d}_\lambda^p \leq (C+2)^{dp} \sum_{\lambda\in\Lambda_j} d_\lambda^p. \tag{3.23}$$

If $j' < j$, applying Taylor's formula to $\Delta_h^n f_{j'}$, it follows that, if x and $x+nh$ belong to B_λ,

$$|\Delta_h^n f_{j'}| \leq C|h|^n \sup_{x\in B_\lambda} |\partial^\alpha f_{j'}(x)|,$$

where the supremum bears on the $|\alpha| = n$. But, if $|\alpha| = n$,

$$|\partial^\alpha f_{j'}(x)| \leq C 2^{nj'} \sup_{C\lambda'\cap B_\lambda\neq\emptyset} |c_{\lambda'}|.$$

Since $|h| \leq C2^{-j}$, it follows that

$$\mathbf{OS}^n_{f_{j'}}(B_\lambda) \leq C2^{-n(j-j')} \sup_{C\lambda'\cap B_\lambda\neq\emptyset} |c_{\lambda'}|.$$

Therefore, if $\lambda^{j'}$ denotes the dyadic cube of scale j' which includes λ,

$$\mathbf{OS}^n_{f_{j'}}(B_\lambda) \leq C2^{-n(j-j')} \tilde{d}_{\lambda^{j'}}. \tag{3.24}$$

Since $f = \sum_{j'=-1}^{\infty} f_{j'}$, we have $\mathbf{OS}^n_f(B_\lambda) \leq \sum_{j'=-1}^{\infty} \mathbf{OS}^n_{f_{j'}}(B_\lambda)$. Using (3.21), it follows that

$$\sum_{j'=j^2}^{\infty} \mathbf{OS}^n_{f_{j'}}(B_\lambda) \leq C2^{-\epsilon j^2},$$

and using (3.22),we get

$$\sum_{j'=j}^{j^2} \mathbf{OS}^n_{f_{j'}}(B_\lambda) \leq Cj^2 \tilde{d}_\lambda.$$

Therefore

$$\mathbf{OS}^n_f(B_\lambda) \leq \sum_{j'=-1}^{j-1} \mathbf{OS}^n_{f_{j'}}(B_\lambda) + Cj^2\tilde{d}_\lambda + C2^{-\epsilon j^2}.$$

Since

$$\forall p > 0, \quad \left(\sum_{i=1}^n a_i\right)^p \leq (n \sup a_i)^p \leq n^p \sum_{i=1}^n (a_i)^p,$$

it follows that

$$\left(\mathbf{OS}^n_f(B_\lambda)\right)^p \leq C(j+3)\left(\sum_{j'=-1}^{j-1} \left[\mathbf{OS}^n_{f_{j'}}(B_\lambda)\right]^p + j^{2p}(\tilde{d}_\lambda)^p + 2^{-\epsilon p j^2}\right).$$

Thus, using (3.24), it follows that

$$\left(\mathbf{OS}^n_f(B_\lambda)\right)^p \leq C(j+3)\left(\sum_{j'=-1}^{j-1} 2^{-np(j-j')}\left(\tilde{d}_{\lambda^{j'}}\right)^p + j^{2p}(\tilde{d}_\lambda)^p + 2^{-\epsilon p j^2}\right).$$

When we sum on λ, each term $\left(\tilde{d}_{\lambda^{j'}}\right)^p$ appears $2^{d(j-j')}$ times, so that

$$\sum_{\lambda\in\Lambda_j} (\mathbf{OS}^n_f(B_\lambda))^p \leq$$

$$C(j+3)\left(\sum_{j'=-1}^{j-1} 2^{(d-np)(j-j')} \sum_{\lambda'\in\Lambda_{j'}} \left(\tilde{d}_{\lambda^{j'}}\right)^p + j^{2p}\sum_{\lambda\in\Lambda_j}(\tilde{d}_\lambda)^p + C2^{dj}2^{-\epsilon pj^2}\right)$$

(in the derivation of the last term we use the fact that f is compactly supported, so that the sum bears on at most $C2^{dj}$ dyadic cubes of scale j). Using (3.23), we can replace $\tilde{d}_\lambda$ by d_λ. By hypothesis,

$$\forall j' \sum_{\lambda'\in\Lambda_{j'}} (d_{\lambda'})^p \leq C2^{(d-sp+\epsilon)j'};$$

so that

$$\sum_{\lambda\in\Lambda_j} (\mathbf{OS}^n_f(B_\lambda))^p$$

$$\leq c(j+3)\left(\sum_{j'=-1}^{j-1} 2^{-(np-d)(j-j')}2^{(d-sp+\epsilon)j'} + j^{2p}2^{(d-sp+\epsilon)j} + C2^{dj}2^{-\epsilon pj^2}\right),$$

which is bounded by $2^{(d-sp+2\epsilon)j'}$ if n is picked large enough so that $np-d > sp-d$, i.e. $n > s$.

The result follows for any function f (non necessarily compactly supported), using a partition of unity: Let θ be a C^∞ compactly supported function, satisfying $\sum_{k\in\mathbb{Z}} \theta(x-k) = 1$. We apply the result to each $f_k(x) = f(x)\theta(x-k)$ and use

$$\| f \|^p_{\mathcal{O}^s_p} \sim \sum_k \| f_k \|^p_{\mathcal{O}^s_p} .$$

A similar argument also works for the $\mathcal{V}^s_p$.

Let us now check that the $\sum \left|\mathbf{OS}^n_f(B(\mu_\lambda, A\lambda))\right|^p$ are independent of the (large enough) constant A. We will check that, if

$$A_j = \sum_{\lambda\in\Lambda_j} \left|\mathbf{OS}^n_f(3\lambda)\right|^p$$

and

$$B_j = \sum_{\lambda\in\Lambda_j} \left|\mathbf{OS}^n_f(A\lambda)\right|^p$$

where $A \geq 3$, then

$$\liminf_{j\to+\infty} \frac{\log(A_j)}{\log(2^{-j})} = \liminf_{j\to+\infty} \frac{\log(B_j)}{\log(2^{-j})}.$$

Of course, $A_j \leq B_j$. Let l be such that

$$\frac{1}{2}2^l < A \leq 2^l.$$

Then

$$B_j \leq \sum_{\lambda\in\Lambda_j} \left|\mathbf{OS}^n_f(2^l\lambda)\right|^p \leq 3^d 2^{dl} \sum_{\lambda'\in\Lambda_{j-l}} \left|\mathbf{OS}^n_f(3\lambda')\right|^p$$

because each cube of scale j is contained in a cube $3\lambda'$, where λ' is a dyadic cube of scale $l-j$, and each cube of this type contains at most 3^d2^{dl} cubes of scale j.

In order to end the proof of Theorem 3, we have to check that

$$\epsilon^{d-sp} \sum_{l\in\epsilon\mathbb{Z}^d} |\mathbf{OS}^n_f(B(l,3\sqrt{d}\epsilon))|^p \leq C2^{(sp-d)j} \sum_{\lambda\in\Lambda_j} |\mathbf{OS}^n_f(A\lambda)|^p,$$

wher we can pick any $A > 3$. we pick j such that

$$2^{-j} \leq \epsilon < 2\cdot 2^{-j},$$

and A large enough so that any ball $B((l, 3\sqrt{d}\epsilon)$ is entirely included in a cube $A\lambda$ ($\lambda\in\Lambda_j$). The estimate follows.

3.5. Notes. We mentioned that the multifractal formalism may not yield the correct result when applied to chirps; this problem is discussed in [**6**] and [**86**]. A chirp at x_0 displays strong oscillations near x_0 similar to those of

$$|x-x_0|^\alpha \sin\left(\frac{1}{|x-x_0|^\beta}\right).$$

Their study was initiated by P. Tchamitchian and B. Torresani in [**98**], see also [**24**]; there exist two slightly different ways to model such such a behavior, either mathematical chirps, see [**60, 88**], or oscillating singularities, see [**6, 8**]. In both cases, the couple (α,β) is used to label either chirps or oscillating singularities. Therefore, one can associate to such singularities a new spectrum $d(\alpha,\beta)$ indexed by two parameters which is, as expected, the Hausdorff dimension of these singularities. Such two-variables spectra have been determined either for stochastic processes (random wavelet series), see [**9**]), or for quasi-all functions in Besov or Sobolev spaces, see [**86**].

The justification of the multifractal formalism that we proposed is an adaptation of the one initially introduced by G. Parisi and U. Frisch in [**92**] (see also [**39**]). One difference is that the "local quantities" which were their starting point differ from the $T_{p,\epsilon}$: Let

$$S_p(l) = \int |f(x+l) - f(x)|^p\, dx;$$

the corresponding scaling function used in the former formalism is

$$\zeta_f(p) = \sup\{\tau :\ \ |S_p(l)| \leq C|l|^\tau\}.$$

Recall that, if

$$d\left(\frac{1}{p}-1\right)_+ < s < 1, \tag{3.25}$$

the Besov space $B^{s,q}_p$ can be characterized by the condition

$$\int_0^t \sup_{0\leq|h|\leq t} \left(\int |f(t+h)-f(t)|^p dt\right)^{q/p} \frac{dt}{t^{sq+1}} \leq C,$$

see [**100**]. This characterization clearly implies that the corresponding scaling function is

$$\zeta_f(p) = \sup\{s :\ \ f\in B^{s/p,q}_{p,loc}\}, \tag{3.26}$$

at least as long as $d\left(\frac{1}{p}-1\right)_+ < \frac{\zeta_f(p)}{p} < 1$ (we can choose any $q>0$); in general, one should rather take (3.26) as a definition of the scaling function. One easily checks that ζ_f is also concave. The corresponding multifractal formalism asserts that

$$d_f(H) = \inf_p \; (Hp - \zeta_f(p) + d)\,. \tag{3.27}$$

We will show in the following section (Proposition 9) that, if $s > \frac{d}{p}$, $\mathcal{O}^s_p = B^{s,\infty}_p$. It follows that, when the infimum in (3.27) is attained for values of p such that $\zeta(p) > d$, this formula coincides with (3.8). However, the formulation of the multifractal formalism based on oscillation spaces has a wider range of validity, see [**6**]. In the applications, one prefers to use wavelet coefficients in the formulation of the multifractal formalism, rather than increments, which are numerically less stable; indeed, increments (or oscillations) are very sensitive to noise; by contrast, since the wavelet coefficients are averages (integrals against smooth functions), they are numerically much less sensitive to noise. Note also that the extension of the corresponding scaling function $\zeta_f(p)$ to $p<0$ has no theoretical backing, whereas we will see in the next section that the scaling function based on oscillation spaces extends in a 'canonical' way to $p<0$ (see Section 4.2). Wavelet methods for the numerical computation of spectra of singularities were developed by A. Arneodo, E. Bacry, J.-F. Muzy and their collaborators, see [**4, 5, 7**].

The validity of the multifractal formalism for functions has been the subject of many papers. We won't give a detailed review of this topic; however, let us just mention that this validity has been proved under assumptions of selfsimilarity for the function, either exact [**50**], approximate [**3, 20**] or statistical [**54**]. These results followed a similar line of research for multifractal measures. We will see in Section 5 how generic results of validity can be obtained.

Assuming that the multifractal formalism holds, a function f will be monofractal if and only if its scaling function is affine. This led to a confusion and many misunderstandings between mathematicians and signal processers: In the applied community, a multifractal function is usually defined as a function f whose scaling function (either ω_f or ζ_f) is **not** an affine function. Needless to say, such a definition by the negative insures that most signals met turn out to be multifractal... This partly explains the flourishing literature on multifractality which has spread to a large numbers of applied fields.

4. Properties of the scaling function

Our purpose in this section is to derive some properties of the scaling function that have been used in the derivation of the multifractal formalism. The multifractal formalism given by (3.9) relates the scaling function (which has been constructed with the help of wavelet coefficients) to the spectrum of singularities (which has been defined without any reference to a particular wavelet basis). Therefore, an obvious consistency requirement is to check that the scaling function is independent of the choice of the wavelet basis.

THEOREM 4. Let $f \in L^\infty$. If the wavelets used belong to the Schwartz class, then, for any $p \neq 0$, the scaling function of f is independent of the wavelet basis chosen.

The proof of Theorem 4 will be split into two cases, depending on whether p is positive or negative. When p is positive, we already saw that the scaling function indicates which oscillation spaces f belongs to. We will show in Section 4.1 that their definition is independent of the wavelet basis chosen, thus proving Theorem 4 when p is positive. The case $p < 0$ will be examined in the Section 4.2. The function space interpretation of the scaling function is important for another reason: One cannot expect (3.8) or (3.9) to hold for any function; indeed, it is very easy to construct counterexamples to the multifractal formalism; the central question of multifractal analysis is to determine the domain of validity of (3.9). Up to now, most results concerned specific functions or random processes. A way to see how general results of validity can be obtained is to remark that, since the scaling function for $p > 0$ implies that f belongs to a certain intersection of oscillation spaces, we can wonder if there is a "generic" result of multifractality that would hold for 'most' functions of this function space. We will address this problem in Section 5.

4.1. Oscillation spaces. Oscillation spaces were introduced in Definition 15; they are closely related to Besov spaces. Let us recall the definition of Besov spaces. If s is large enough, Besov spaces can be defined by conditions on the finite differences $\Delta_h^M f$ already considered in Section 2.5: Let p, q and s be such that $0 < p \leq +\infty$, $0 < q \leq +\infty$ and $s > d(\frac{1}{p} - 1)_+$; then $f \in B_p^{s,q}$ if $f \in L^p$ and if, for $M > s$,

$$\int_0^t \sup_{0<|h|\leq t} \| (\Delta_h^M f) \|_p^q \frac{dt}{t^{sq+1}} \leq C. \tag{4.1}$$

The following wavelet characterization (which can be taken as a definition) has been proved in [**87**].

Definition 17. Let $s \in \mathbb{R}$ and $p, q > 0$. A distribution f belongs to $B_p^{s,q}$ if the sequence c_k belongs to l^p and if

$$\sum_{j\geq 0} \left(\sum_{k,i} \left| c_\lambda 2^{(s-\frac{d}{p})j} \right|^p \right)^{q/p} < +\infty. \tag{4.2}$$

Remark: This characterization is of course reminiscent of (3.10); indeed, if $q = +\infty$, $f \in B_p^{s,\infty}$ if $(c_k) \in l^p$ and if

$$\forall j \quad 2^{(sp-d)j} \sum_{k,i} |c_\lambda|^p < \infty. \tag{4.3}$$

Since $|c_\lambda| \leq |d_\lambda|$, it follows that $\mathcal{O}_p^s \hookrightarrow B_p^{s,\infty}$.

When $p \geq 1$, Besov spaces are also closely related with the Sobolev spaces.

Definition 18. Let $s \geq 0$ and $p \geq 1$. A function f belongs to the Sobolev space $L^{p,s}(\mathbb{R}^d)$ if $f \in L^p$ and if $(Id - \Delta)^{s/2} f \in L^p$, where the operator $(Id - \Delta)^{s/2}$ is defined as follows: $g = (Id - \Delta)^{s/2} f$ means that

$$\widehat{g}(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$$

(the function $(1 + |\xi|^2)^{s/2}$ being C^∞ with polynomial increase, $(1 + |\xi|^2)^{s/2} \hat{f}(\xi)$ is well defined if f is a tempered distribution).

This definition amounts to say that f and its fractional derivatives of order at most s belong to L^p, see [**87**]. If $s \geq 0$ and $p \geq 1$, then

$$B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty}.$$

Thus $B_p^{s,q}$ is very close to $L^{p,s}$. The following proposition shows that oscillation and Besov spaces sometimes coincide.

PROPOSITION 9. If $s > \frac{d}{p}$, then $\mathcal{O}_p^s = B_p^{s,\infty}$.

Proof of Proposition 9: We already saw that $\mathcal{O}_p^s \hookrightarrow B_p^{s,\infty}$. In order to prove the converse embedding, note that $d_\lambda^p = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|^p$ can be bounded by $\sum_{j' \geq j} \sum_{\lambda' \subset \lambda} |c_{\lambda'}|^p$ (where the sum over λ' is taken on the subcubes of λ of scale j'), so that

$$\begin{aligned} \sum_k d_\lambda^p 2^{(sp-d)j} &\leq \sum_k \sum_{j' \geq j} \sum_{\lambda' \subset \lambda} |c_{\lambda'}|^p 2^{(sp-d)j} \\ &= \sum_{j' \geq j} \sum_{k'} |c_{\lambda'}|^p 2^{(sp-d)j} \\ &= \sum_{j' \geq j} \sum_{k'} |c_{\lambda'}|^p 2^{(sp-d)j'} 2^{-(sp-d)(j'-j)}. \end{aligned} \tag{4.4}$$

If $s > \frac{d}{p}$, and $f \in B_p^{s,\infty}$, then $\sum_{k'} |c_{\lambda'}|^p 2^{(sp-d)j'} \leq C$; (4.4) is therefore bounded; hence $B_p^{s,\infty} \hookrightarrow \mathcal{O}_p^s$.

The following embeddings will be useful in order to derive the properties of the scaling function.

PROPOSITION 10. (1) If $s_1 \leq s_2$, $\mathcal{O}_p^{s_2} \hookrightarrow \mathcal{O}_p^{s_1}$;

(2) if $q \geq p$, $\mathcal{O}_p^s \hookrightarrow \mathcal{O}_q^{s-\frac{d}{p}+\frac{d}{q}}$;

(3) If f is compactly supported, then

$$f \in \mathcal{O}_p^s \implies \forall q < p, \;\; f \in \mathcal{O}_q^s. \tag{4.5}$$

Remark: The third embedding implies that $\forall q < p$, $\mathcal{O}_{p,loc}^s \hookrightarrow \mathcal{O}_{q,loc}^s$.

Proof of Proposition 10: The first embedding is straightforward; as regards to the second one, if $f \in \mathcal{O}_p^s$, for any j, the sequences $2^{(s-\frac{d}{p})j} d_\lambda$ are bounded in l^p uniformly in j, therefore, they are also bounded in l^q, uniformly in j, if $q \geq p$. Since

$$2^{(s-\frac{d}{p})j} d_\lambda = 2^{((s-\frac{d}{p}+\frac{d}{q})-\frac{d}{q})j} d_\lambda,$$

the second embedding is proved. As regards to the third one, if f is compactly supported, and if the wavelets are also compactly supported, then, at each scale j, at most $C2^{dj}$ of the d_λ do not vanish. Let $q < p$; applying Hölder's inequality to the sequence d_λ^q, and to the sequence identically equal to 1 if $d_\lambda \neq 0$, and vanishing elsewhere, we obtain

$$\sum_{\lambda \in \Lambda_j} d_\lambda^q \leq \| \, d_\lambda^q \, \|_{p/q} \| \, 1 \, \|_r,$$

where r is the exponent conjugate to p/q. This inequality implies that

$$\sum_{\lambda\in\Lambda_j} d_\lambda^q \le \left(\sum_{\lambda\in\Lambda_j} d_\lambda^p\right)^{q/p} \left(C2^{dj}\right)^{1-\frac{q}{p}}.$$

Therefore

$$\sum_{\lambda\in\Lambda_j} 2^{(sq-d)j} d_\lambda^q \le C\left(\sum_{\lambda\in\Lambda_j} 2^{(sp-d)j} d_\lambda^p\right)^{q/p}.$$

So that $f\in\mathcal{O}_p^s \Longrightarrow f\in\mathcal{O}_q^s$.

The spaces $\mathcal{O}_p^s$ are defined by conditions on the wavelet coefficients, therefore the first point to check is that this definition is independent of the wavelet basis chosen. In practice, one usually checks a stronger (but simpler) requirement which implies that the condition considered has some additional stability; indeed, the matrix of the operator which maps an orthonormal wavelet basis onto another orthonormal wavelet basis is invariant under the action of infinite matrices which belong to algebras $\mathcal{M}^\gamma$ that will be defined below; therefore, one can check that Condition (3.10) is also invariant under this action, which is the purpose of Proposition 11 below; these algebras are defined as follows:

Definition 19. An infinite matrix $A(\lambda,\lambda')$ indexed by the dyadic cubes belongs to $\mathcal{M}^\gamma$ if

$$|A(\lambda,\lambda')| \le \frac{C\, 2^{-(\frac{d}{2}+\gamma)(j-j')}}{(1+(j-j')^2)(1+2^{\inf(j,j')}dist(\lambda,\lambda'))^{d+\gamma}}.$$

Matrices of operators which map a smooth wavelet basis onto another one belong to these algebras, and more generally matrices (on wavelet bases) of pseudodifferential operators of order 0, such as the Hilbert transform in dimension 1, or the Riesz transforms in higher dimensions, belong to these algebras, see [**87**]. We denote by $\mathcal{O}p(\mathcal{M}^\gamma)$ the space of operators whose matrix on a wavelet basis belongs to $\mathcal{M}^\gamma$.

Proposition 11. If $\gamma\ge|s|$, the operators which belong to $\mathcal{O}p(\mathcal{M}^\gamma)$ are continuous on $\mathcal{O}_p^s(\mathbb{R}^d)$.

Remark: Since $\mathcal{O}p(\mathcal{M}^\gamma)$ is an algebra, this result clearly implies that the definition of oscillation spaces is independent of the (smooth enough) wavelet basis chosen.

Proof of Proposition 11: Recall that we write $\lambda=\lambda(i,j,k)$, $\lambda'=\lambda(i',j',k')$,... see Section 2.3. Let $e_{\lambda'}=\sum_{\lambda''}A(\lambda',\lambda'')c_{\lambda''}$ and

$$\omega_{\lambda,\lambda'} = \frac{2^{-(\frac{d}{2}+\gamma)|j-j'|}}{(1+(j-j')^2)(1+2^{\inf(j,j')}dist(\lambda,\lambda'))^{d+\gamma}}.$$

The $e_{\lambda'}$ satisfy $|e_{\lambda'}|\le C\sum_{\lambda''}C\omega_{\lambda',\lambda''}|c_{\lambda''}|$; assume that the c_λ satisfy (3.10); we want to bound

$$2^{(sp-d)j}\sum_{|\lambda|=2^{-j}}\sup_{\lambda'\subset\lambda}|e_{\lambda'}|^p \le C2^{(sp-d)j}\sum_{|\lambda|=2^{-j}}\sup_{\lambda'\subset\lambda}\left(\sum_{\lambda''}\omega_{\lambda',\lambda''}|c_{\lambda''}|\right)^p.$$

We split the sum over λ'' into two terms, depending on whether $\lambda'' \subset 3\lambda$ or $\lambda'' \not\subset 3\lambda$.

First case: $\lambda'' \subset 3\lambda$. Then,

$$\sum_{\lambda''\subset 3\lambda} \omega_{\lambda',\lambda''}|c_{\lambda''}| \leq \left(\sup_{\lambda''\subset 3\lambda} |c_{\lambda''}|\right) \sum_{\lambda''} \omega_{\lambda',\lambda''} \leq C \sup_{\lambda''\subset 3\lambda} (|c_{\lambda''}|)\,;$$

therefore

$$\begin{aligned}\sum_{|\lambda|=2^{-j}} \sup_{\lambda'\subset\lambda} \left(\sum_{\lambda''\subset 3\lambda} \omega_{\lambda',\lambda''}|c_{\lambda''}|\right)^p &\leq C \sum_{|\lambda|=2^{-j}} \sup_{\lambda''\subset 3\lambda} |c_{\lambda''}|^p \\ &\leq C\, 3^d \sum_{|\lambda|=2^{-j+1}} \sup_{\lambda''\subset\lambda} |c_{\lambda''}|^p.\end{aligned}$$

Second case: $\lambda'' \not\subset 3\lambda$. Let us first prove that, in this case,

$$\omega_{\lambda',\lambda''} \leq C\omega_{\lambda,\lambda''}. \tag{4.6}$$

In order to prove (4.6), let us first assume that $j'' \leq j$. In that case, $dist(\lambda', \lambda'') \geq \frac{1}{3} dist(\lambda, \lambda'')$ and $|j' - j''| \leq |j - j''|$; (4.6) follows directly from these two results.

If $j'' > j$, then, since $\lambda' \subset \lambda$

$$\omega_{\lambda',\lambda''} \leq |\frac{C\, 2^{-(\frac{d}{2}+\gamma)|j'-j''|}}{(1+(j'-j'')^2)dist(\lambda,\lambda'')^{d+\gamma}2^{\inf(j',j'')(d+\gamma)}}.$$

We now separate two cases:

If $j \leq j'' \leq j'$, then

$$\begin{aligned}\omega_{\lambda',\lambda''} &\leq \frac{C\, 2^{-(\frac{d}{2}+\gamma)|j-j''|}}{(1+(j'-j'')^2)dist(\lambda,\lambda'')^{d+\gamma}2^{j''(d+\gamma)}2^{-(\frac{d}{2}+\gamma)(j''-j)}} \\ &\leq \frac{C\, 2^{-(\frac{d}{2}+\gamma)|j-j''|}}{(1+(j'-j'')^2)2^{\frac{d}{2}(j''-j)}dist(\lambda,\lambda'')^{d+\gamma}2^{(d+\gamma)j}} \\ &\leq \frac{C\, 2^{-(\frac{d}{2}+\gamma)(j''-j)}}{(1+(j-j'')^2)dist(\lambda,\lambda'')^{d+\gamma}2^{(d+\gamma)j}} \\ &\leq C\omega_{\lambda,\lambda''}.\end{aligned}$$

We finally consider the case $j \leq j' \leq j''$. Then

$$\begin{aligned}\omega_{\lambda',\lambda''} &\leq \frac{C\, 2^{-(\frac{d}{2}+\gamma)|j-j''|}}{(1+(j'-j'')^2)dist(\lambda,\lambda'')^{d+\gamma}2^{j'(d+\gamma)}2^{-(\frac{d}{2}+\gamma)(j'-j)}} \\ &\leq \frac{C\, 2^{-(\frac{d}{2}+\gamma)|j-j''|}}{(1+(j'-j'')^2)2^{-\frac{d}{2}(j-j')}dist(\lambda,\lambda'')^{d+\gamma}2^{(d+\gamma)j}} \\ &\leq C\omega_{\lambda,\lambda''}.\end{aligned}$$

It follows from (4.6) that

$$\begin{aligned}\sum_{|\lambda|=2^{-j}} \sup_{\lambda'\subset\lambda} \left(\sum_{\lambda''\not\subset 3\lambda} \omega_{\lambda',\lambda''}|c_{\lambda''}|\right)^p &\leq C \sum_{|\lambda|=2^{-j}} \left(\sum_{\lambda''} \omega_{\lambda,\lambda''}|c_{\lambda''}|\right)^p \\ &\leq C \sup_{j'} \sum_{|\lambda'|=2^{-j'}} |c_{\lambda''}|^p\end{aligned}$$

(the last inequality holds by continuity of the operators in $\mathcal{O}p(M^\gamma)$ on the Besov spaces $B_p^{s,\infty}$, see [**87**]); hence Proposition 11 follows.

4.2. Oscillation spaces for $p < 0$. The multifractal formalism asserts that the spectrum of singularities of a function f can be deduced from its scaling function $\omega_f(p)$ by a Legendre transform. We saw that, if p is positive, the scaling function $\omega_f(p)$ determines which oscillation spaces f belongs to. In particular, this interpretation shows that $\omega_f(p)$ is independent of the particular wavelet basis chosen. Now we want to determine if this interpretation can be extended to negative values of p: Can one define $\mathcal{O}_p^s$ spaces for p negative? And, if so, is the definition independent of the wavelet basis chosen? We will need here a different requirement than the one used in Proposition 11.

DEFINITION 20. An infinite matrix $A(\lambda, \lambda')$ is quasidiagonal if A is invertible, and if A and A^{-1} belong to $\mathcal{M}^\gamma$ for any $\gamma > 0$.

The matrix of an operator which maps a C^∞ orthonormal wavelet basis onto another C^∞ orthonormal wavelet basis is quasidiagonal, see [**87**]. Therefore, in order to check that a condition defined on the wavelet coefficients is independent of the wavelet basis (in the Schwartz class) used, one can check the stronger property that it is invariant under the action of quasidiagonal matrices.

DEFINITION 21. Let $C = \{c_\lambda\}_{\lambda \in \Lambda}$ be a collection of coefficients indexed by the dyadic cubes. A property $\mathcal{P}$ is **robust** if the following condition holds: If $\mathcal{P}(C)$ holds then, for any quasidiagonal operator $\mathbf{M}$, $\mathcal{P}(\mathbf{M}C)$ holds.

Similarly, we will say that a function space, which is defined by conditions on the wavelet coefficients, is robust if this definition is independent of the (smooth enough) wavelet basis which is chosen.

Let us now motivate the formulation of the definition of oscillation spaces that we will adopt for $p < 0$. Consider Condition (3.10) for $p > 0$. As we already remarked, applying this condition for $j = 0$, we obtain that, for any $j' \geq 0$, $|c_{\lambda'}| \leq C$ so that $f \in B_\infty^{0,\infty}$. Conversely the condition $f \in B_\infty^{0,\infty}$ is necessary to make sure that the suprema in (3.10) are finite, therefore we include this global regularity condition in the definition when $p < 0$.

Since we are interested in local properties of functions, we will first define these spaces for periodic functions (or distributions). We will use periodized wavelets defined as follows. The functions

$$\sum_{l \in \mathbb{Z}^d} 2^{dj/2} \psi^{(i)}[2^j(x-l) - k], \quad j \geq 0, \quad k \in \{0, \dots, 2^j - 1\}^d,$$

together with the function $\Omega(x) = 1$ form an orthonormal basis of the space of square integrable periodic functions. We keep the same notations as before for periodic wavelets and the corresponding periodic wavelet coefficients, which will bring no confusion.

DEFINITION 22. Let $p < 0$, and $s \in \mathbb{R}$. A periodic distribution f belongs to $\mathcal{O}_p^s$ if the two following conditions hold:

- f belongs to $B_\infty^{0,\infty}$,
- for any $\epsilon > 0$, there exists exists $C(\epsilon)$ and $J(\epsilon)$ such that

$$\forall j \geq J(\epsilon) \qquad \sum_k 2^{(sp-d)j} d_\lambda^p \leq C(\epsilon) 2^{\epsilon j}. \tag{4.7}$$

Definition 22 does not define a vector space (the d_λ^p are infinite for $f = 0$). Its remarkable property is that it is independent of the orthonormal wavelet basis (in the Schwartz class) which is chosen.

THEOREM 5. For any $s \in \mathbb{R}$ and for any $p < 0$, the space $\mathcal{O}_p^s$ is robust.

Let c_λ denote a collection of wavelet coefficients satisfying $|c_\lambda| \leq C$. Let (ψ_λ) be a given orthonormal wavelet basis in the Schwartz class. We denote by f the distribution $\sum c_\lambda \psi_\lambda$. If $A = (A(\lambda, \lambda'))$ is an infinite matrix indexed by the dyadic cubes, g will denote the distribution whose wavelet coefficients are

$$e_\lambda = \sum_{\lambda'} A(\lambda, \lambda') c_{\lambda'}$$

(whenever these sums converge), and T will denote the operator whose matrix in the basis ψ_λ is A; thus T maps f onto g.

In order to prove Theorem 5, some preliminary remarks will be useful. We start by recalling a classical lemma.

LEMMA 3. Let $\gamma \geq |\alpha|$ and $A \in \mathcal{M}^\gamma$. There exists a constant $\tilde{C}(d)$ (which depends only on the dimension d) such that

$$\forall j, k \quad |c_{j,k}^i| \leq C 2^{-\alpha j} \qquad \Longrightarrow \qquad \forall \lambda \quad |e_\lambda| \leq \tilde{C}(d) \parallel A \parallel_\gamma C 2^{-\alpha j}.$$

This lemma expresses the fact that operators whose matrix in a wavelet basis belongs to $\mathcal{M}^\gamma$ are continuous on C^α if $|\alpha| < \gamma$. It is a straightforward consequence of Schur's lemma (see [**87**]).

DEFINITION 23. Let $\epsilon > 0$ and λ' be a dyadic cube. The ϵ-neighbourhood of λ' (which we denote by $N^\epsilon(\lambda')$) is the set of dyadic cubes λ such that

$$\begin{cases} |j - j'| \leq \epsilon j' \\ \left| \dfrac{k}{2^j} - \dfrac{k'}{2^{j'}} \right| \leq 2^{2\epsilon j'} 2^{-j'}. \end{cases}$$

Let λ' be a fixed dyadic cube, and let j be such that $|j - j'| \leq \epsilon j'$. The number of dyadic cubes λ of size 2^{-j} which belong to the ϵ-neighbourhood of λ' is bounded by

$$(2^{j+2} 2^{2\epsilon j'} 2^{-j'})^d. \tag{4.8}$$

Note also that, if λ does not belong to the ϵ-neighbourhood of λ' and if $\gamma \geq 1/\epsilon^2$, then

$$\omega_{2\gamma}(\lambda, \lambda') = \frac{2^{-(2\gamma+d+1)|j-j'|}}{(1 + 2^{\inf(j,j')} dist(\lambda, \lambda'))^{2\gamma+d+1}} \leq \omega_\gamma(\lambda, \lambda') 2^{-j'/\epsilon}. \tag{4.9}$$

Furthermore, since $f \in B_0^{0,\infty}$, $|c_{\lambda'}| \leq C_1$; thus, if $\gamma \geq 2/\epsilon$, it follows from Lemma 3 that

$$\left| \sum_{\lambda' \notin N^\epsilon(\lambda)} A(\lambda, \lambda') c_{\lambda'} \right| \leq \tilde{C}(d) \parallel A \parallel_\gamma C_1 2^{-j/\epsilon}. \tag{4.10}$$

Now, we can prove Theorem 5. Suppose that the wavelet coefficients c_λ of f satisfy (4.7) and that A is quasidiagonal. First, we notice that, since

$$\sum_k 2^{(sp-d)j} d_\lambda^p \leq C(\epsilon) 2^{\epsilon j},$$

it follows that

$$\forall \lambda, \;\; 2^{(sp-d)j} d_\lambda^p \leq C,$$

so that, since $p<0$,

$$\forall \lambda, \;\; d_\lambda \geq C(\epsilon)^{1/p} 2^{-(s-d/p)j} 2^{\epsilon j/p}. \tag{4.11}$$

Let λ be a given dyadic cube. There exists $\lambda' \subset \lambda$ such that

$$|c_{\lambda'}| \geq \frac{1}{2} d_\lambda. \tag{4.12}$$

For each λ we pick one such λ' which we denote by $\lambda'(\lambda)$. We will first prove the following lemma.

Lemma 4. Suppose that f belongs to $\mathcal{O}_p^s$, and let ϵ such that

$$0 < \epsilon \leq \frac{1}{2(|s-d/p|+1)}. \tag{4.13}$$

There exists $C(\epsilon)>0$ and $J(\epsilon, A)$ such that for any $j \geq J(\epsilon)$ and for any λ such that $|\lambda| = 2^{-j}$,

$$\exists \lambda'' \in N^\epsilon(\lambda'(\lambda)) : \qquad |e_{\lambda''}| \geq \frac{d_\lambda}{4\tilde{C}(d) \parallel A^{-1} \parallel}.$$

We will denote this cube by $\lambda''(\lambda)$.

Proof of Lemma 4: If Lemma 4 were wrong, all cubes $\lambda'' \in N^\epsilon(\lambda'(\lambda))$ would satisfy

$$|e_{\lambda''}| \leq \frac{d_\lambda}{4\tilde{C}(d) \parallel A^{-1} \parallel}.$$

Let

$$\begin{aligned} e^1_{\lambda''} &= e_{\lambda''} && \text{if } \lambda'' \in N^\epsilon(\lambda'(\lambda)) \\ &= 0 && \text{otherwise,} \end{aligned}$$

$e^2_{\lambda''} = e_{\lambda''} - e^1_{\lambda''}$, $(c^1_\lambda) = A^{-1}(e^1_\lambda)$ and $(c^2_\lambda) = A^{-1}(e^2_\lambda)$. Since A^{-1} is almost diagonal, applying Lemma 3, we obtain

$$\forall \lambda'' \quad |c^1_{\lambda''}| \leq \frac{d_\lambda}{4}. \tag{4.14}$$

We have $c^2_{\lambda'(\lambda)} = \sum_{\lambda_1 \notin N^\epsilon(\lambda'(\lambda))} A^{-1}(\lambda'(\lambda), \lambda_1) e_{\lambda_1}$. Since $f \in B_0^{0,\infty}$ and A is almost diagonal, Lemma 3 implies that

$$|e_{\lambda_1}| \leq C(f)\tilde{C}(d) \parallel A \parallel$$

(where 2^{-j_1} is the width of λ_1); (4.10) implies that, if $\gamma \geq 2/\epsilon$,

$$|C^2_{\lambda'(\lambda)}| \leq \tilde{C}(d)^2 \parallel A^{-1} \parallel_\gamma \parallel A \parallel C(f) 2^{-j'/\epsilon}.$$

Since ϵ is such that $1/(2\epsilon) \geq |s-d/p|+1$, using (4.11) and $j' \geq j$, we obtain

$$2^{-(1/2\epsilon)j'} \leq 2^{-(s+1-d/p)j} \leq d_\lambda C^{-1/p};$$

so that

$$|c^2_{\lambda'(\lambda)}| \leq \left(\frac{\tilde{C}(d)^2 \parallel A^{-1} \parallel_\gamma \parallel A \parallel C(f)}{C^{1/p}} \right) d_\lambda 2^{-j'/2\epsilon}.$$

Thus, since $1/(2\epsilon) \geq 1$,

$$|c^2_{\lambda'(\lambda)}| \leq \left(2^{-j} \ \frac{\tilde{C}(d)^2 \parallel A^{-1} \parallel_\gamma \parallel A \parallel C(f)}{C^{1/p}} \right) d_\lambda. \tag{4.15}$$

Choosing j large enough makes the term between parentheses arbitrarily small; thus, since $c_{\lambda'(\lambda)} = c^1_{\lambda'(\lambda)} + c^2_{\lambda'(\lambda)}$, (4.14) together with (4.15) contradicts (4.12).

Let us come back to the proof of Theorem 5. Let ϵ satisfying (4.13) and $j \geq J(\epsilon, A)$, where $J(\epsilon, A)$ is as in Lemma 4; finally, let $l = [j(1-3\epsilon)]$. For each cube μ of size 2^{-l}, there exists $\lambda \subset \mu$ such that μ also contains the cube $\lambda''(\lambda)$ suplied by Lemma 4. We denote these two subcubes $\lambda = \lambda(\mu)$ and $\lambda''(\lambda) = \lambda_1(\mu)$. Denote by 2^{-j_1} the width of $\lambda''(\lambda)$. Finally, let

$$\tilde{R}(\mu) = \sup_{\lambda'' \subset \mu} |e_{\lambda''}|.$$

We have

$$\tilde{R}(\mu) \geq \left| e_{\lambda_1(\mu)} \right|,$$

so that, using Lemma 4,

$$\begin{aligned} \sum_\mu 2^{(sp-d)l} \tilde{R}(\mu)^p &\leq \sum_\mu 2^{(sp-d)l} \left| e_{\lambda_1(\mu)} \right|^p \\ &\leq C(A) \sum_\mu 2^{(sp-d)l} d^p_{\lambda(\mu)} \end{aligned}$$

(where the sums are taken on all dyadic cubes μ of width 2^{-j_1}). Since

$$j \in \{l, \ldots, l + [6\epsilon l]\},$$

this quantity is bounded by

$$C(A) 2^{(sp-d)l} \sum_{j=l}^{l+[6\epsilon l]} \sum_\lambda d^p_\lambda$$

(where $|\lambda| = 2^{-j}$)

$$= C(A) \sum_{j=l}^{l+[6\epsilon l]} \sum_\lambda d^p_\lambda 2^{(sp-d)j} 2^{(-sp+d)(j-l)}.$$

Thus

$$\sum_\mu 2^{(sp-d)l} \tilde{R}(\mu)^p \leq C(A) 2^{6|-sp+d|\epsilon l} \sum_{j=l}^{l+[6\epsilon l]} \sum_\lambda 2^{(sp-d)j} d^p_\lambda.$$

Since ϵ can be chosen arbitrarily small, Theorem 5 follows.

We have proved that the definition of the spaces $\mathcal{O}^s_p$ for $p < 0$ is independent of the wavelet basis chosen. These spaces are defined for periodic functions (or distributions). If f is defined on $\mathbb{R}^d$, we adopt the following definition.

DEFINITION 24. If $p<0$, a distribution f belongs to $\mathcal{O}^s_{p,\,loc}$ if $f\in B^{0,\infty}_\infty$ and if its wavelet coefficients satisfy

$$\forall K \quad \text{compact},\ \forall\epsilon>0,\ 2^{(sp-d-\epsilon)j}\sum_{\lambda\in\Lambda_j\cap K}|d_\lambda|^p\le C(K,\epsilon).$$

The proof that this definition is robust is the same as in the periodic case; it follows in particular that the definition of $\omega_f(p)$ is robust for $p<0$.

4.3. Concavity of the scaling function. We will now prove that the function ω_f, which is defined on $\mathbb{R}^*$, can be extended on $\mathbb{R}$ into a concave function, and we will also establish some additional properties. These results are expressed and proved more easily using an auxilliary function, the *oscillation exponent* $s_f(p')$, defined as follows.

DEFINITION 25. Let $p'\neq 0$. The oscillation exponent of f is defined by

$$s_f(p')=p'\omega_f\left(\frac{1}{p'}\right).$$

Coming back to the definition of $\omega_f(p)$, it is clear that

$$\text{if}\quad p'\in(0,+\infty[\qquad s_f(p')=\sup\{s:\ \ f\in\mathcal{O}^s_{1/p',loc}\}.$$

DEFINITION 26. The semi-local Hölder exponent of f is

$$H_f=\sup\{\alpha: f\in C^\alpha_{loc}\}. \tag{4.16}$$

THEOREM 6. Let $f\in L^\infty$; the oscillation exponent $s_f(p')$ is a concave function on $\mathbb{R}^+$; therefore, it is right and left differentiable at every point. Its right and left derivatives belong to L^∞ and satisfy

$$\forall p'\in[0,+\infty[,\quad (s_f')_r(p')\le d\quad\text{and}\quad (s_f')_l(p')\le d.$$

Furthermore, $s_f(p')$ is increasing on $\mathbb{R}^+$.

The scaling function $\omega_f(p)$ can be extended on $\mathbb{R}$ into a concave and increasing function; its derivative ω_f' is positive and decreasing, and satisfies

$$\lim_{p\to+\infty}\omega_f'(p)=H_f. \tag{4.17}$$

The fact that $s_f(p')$ is increasing on $\mathbb{R}^+$ is a direct consequence of the third statement of Proposition 10.

In order to prove that ω_f is concave on $\mathbb{R}$, we will first prove that ω_f is concave on $\mathbb{R}^+$ and on $\mathbb{R}^-$. Two cases can then occur: Either $\omega_f(p)=-\infty$ on $\mathbb{R}^-$ or $\omega_f(p)$ extends continuously at 0, and $(\omega_f)'_d(0)\ge(\omega_f)'_d(0)$; in both cases, one can conclude that ω_f is concave on $\mathbb{R}$. The concavity of ω_f on $\mathbb{R}^+$ will follow from the concavity of s_f; in order to prove it, we will need the following inequality which follows directly from Hölder's inequality. If $0<p<r<q$, and if (d_k) is a sequence of real numbers,

$$\left(\sum|d_k|^r\right)^{1/r}\le\left(\sum|d_k|^p\right)^{\alpha/p}\left(\sum|d_k|^q\right)^{(1-\alpha)/q} \tag{4.18}$$

where α is defined by

$$\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}. \tag{4.19}$$

LEMMA 5. The functions s_f and ω_f are concave on $\mathbb{R}^+$.

Proof: Let $p'=1/p$, $q'=1/q$ and $r'=1/r$. By definition of $s_f(p')$ and $s_f(q')$, $\forall \epsilon>0$, $\exists C>0$ such that, for j large enough,

$$\sum_\lambda d_\lambda^p \leq 2^{dj}2^{-ps_f(p')j}2^{\epsilon j} \quad \text{and} \quad \sum_\lambda d_\lambda^q \leq 2^{dj}2^{-qs_f(q')j}2^{\epsilon j}.$$

Therefore

$$\sum_\lambda d_\lambda^r \leq \left(2^{-ps_f(p')j}\right)^{\alpha r/p}\left(2^{-qs_f(q')j}\right)^{(1-\alpha)r/q}2^{(d+\epsilon)j}.$$

By definition of $s_f(r')$, there exists a sequence $j_n\to\infty$ such that

$$\sum_{|\lambda|=2^{-j_n}} d_\lambda^r \geq 2^{-rs_f(r')j_n}2^{(d-\epsilon)j_n};$$

it follows that

$$rs_f(r')+\epsilon \geq \alpha r s_f(p')+(1-\alpha)rs_f(q')-\epsilon.$$

Since this inequality holds for any $\epsilon>0$, and since (4.19) can be rewritten

$$r'=\alpha p'+(1-\alpha)q',$$

we actually proved that

$$\forall \alpha\in]0,1[\quad \text{if} \quad r'=\alpha p'+(1-\alpha)q', \quad \text{then} \quad s_f(r')\geq \alpha s_f(p')+(1-\alpha)s_f(q');$$

which means that $s_f(p')$ is concave. But a function is concave if and only if its second derivative (in the sense of distributions) is negative. Since $s_f(p')=p'\omega_f(\frac{1}{p'})$, then $s_f''(p)=\frac{1}{(p')^2}\omega_f''(\frac{1}{p'})$, therefore ω_f'' is a negative measure, and ω_f is concave on $\mathbb{R}^+$.

Lemma 6. The functions s_f and ω_f are concave on $\mathbb{R}^-$.

Proof: Let p, q, $r>0$ be such that

$$\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}.$$

applying (4.18) to the sequences $\dfrac{1}{d_k}$ with exponents $-p$, $-q$ et $-r$, we get

$$\sum |d_k|^r \geq \left(\sum |d_k|^p\right)^{r\alpha/p}\left(\sum |d_k|^q\right)^{r(1-\alpha)/q}$$

and therefore

$$-rs_f\left(\frac{1}{r}\right)\geq -ps_f\left(\frac{1}{p}\right)\frac{r\alpha}{p}-qs_f\left(\frac{1}{q}\right)\frac{r(1-\alpha)}{q}$$

so that, since $-r>0$,

$$s_f\left(\frac{1}{r}\right)\geq \alpha s_f\left(\frac{1}{p}\right)+(1-\alpha)s_f\left(\frac{1}{q}\right)$$

and s_f is concave on $\mathbb{R}^-$. It follows as before that ω_f is also concave on $\mathbb{R}^-$.

We will now prove the concavity of ω_f on $\mathbb{R}$.

First case: $\forall A>0$, $\exists\Omega$ bounded and $\lambda_n\in\Omega$ such that

$$d_{\lambda_n}\leq 2^{-Aj_n}.$$

Then, if $p < 0$,

$$\sum_{\lambda \in \Lambda_n \cap \Omega} |d_\lambda|^p \geq 2^{-Apj_n}$$

so that

$$-\omega_f^\Omega(p) \geq -Ap$$

i.e. $\omega_f^\Omega(p) \leq Ap$. Since $\omega_f(p) = \inf_\Omega \omega_f^\Omega(p)$, it follows that

$$\forall A > 0, \quad \omega_f(p) \leq Ap;$$

so that $\omega_f(p) = -\infty \; \forall p < 0$.

Second case: $\exists A \; \forall \lambda$, $d_\lambda \geq 2^{-Aj}$. We will first prove that ω_f is continuous at 0. Since the function f is uniformly Hölder, it follows that $d_\lambda \leq 1$ for j large enough. Assume that $p > 0$; for any bounded Ω,

$$2^{-dj} \sum_{\lambda \in \Lambda_j \cap \Omega} |d_\lambda|^p \leq C;$$

so that $\omega_f(p) \geq 0$. On the other hand, it follows from the upper bound $\forall \lambda$, $d_\lambda \geq 2^{-Aj}$, that

$$2^{-dj} \sum_{\lambda \in \Lambda_j \cap \Omega} |d_\lambda|^p \geq 2^{-Apj};$$

so that $\omega_f(p) \leq Ap$ and therefore $\lim_{p \to 0^+} \omega_f(p) = 0$.

By the same argument, we get that, if $p < 0$, $\omega_f(p) \leq 0$ and $\omega_f(p) \geq Ap$. therefore ω_f can be extended at 0 into a continuous function satisfying

$$\omega_f(0) = 0.$$

Let us now prove that $(\omega_f)'_d(0) \geq (\omega_f)'_g(0)$. Let $p > 0$ and $q < 0$. Let $r = \dfrac{p-q}{-q}$ and $r = \dfrac{p-q}{p}$; r and r' are positive and conjugate. Let $\omega = \dfrac{-pq}{p-q}$. let Ω be a regular bounded domain. Since no d_λ vanishes, applying Hölder's inequality, we obtain

$$\begin{aligned} 2^{-dj} card(\Lambda_j \cap \Omega) &= 2^{-dj} \sum_{\lambda \in \Lambda_j \cap \Omega} (d_\lambda)^\omega \left(\frac{1}{d_\lambda}\right)^\omega \\ &\leq \left(2^{-dj} \sum (d_\lambda)^{\omega r}\right)^{1/r} \left(2^{-dj} \sum \left(\frac{1}{d_\lambda}\right)^{\omega r'}\right)^{1/r'} \\ &= \left(2^{-dj} \sum {d_\lambda}^p\right)^{1/r} \left(2^{-dj} \sum {d_\lambda}^q\right)^{1/r'}. \end{aligned}$$

Since $Card(\Lambda_j \cap \Omega) = 2^{dj} Vol(\Omega) + o(2^{dj})$, it follows that

$$\frac{\omega_f(p)}{r} + \frac{\omega_f(q)}{r'} \leq 0;$$

since $r = p/\omega$ and $r' = -q/\omega$, therefore

$$\frac{\omega_f(p)}{p} - \frac{\omega_f(q)}{q} \leq 0.$$

If $p \to 0$ and $q \to 0$, it follows that $(\omega_f)'_d(0) - (\omega_f)'_g(0) \le 0$.

Let us now prove the upper bound for s'_f. By definition of $s_f(p')$,

$$\forall s < s_f(p'), \quad f \in \mathcal{O}^s_{1/p'},$$

therefore (using the second embedding between oscillation spaces stated in Proposition 10)

$$\text{if} \quad q' < p', \quad f \in \mathcal{O}^{s-dp'+dq'}_{1/q'};$$

thus, by definition of $s_f(q')$,

$$s_f(q') \ge s - dp' + dq'.$$

Therefore

$$s_f(q') \ge s_f(p') - dp' + dq',$$

which can be rewritten

$$\frac{s_f(p') - s_f(q')}{p' - q'} \le d;$$

hence the upper bound on the right and left derivatives of s'_f.

Let us now prove that $\omega_f(p)$ is non-decreasing. Since

$$\omega'_f(p) = s_f\left(\frac{1}{p}\right) - \frac{1}{p} s'_f\left(\frac{1}{p}\right) \tag{4.20}$$

we only have to prove that

$$\forall p' > 0 \quad s_f(p') - p' s'_f(p') \ge 0.$$

At $p' = 0$, the inequality holds because f is uniformly Hölder, so that $s_f(0) \ge 0$, and, on the other hand, s'_f is bounded in a neighbourhood of 0; furthermore, the derivative of $s_f(p') - p' s'_f(p')$ is $-p' s''(p')$ which is nonnegative. Since ω_f is concave on $\mathbb{R}$, if it is increasing on $\mathbb{R}^+$, it must also be increasing on $\mathbb{R}^-$. Therefore, it is increasing on $\mathbb{R}$.

We checked that ω'_f is nonnegative. Since ω''_f is nonnegative, it follows that ω'_f is nondecreasing; therefore, it has a limit when $p \to +\infty$. Since s'_f is bounded in a neighbourhood of 0, (4.20) implies that

$$\lim_{p \to +\infty} \omega'_f(p) = \lim_{p \to 0} s_f(p).$$

Let $s_f(0) = \lim\limits_{p \to 0} s_f(p)$; if $s < H_f$, then, for any bounded Ω, $f \in C^s(\Omega)$ so that, if Ω' is such that $\overline{\Omega'} \subset \Omega$,

$$\sup_{\lambda' \in \Lambda_{j'} \cap \Omega'} |c_{\lambda'}| \le C 2^{-sj'}.$$

Therefore

$$2^{(sp-d)j} \sum_{\lambda \in \Lambda_j \cap \Omega'} {d_\lambda}^p < \infty,$$

so that $s_f(\frac{1}{p}) \ge s$, and $s_f(0) \ge H_f$.

Conversely, if $s > H_f$, $\exists \Omega$ bounded and $j_n \to +\infty$ such that

$$\sup_{\lambda \in \Lambda_{j_n} \cap \Omega} |c_\lambda| \ge A_{j_n} 2^{-sj_n}$$

with $A_{j_n} \to \infty$; therefore,

$$2^{(s'p-d)j_n} \sum_{\lambda \in \Lambda_{j_n} \cap \Omega'} d_\lambda{}^p \leq 2^{(s'p-d-sp)j_n} A_{j_n}^p$$

which tends to $+\infty$ if $s'p - d - sp = 0$, hence if $s' = \frac{d}{p} + s$. Therefore $s_f(\frac{1}{p}) \leq s + \frac{d}{p}$ $\forall s > H_f$. Letting $p \to +\infty$, it follows that $s_f(0) \leq H_f$; hence (4.20) holds.

4.4. Notes. The problem of finding a correct definition for the scaling function when p is negative has attracted a lot of attention among physicists, especially in the context of fully developed turbulence, see for instance [**7, 25, 90, 97**] where possible extensions of the scaling function for $p < 0$ are proposed. Up to now, mathematically, the scaling function was not defined in terms of oscillation spaces $\mathcal{O}_p^s$, but in terms of Besov spaces $B_p^{s,\infty}$. Recall that f belongs to $B_p^{s,\infty}$ if its wavelet coefficients satisfy

$$\sup_j 2^{(sp-d)j} \sum_{|\lambda|=2^{-j}} |c_\lambda|^p \leq C; \tag{4.21}$$

using this definition for $p < 0$ is clearly absurd; indeed the quantity (4.21) is totally unstable for $p < 0$ because one wavelet coefficient can take an arbitrarily small value 'accidentally'. This explains why (to our knowledge) mathematical extensions of Besov spaces to negative values of p have never been proposed. One way to eliminate this source of instability is to replace in (4.21) the single value $|c_\lambda|$ by a supremum of the $|c_{\lambda'}|$ where λ' is close to λ. This is consistent with the purpose of deriving spectra of singularities; indeed, a very small wavelet coefficient is not the signature of a large Hölder exponent if it has a large coefficient in its immediate vicinity. On the contrary, a small value of the supremum means that, indeed, a whole set of wavelet coefficients close to each other takes small values, which is the signature of a smooth zone. Therefore, the idea of taking suprema is a sensible way to 'renormalize' the divergence of (4.21) by disregarding the wavelet coefficients which are small 'by accident' (which means here that there is a large wavelet coefficient in the neighbourhood). This is the starting point of the use of the *Wavelet Maxima Method* in this context, see [**7, 80, 97**]: The pointwise value of the continuous wavelet transform is replaced by a supremum on all lines of maxima ending at the point considered. The wavelet maxima method may be mathematically unstable, see [**50**]; however, we note that, in the discrete setting supplied by orthonormal wavelet bases, it would amount to replacing in (4.21) the coefficient c_λ by $d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$, hence to replacing Besov spaces by oscillation spaces in the definition of the scaling function. Therefore, it is not surprising that, though a scaling function based on Besov spaces has no natural extension for $p < 0$, a scaling function based on oscillation spaces has a natural and robust extension.

The idea of proving that a criterion does not depend on a particular wavelet basis by proving the invariance of this criterion under the application of an element of $\mathcal{M}^\gamma$ was introduced by Y. Meyer in [**87**]. The notion of robustness was introduced in [**56, 57**] where it is systematically used as a tool to determine the largest possible information that can be derived from wavelet coefficients and is independent of the wavelet basis.

5. Prevalent results in multifractal analysis

Many results of multifractality have been obtained for specific functions or random processes. In order to obtain 'generic' results of multifractality, we will first discuss what is meant by generic.

5.1. Genericity in infinite dimensional spaces. In $\mathbb{R}^d$, two notions of genericity are widely used: The notion of Lebesgue *almost everywhere*, and Baire's notion of *quasisure*. These two notions share some basic properties which are a natural requirement for any notion of genericity used in infinite dimensional spaces:

(1) Invariance with respect to dilations and translations,
(2) Stability with respect to countable intersection,
(3) Stability with respect to inclusion (if A is a generic set, $A \subset B \Longrightarrow B$ is a generic set),
(4) A generic set is dense.

Besides Lebesgue's and Baire's notions of genericity, there exists many other examples in $\mathbb{R}^d$; let us mention the one-parameter family defined as follows:

Let $\delta \in [0, d)$. A set A is *Hausdorff-δ generic* if the complement of A has Hausdorff dimension less than δ.

Let ω be a notion of genericity defined on a vector space E (which, to avoid trivialities, is supposed to be of dimension at least 1); we assume that the four requirements listed above are fulfilled, and we denote by G_ω the collection of all subsets which are generic for ω.

The collection of genericities has the structure of *a net*: ω is said to be *stronger* than ω' if $G_\omega \subset G_{\omega'}$ (this means that a set which is generic for ω is also generic for ω'). Equipped with this partial ordering, any family of genericities $\{\omega\}_{\omega \in O}$ has a supremum: The genericity μ defined by

$$A \in G_\mu \quad \text{if } A \in G_\omega \ \ \forall \omega \in O.$$

The two strongest notions of genericity clearly are:

- The *trivial genericity*, where the only generic set is E itself.
- The *countable genericity*, where the generic sets are the complements of the countable sets.

The Hausdorff genericities can also be defined on infinite dimensional spaces. In infinite dimension, another slightly weaker notion is supplied by the *compact genericity*: The generic sets are the complements of the countable unions of compact sets. This notion is clearly weaker than the previous ones, but stronger than Baire genericity, and it is usually too strong to be useful in practice.

In finite dimension, Baire genericity cannot be compared with either Lebesgue genericity or the Hausdorff genericities for the partial order defined above; indeed, there exist residual sets in the sense of Baire which have dimension 0; in space dimension 1, it is the case for instance of the set

$$A = \limsup_{p \in \mathbb{Z}, q > 0} \left(\frac{p}{q} - e^{-q}, \frac{p}{q} - e^{-q} \right).$$

The importance of proving a generic result lies in the fact that it implies a kind of uniqueness: If a property $\mathcal{P}$ is generic, another property $\mathcal{Q}$ incompatible with $\mathcal{P}$ cannot be generic too.

5.2. Prevalence. Lebesgue genericity plays a special role in finite dimension because the generic sets are obtained as the complements of the measure-zero sets, for a "canonical" measure which is both σ-finite and shift-invariant: The Lebesgue measure. Therefore, a natural question is to wonder if such a measure also exists in an infinite dimensional Banach spaces. Unfortunately, the answer is negative: There does not exist a σ-finite translation-invariant measure in any infinite dimensional normed space. Indeed a ball B of radius 1 contains an infinite number of disjoint balls of radius $1/4$; therefore, if the measure of B is finite, the measures of the smaller balls, which are all the same, must vanish. (The property of the unit ball in infinite dimensional that we used is well known; indeed, it is the first step of the standard proof that it is not compact: One construct an infinite sequence of points x_n in this ball such that $\forall n \neq m$, $\| x_n - x_m \| \geq 1/2$.)

However, this remark does not kill any hope for an infinite-dimensional extension of the notion of translation-invariant "Lebesgue measure zero"; indeed, let us consider the following characterization of the Lebesgue measure.

Lemma 7. In $\mathbb{R}^d$, a Borel set S has Lebesgue measure zero if and only if there exists a compactly supported probability measure μ such that

$$\forall x \in \mathbb{R}^d \quad \mu(x+S) = 0. \tag{5.1}$$

The proof of this lemma is straightforward: First, if E has Lebesgue measure 0, one can use for μ the Lebesgue measure on the unit cube, which clearly satisfies (5.1). Conversely, suppose that S is a Borel set, and that (5.1) holds. Then

$$0 = \int \mu(S-x)dx = \int meas(S-y)d\mu(y) = \mu(\mathbb{R}^d)meas(S)$$

(recall that $meas$ denotes the Lebesgue measure), so that S has Lebesgue measure 0.

The characterization of the sets of vanishing Lebesgue measure supplied by Lemma 7 does not refer explicitely to the Lebesgue measure; therefore it can be turned into a definition in infinite dimension spaces; the sets thus defined are called *Haar-null*, and the notion of genericity that it yields is called *prevalence*. From now on, we will only use this notion of genericity. The following definition was introduced by J. Christenssen [**26**].

Definition 27. Let E be a complete metric space. A Borel set $A \subset E$ is *Haar-null* if there exists a compactly supported probability measure μ on E such that

$$\forall x \in \mathbb{R}^d \quad \mu(x+A) = 0. \tag{5.2}$$

A subset A of E is *Haar-null* if it is included in a Haar-null Borel set. The complement of a Haar-null set is called a *prevalent* set.

If (5.2) holds, the measure μ is said to be transverse to A.

Remark: Recall that a probability measure is called *tight* if $\forall \epsilon > 0$ there exists a compact set K_ϵ such that

$$\mu(K_\epsilon) \geq 1 - \epsilon.$$

If E is a separable complete metric space, every probability measure on E is tight, see [**93**]. It follows that, if E is separable, we can drop the assumption that μ is compactly supported in Definition 27.

The basic properties of prevalence and several applications are detailed in [**26, 46**] and we refer to them for additional information. The following results show that prevalence indeed supplies a natural generalisation of the notions of Lebesgue-measure zero and almost everywhere in infinite dimensional complete metric separable spaces. Furthermore, this proposition implies that prevalence satisfies the assumptions for genericity that we listed above.

PROPOSITION 12. (1) If S is Haar-null, $\forall x \in E$, $x + S$ is Haar-null.
(2) If A is Haar-null, $\forall \lambda \neq 0$, λA is Haar-null.
(3) If E is finite-dimensional, S is Haar-null $\Longleftrightarrow$ $meas(S) = 0$.
(4) A is prevalent $\Longrightarrow$ A is dense.
(5) A countable intersection of prevalent sets is a prevalent set.

The first and second points assert that the notion of Haar-null (and therefore of prevalence) is translation and dilation invariant, which immediately follows from Definition 27. The third point is just a restatement of Lemma 7. The fourth point is also straightforward; indeed, we can clearly assume that μ has a support included in a ball of arbitrarily small radius (by considering a finite covering of the support of μ by balls B_i of radius ϵ and then correctly renormalizing one of the $\mu 1_{B_i}$ which has a non-zero mass); therefore, a Haar-null set cannot contain a ball. The stability by countable intersection is proved in [**26, 46**]. However, we will give a simple proof this result in the case where E is separable.

We will say that *almost every* element of E satisfies a property $\mathcal{P}$ if the set of elements satisfying $\mathcal{P}$ is a prevalent set; equivalently, we will say that $\mathcal{P}$ holds *almost everywhere* in E.

Two techniques for proving that a set is Haar-null are used: the *probe* technique, and the *stochastic process* technique. Let us describe them.

If one uses for transverse measure the Lebesgue measure on the unit ball of a finite dimensional subset V, Condition (5.2) becomes

$$\forall x \in E, \quad (x + V) \cap A \quad \text{is of Lebesgue measure zero.}$$

In this case V is called a *probe* for the complement of A.

As a simple illustration, let us prove the following result which shows that prevalence is a weaker notion than compact genericity.

PROPOSITION 13. If E is an infinite dimensional complete metric space, then any compact set K is Haar-null.

Proof of Proposition 13: Let $f : \mathbb{R} \times K \times K \longrightarrow E$ be defined by

$$f(\alpha, x, y) = \alpha(x - y).$$

Let us first check that $Im(f) \neq E$. Indeed,

$$Im(f) = \bigcup f\left([-N, N] \times K \times K\right);$$

thus $Im(f)$ is a countable union of compact sets, which are thus closed and of empty interior; therefore $Im(f)$ is a set of first category which, by Baire's theorem, differs from E. Thus, let $v \notin Im(f)$, and let V be the one-dimensional vector space generated by v. We will check that $\forall x \in E$, $(x+V)\cap K$ has at most one point. Indeed, if this intersection contained two distinct points y_1 and y_2, then

$$\exists \lambda_1, \lambda_2 : \ \ y_1 = \lambda_1 v + x \ \text{ and } \ y_2 = \lambda_2 v + x;$$

so that $v = \dfrac{y_1 - y_2}{\lambda_1 - \lambda_2} \in Imf$; hence a contradiction follows. Therefore

$$meas((x+V)\cap K) = 0.$$

Recall that a random variable X taking values in E is, by definition, a measurable function X defined on a probablility space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in E. Such a random variable defines a probability on E by the formula (if A is a Borel set of E:

$$\mathbb{P}(A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = \mathbb{P}(X \in A).$$

Using this probability measure in the definition of a Haar-null set, we see that, in order to prove that a set $A \in E$ is Haar-null, it is suffucient to check that

$$\forall f \in E, \mathbb{P}(A+f) = 0,$$

i.e. that

$$\forall f \in E, \mathbb{P}(X \in A+f) = 0.$$

Suppose ow that $\mathcal{P}$ is a property satisfied by some elements of E, and that E is a space of functions defined on $\mathbb{R}^d$ (in that case, X is called a stochastic process if $d=1$ and a random field if $d \geq 2$). In order to prove that the set of functions satisfying $\mathcal{P}$ is Haar-null, it is sufficient to exhibit a stochastic process X such that

$$\forall f \in E, \quad \text{a.s. } f + X \quad \text{does not satisfy} \quad \mathcal{P}.$$

We will refer to this way of proving that a set is Haar-null as the *stochastic process* technique.

As an example of application, let us check which regularity results can be obtained using for transverse measure the Wiener measure on the space of continuous functions. The stochastic process X associated with the Wiener measure by the argument described above is the Brownian motion B. It is easy to check that, for every continuous function f, then

$$\forall \epsilon > 0, \quad \text{a. s.} \quad f + B_x \quad \text{is nowhere} \quad C^{1/2+\epsilon}.$$

Applying the line of argument developed above, it follows that $\forall \alpha > 1/2$, for almost every $f \in \mathcal{C}([0,1])$, f is nowhere $C^{1/2+\epsilon}$. The use of the Wiener measure cannot yield a better result; however, if instead of the Brownian motion, one uses a fractional Brownian motion B^β with $0 < \beta < 1/2$, one can show that, for every continuous f, then

$$\forall \epsilon > 0, \quad \text{a. s.} \quad f + B^\beta \quad \text{is nowhere} \quad C^{\beta+\epsilon}.$$

This yields an alternative proof of the following result of B. Hunt concerning the prevalent Hölder regularity of functions in C^s, see [**45**]:

Proposition 14. If $s > 0$, almost every function in $C^s(x_0)$ satisfies

$$\forall x \in \mathbb{R}^d, \quad h_f(x) = s.$$

Let us now prove the stability of prevalent sets by countable intersection in the case where the space E is separable. Of course, it suffices to prove the stability of Haar-null sets by countable union. Let $(A_n)_{n\in\mathbb{N}}$ be a collection of Haar-null sets. To each of them is associated a random variable X_n defined on a probability space Ω_n, taking values in E, and such that

$$\forall f \in E \quad \text{a.s.} \quad f + X_n \notin A_n. \tag{5.3}$$

Let $\Omega = \prod_{n=1}^{\infty} \Omega_n$, endowed with the product measure (which amounts to consider independent copies of the random variables X_n). Since, almost surely, $X_n \in E$, there exist $F_n \subset \Omega_n$ and $N(n)$ such that

$$\forall \Omega \in F_n, \quad \mathbb{P}(dist(X_n, 0) \geq N(n)) \leq \frac{1}{n^2}.$$

Let $Y = \sum_{n\in\mathbb{N}} \frac{1}{(n+N(n))^3} X_n$. By the Borel-Cantelli lemma, this series converges almost surely in E. Let now $F \in E$ and $m \in \mathbb{N}$ be given. Then

$$f + Y = g_m + \frac{1}{(m+N(m))^3} X_m \quad \text{where} \quad g_m = f + \sum_{n\neq m} \frac{1}{(n+N(n))^3} X_n.$$

But g_m belongs to E; since (5.3) holds for $\frac{1}{(m+N(m))^3} X_m$ instead of X_m, applying (5.3) to $f+Y$, we obtain that, for almost every $\omega \in \prod_{n\neq m} \Omega_n$, $f + Y \notin A_m$. Thus, on the product space $\Omega = \Omega_M \times \prod_{n\neq m} \Omega_n$,

$$\forall f \in E, \ \forall m \notin \mathbb{N}, \quad \text{a. s.} \quad f + Y \notin A_m,$$

which means that the union of the A_m is a Haar-null set.

We will prove the following result, which describes the prevalent Hölder regularity of functions of $\mathcal{O}_p^s$ when $s > d/p$ (which coincides which the space $B_p^{s,\infty}$ in this case).

THEOREM 7. Let $s > d/p$; then

- the Hölder exponent of almost every function f of the space $\mathcal{O}_p^s$ takes values in $[s - d/p, s]$ and

$$\forall H \in [s-d/p, s], \qquad d_f(H) = Hp - sp + d; \tag{5.4}$$

  furthermore, for almost every x, $h_f(x) = s$;
- let x_0 be an arbitrary given point in $\mathbb{R}^d$; then, for almost every function in $\mathcal{O}_p^s$, $h_f(x_0) = s - d/p$.

The idea of the proof of Theorem 7 is to find appropriate probes for Oscillation spaces. We will explicitly construct bases of these probes by defining their wavelet coefficients.

5.3. The prevalent spectrum. In this section, we prove the first point of Theorem 7. Thanks to the Sobolev embeddings, if $s - d/p > 0$, functions in $\mathcal{O}_p^s$ cannot have Hölder exponents less than $s - d/p$.

Let $l \in \mathbb{N}$ and $M = 2^{dl}$. We will construct an M-dimensional probe in $\mathcal{O}_p^s$. The M generators of this space are defined by their wavelet coefficients. Let $j \geq 1$ and $k \in \{0, ..., 2^{dj} - 1\}$; K and $J \leq j$ are defined by

$$\frac{k}{2^j} = \frac{K}{2^J} \qquad \text{where} \ K \in \mathbb{Z}^d - 2\mathbb{Z}^d.$$

Each dyadic cube λ is now split into M subcubes with side 2^{-j-l}. For each index $i \in \{1, ..., M\}$ we choose a different subcube $i(\lambda)$. Let

$$a = \frac{1}{p}. \tag{5.5}$$

The probe is spanned by M functions g_i with the following wavelet coefficients

$$\begin{cases} \forall\lambda, \ \ d^i_{i(\lambda)} & = j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J} \\ \qquad\ \ d^i_{\lambda'} & = 0 \ \text{ if } \lambda' \text{ is not of the form } i(\lambda). \end{cases} \tag{5.6}$$

Since $\mathcal{O}^s_p$ coincides with the Besov spaces $B^{s,\infty}_p$ (because $s - \frac{d}{p} > 0$), it follows that the functions $\{g_i\}_{i=\{1,\ldots,M\}}$ belong to $\mathcal{O}^s_p$.

Let $\alpha \geq 1$; a point $x_0 \in \mathbb{R}^d$ is said to be α-approximable (by dyadics) if there exists a sequence $(J_n, K_n) \in \mathbb{N}^d \times \mathbb{Z}^d$ such that

$$\left| x_0 - \frac{K_n}{2^{J_n}} \right| \leq \frac{1}{2^{\alpha J_n}} \tag{5.7}$$

(clearly, we can assume that $K_n \in \mathbb{Z}^d - 2\mathbb{Z}^d$).

Lemma 8. If x_0 is α-approximable, then there exists a sequence $(d^i_{j,k})$ in the cone of influence of width 2^l above x_0 such that

$$\forall i \quad |d^i_{j,k}| \geq c(M) j^{-a} 2^{-Hj}, \tag{5.8}$$

with $H(= H(\alpha)) = s - \frac{d}{p} + \frac{d}{\alpha p}$.

Proof: Suppose that x_0 is α-approximable and let λ be the dyadic cube such that $\frac{k}{2^j} = \frac{K_n}{2^{J_n}}$ and $j = [\alpha J_n]$ (where J_n and K_n are given by (5.7)). The wavelet coefficient indexed by $i(\lambda)$ has size

$$d^i_{i(\lambda)} = j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J_n} \geq c(M) j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p\alpha}j}.$$

Let f be an arbitrary function in $\mathcal{O}^s_p$ with wavelet coefficients c_λ. Consider the affine subspace of dimension M composed of the functions

$$f_\beta = f + \sum_{i=1}^{M} \beta^i g^i,$$

where $\beta = (\beta^1, ..., \beta^M)$. Let $x_0 \in \mathbb{R}^d$ and $\gamma > 0$. If f_β is (C, γ) smooth at x_0 then, inside the cone of width 2^l above x_0,

$$\left| c_\lambda + \sum_{i=1}^{M} \beta^i d^i_\lambda \right| \leq C \ c(M) 2^{-\gamma j}. \tag{5.9}$$

Denote by E^α_j the set of points x_0 such that

$$\exists k : \quad \left| x_0 - \frac{k}{2^j} \right| \leq \frac{1}{2^{\alpha j}}.$$

(Note that x_0 is α-approximable if $x_0 \in E^\alpha = \limsup\limits_{j\to\infty} E^\alpha_j$).

The set E_j^α is the union of 2^{dj} cubes of width $2\cdot 2^{-\alpha j}$. Suppose that x and y are two points in the same cube and suppose furthermore that f_β is (C,γ) smooth at x and $f_{\tilde\beta}$ is (C,γ) smooth at y. Then $|x-y|\le 2\cdot 2^{-\alpha j}$ and $\forall i=1,...,M$

$$(5.10)\qquad \begin{cases} \left| c_{i(\lambda')} + \sum_{i=1}^{M} \beta^i d^i_{i(\lambda')} \right| \le Cc(M)2^{-\gamma j'} \\ \\ \left| c_{i(\lambda')} + \sum_{i=1}^{M} \tilde\beta^i d^i_{i(\lambda')} \right| \le Cc(M)2^{-\gamma j'} \end{cases}$$

for any dyadic cubes λ' at scale j' inside the cone of width 2 above x and y. But, since $|x-y|\le 2\cdot 2^{-\alpha j}$, we can find such a λ' satisfying $j'=[\alpha j]$. Using Lemma 8 and (5.8) it follows from (5.10) that

$$\|\beta-\tilde\beta\| \le 2Cc(M)2^{-(\gamma-H)j'}(j')^a (:= A(j'))$$

(where $\|\beta\| = \sup_{i=1,...,M} |\beta_i|$). Therefore the set of β satisfying

$$\exists x\in E_j^\alpha \quad \text{such that } \ f_\beta \text{ is } (C,\alpha) \text{ smooth at x}$$

is included in the union of 2^{dj} balls with radii $A(j')$. It follows that the Lebesgue measure of the M-uples β satisfying

$$\exists x\in E^\alpha \quad \text{such that } \ f_\beta \text{ is } (C,\alpha) \text{ smooth at x}$$

is bounded by

$$(5.11)\qquad \sum_{j=J}^{\infty} (Cc(M)[\alpha j]^a)^M 2^{-(\gamma-H)M[\alpha j]} 2^{dj}$$

(where J can be chosen arbitrary large); we can choose M large enough so that $d-(\gamma-H)M\alpha<0$; thus, when J tends to ∞, (5.11) goes to 0, so that it vanishes. Therefore, the set of M-uples $\beta=(\beta_1,...,\beta_M)$ such that $f+\sum\beta^i g^i$ is (C,γ) smooth at a point in E^α has measure zero. Since it is true for all $C>0$, the set of β such that

$$\exists x\in E^\alpha: \quad f+\sum \beta^i g^i \ \text{ is } C^\gamma$$

also has measure zero. Therefore

$$\forall\alpha>1,\ \ \forall\gamma>H(\alpha), \quad \text{a.s. in } \mathcal{O}_p^s, \ \ \forall x\in E^\alpha \ \ h_f(x)\le\gamma.$$

Taking $\gamma_n\to H(\alpha)$ (with $\gamma_n>H(\alpha)$) it follows by countable intersection, that

$$(5.12)\qquad \forall\alpha>0, \quad \text{a.s. } \forall x\in E^\alpha \ \ h_f(x)\le H(\alpha).$$

Therefore, if α_n is a dense sequence in $(1,\infty)$, using the same argument, one obtains that

$$\text{a.s. in } \ \mathcal{O}_p^s,\ \forall n,\ \forall x\in E^{\alpha_n}, \quad h_f(x)\le H(\alpha_n) \qquad (\mathcal{P}).$$

Let f be a function such that $(\mathcal{P})$ holds. Let α be fixed and $\alpha_{\varphi(n)}$ a subsequence of α_n such that $\alpha_{\varphi(n)}$ is non decreasing and tends to α. The subsets $E^{\alpha_{\varphi(n)}}$ are decreasing and their intersection $(:=\tilde E^\alpha)$ contains E^α. Therefore any $x\in\tilde E^\alpha$ satisfies $h_f(x)\le H(\alpha)$ and thus

$$(5.13)\qquad \forall x\in E^\alpha,\ \ h_f(x)\le H(\alpha).$$

But (see [**52**]) there exists a measure m_α supported on E^α such that any set E of dimension less than d/α satisfies $m_\alpha(E) = 0$, and $m_\alpha(E^\alpha) > 0$. Moreover, if $G_H = \{x:\ h_f(x) \leq H\}$, then

$$\forall f \in \mathcal{O}_p^s \quad \dim_H(G_H) \leq Hp - sp + d. \tag{5.14}$$

In particular, if F_α denotes the set of points where $h_f(x) < H(\alpha)$, F_α can be written as a countable union of sets with dimension less than d/α (it is the case because $H(\alpha) = s - \frac{d}{p} + \frac{d}{\alpha p}$, so that $\frac{d}{\alpha} = pH(\alpha) - sp + d$). It follows that $m_\alpha(F_\alpha) = 0$ and $m_\alpha(E^\alpha - F^\alpha) = 0$; but $E^\alpha - F^\alpha$ is a set of point where the Hölder exponent is exactly $H(\alpha)$. Thus

$$\forall H \in \left[s - \frac{d}{p}, s\right] \quad d_f(H) = Hp - sp + d,$$

and (5.4) holds on a a prevalent set.

Moreover $E^1 = [0,1]^d$, so that we can take m_1 equal to the Lebesgue measure, and (5.13) yields, if $\alpha = 1$,

$$\text{a.s.} \quad \forall x \in [0,1]^d,\ h_f(x) \leq s. \tag{5.15}$$

Furthermore, as before, almost every function f of $\mathcal{O}_p^s$ satisfies

$$meas(\{x:\ \ h_f(x) < s\}) = 0;$$

so that

$$\text{a.s. for almost every } x \text{ in } [0,1]^d \quad h_f(x) = s. \tag{5.16}$$

Results (5.15) and (5.16) are not specific to the unit cube, but they also hold for any cube. By countable intersection, it follows that, almost surely, $\forall x \in \mathbb{R}^d$, $h_f(x) \leq s$ and, almost surely, a.e., $h_f(x) = s$. Therefore the first point of Theorem 7 holds.

5.4. Regularity at a fixed point. In order to prove the second point of Theorem 7, we will use the following regularity condition, which is a slight variant of the usual pointwise Hölder condition:

f is $C^\alpha_{(log)^{-1}}(x_0)$ if there exists $C > 0$, $\delta > 0$ and a polynomial P of degree at most $[\alpha]$ such that

$$\text{if } |x - x_0| \leq \delta, \qquad |f(x) - P(x - x_0)| \leq C \frac{|x - x_0|^\alpha}{\log(1/|x - x_0|)}.$$

A straightforward adaptation of the proof of the the direct part of Theorem 1 shows that, if $f \in C^\alpha_{(log)^{-1}}(x_0)$, then there exists $C > 0$ such that

$$\forall j \geq 2, \qquad d_j(x_0) \leq C \frac{2^{-\alpha j}}{j}. \tag{5.17}$$

Let $\beta = s - d/p$; we define a function g by its wavelet coefficients $d_{j,k}$ as follows: For each j, only one $d_{j,k}$ does not vanish, and the corresponding k_j is such that $|2^j x_0 - k_j| \leq 2$, in which case

$$d_{j,k_j} = 2^{-\beta j}.$$

Clearly, $g \in \mathcal{O}_p^s$. The probe used is the one-dimensional subspace spanned by g. Let f be an arbitrary function in $\mathcal{O}_p^s$. Let us assume that there exist λ_1 and

λ_2 such that $f+\lambda_1 g$ and $f+\lambda_2 g$ both belong to $C^{\beta}_{(log)^{-1}}(x_0)$. Using (5.17), the wavelet coefficients of $f+\lambda_1 g$ and of $f+\lambda_2 g$ inside the cone $|2^j x_0 - k| \leq 2$ satisfy

$$|c_{j,k}+\lambda_1 d_{j,k}| \leq c\frac{2^{-\beta j}}{j}$$

$$|c_{j,k}+\lambda_2 d_{j,k}| \leq c'\frac{2^{-\beta j}}{j}.$$

In particular

$$|\lambda_1-\lambda_2||d_{j,k_j}| \leq (c+c')\frac{2^{-\beta j}}{j}.$$

It follows that $\forall n \geq 0$, $|\lambda_1-\lambda_2| \leq 1/j$ so that $\lambda_1=\lambda_2$. Thus, each line $f+\lambda g$ has at most one λ such that $f+\lambda g$ belongs to $C^{\beta}_{(log)^{-1}}(x_0)$. Hence, the second point of Theorem 7 holds.

5.5. Notes. The notion of Haar-null sets was introduced as early as 1972 by J. Christenssen in [**26**]; however this very promising notion did not receive the full attention it deserved untill it was reintroduced in 1992 by B. Hunt, T. Sauer and J. Yorke, see [**46**]. Several alternative notions of genericity have been introduced since then, see for instance [**68**] and the references mentioned there. The results exposed in this section are part of a joint work with Aurelia Fraysse, see [**36**]. Up to now it was commonly believed among mathematicians and physicists that multifractality was the signature of very peculiar properties of the function considered (such as self-similarity for instance); therefore Theorem 7 reverses the common point of view in this field. The reader should pay special attention to the position of "almost every" in the statements of Theorem 7; indeed Fubini's theorem does not apply in prevalence: If one considers a "generic" function in $\mathcal{O}^s_p$, its Hölder exponent is almost everywhere s, but when a point x_0 is fixed, the regularity at x_0 of almost every function f will be as bad as possible, i.e. $s-d/p$. Note that, in the previous case, this exponent was the one taken the most exceptionally (on a set of dimension zero). The first and second points of Theorem 7 coincide with the Baire-type results of [**55**]. However, here is an example where prevalent and quasi-sure results differ: Let $p<\infty$; in $\mathcal{O}^s_p$,

- almost every function satisfies almost everywhere $h_f(x)=s$,
- quasi every function satisfies quasi everywhere $h_f(x)=s-d/p$,

which follows immediately from juxtaposing the results of Theorem 7 with the corresponding results for Baire genericity in [**55**]. We saw in Section 3.2 that, for every function of $\mathcal{O}^s_p$,

$$d_f(H) \leq Hp - sp + d.$$

Therefore, Theorem 7 shows that a 'generic' function in $\mathcal{O}^s_p$ is as irregular as possible.

We obtained a generic result of multifractality in a fixed oscillation space, and when $s > d/p$. It is interesting to notice that the information supplied by the scaling function for $p>0$ can be rewritten as stating that f does indeed belong to an intersection of oscillation spaces. Indeed, it follows from Definition 15 that, if $p>0$, the scaling function can be given the following interpretation: for a given p, if $\omega_f(p)=\omega$, then

$$\forall \epsilon>0, \quad f \in \mathcal{O}^{\omega/p-\epsilon}_{p,loc} \quad \text{and} \quad f \notin \mathcal{O}^{\omega/p+\epsilon}_{p,loc}.$$

Conversely, if $\omega(p)$ is a scaling function, let us define the function space $\mathcal{O}^\omega$ by

$$\mathcal{O}^\omega = B^{0,\infty}_\infty \cap \left(\bigcap_{p,\ \epsilon>0} \mathcal{O}^{\omega(p)/p-\epsilon}_{p,loc} \right). \tag{5.18}$$

The space $\mathcal{O}^s_p$ is clearly a Banach space if $p \geq 1$; it is a complete metric space if $0 < p < 1$ for the metric defined by (3.11). Because of the concavity of ω, the intersection in (5.18) can be written as a countable intersection; therefore $\mathcal{O}^\omega$ is a complete metric vector space. It would be interesting to determine if there is also a generic result of multifractality in $\mathcal{O}^\omega$, and if the multifractal formalism holds in this space.

The reader should note that prevalent results cannot be obtained in the spaces $\mathcal{O}^s_p$ when p is negative because, in this case, it is no more a vector space; this is clear if one adopts for $\mathcal{O}^s_p$ the definition supplied by Definition 22, since we noticed that the function 0 does not satisfy this definition. One could however pick the following alternative definition when p is negative: $f \in \mathcal{O}^s_p$ if $f \in B^{0,\infty}_\infty$ and if

$$\forall \epsilon > 0\ \exists C(\epsilon), J(\epsilon): \quad \forall j \geq J(\epsilon), \quad \left(\sum_k 2^{(sp-d+\epsilon)j} d^p_\lambda \right)^{1/p} \leq C(\epsilon). \tag{5.19}$$

This definition is pertinent only if $s > 0$; indeed, if $f \in B^{0,\infty}_\infty$, $d_\lambda \leq C$, so that

$$\left(\sum_k 2^{(sp-d+\epsilon)j} d^p_\lambda \right)^{1/p} \leq C 2^{(s+\epsilon/p)j}$$

which is therefore bounded if $s \leq 0$. This condition coincides with Definition 22 when p is positive, and therefore also supplies a natural extension of oscillation spaces when p is negative. This time, the function 0 belongs to the space, and this condition clearly defines a cone; however, it is not a convex cone (and, a fortiori, not a vector space). Indeed, let us assume that $s > 0$ and let α be such that $0 < \alpha < s$. In the periodic one-dimensional setting, we define f_1 by its wavelet coefficients as

$$\begin{aligned} c_{j,k} &= 0 && \text{if } \ 0 \leq \tfrac{k}{2^j} \leq \tfrac{1}{2} \\ &= 2^{-\alpha j} && \text{if } \ \tfrac{1}{2} < \tfrac{k}{2^j} < 1; \end{aligned}$$

and f_2 by

$$\begin{aligned} c_{j,k} &= 2^{-\alpha j} && \text{if } \ 0 \leq \tfrac{k}{2^j} \leq \tfrac{1}{2} \\ &= 0 && \text{if } \ \tfrac{1}{2} < \tfrac{k}{2^j} < 1. \end{aligned}$$

The functions f_1 and f_2 have vanishing d_λ as soon as $j \geq 2$, and therefore belong to all spaces $\mathcal{O}^s_p$ for $p < 0$, but $f_1 + f_2$ does not belong to $\mathcal{O}^s_p$ since $\alpha < s$.

Even if no prevalent result could make sense in this setting, one could wonder if there can be a generic result in some other sense; we will check that it is not the case for any of the notions we considered; indeed all these notions are based, at least, on the use of a topology on the space considered; and we will show that (5.19) (put together with the condition $f \in B^{0,\infty}_\infty$) does not allow to define a finer topology than the topology of $B^{0,\infty}_\infty$. Indeed, any given neighbourhood of 0 for this finer topology would contain a set of the form $B_\epsilon \cap E_\eta$, where B_ϵ denotes the ball centered at 0 in $B_\infty^{0,\infty}$, i.e. is the set of functions satisfying $\forall\lambda, d_\lambda \le \epsilon$; and E_η denotes the set of functions satisfying

$$\forall j \ge 2 \qquad \left(\sum_k 2^{spj} d_\lambda^p \right)^{1/p} \le \eta$$

(here, we assume again that we are in the periodic, one-dimensional setting). Let V be an open neighbourhood of 0 for this finer topology. Let

$$\tilde{B}_r = \left\{ f \in B_r \quad \text{such that} \quad c_{j,k} = 0 \quad \text{if} \quad \frac{k}{2^j} > \frac{1}{2} \right\}.$$

Clearly, $\tilde{B}_{1/n} \subset B_{1/n} \cap E_0$ so that it is included in V for n large enough, and, since V is open, there exist ϵ' and η' such that, for any n large enough, $\tilde{B}_{1/n} + B_{\epsilon'} \cap E_{\eta'}$ will be included in V and will contain 0. Let

$$\bar{B}_r = \left\{ f \in B_r \quad \text{such that} \quad c_{j,k} = 0 \quad \text{if} \quad \frac{k}{2^j} \le \frac{1}{2} \right\}.$$

For the same reasons, for m large enough, $\bar{B}_{1/m}$ is included in $B_{\epsilon'} \cap E_{\eta'}$; therefore, for n and m large enough, $\tilde{B}_{1/n} + \bar{B}_{1/m} \subset V$. But, clearly, $B_{1/n} = \tilde{B}_{1/n} + \bar{B}_{1/n}$ (by splitting any function f as the sum of a function whose wavelet coefficients vanish for $\frac{k}{2^j} > \frac{1}{2}$ and a function whose wavelet coefficients vanish for $\frac{k}{2^j} \le \frac{1}{2}$). Thus, for n large enough $B_{1/n}$ is included in V. Therefore, the finer topology actually is not stronger than the $B_\infty^{0,\infty}$ topology.

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Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris XII, Faculté des Sciences et Technologie, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, FRANCE

E-mail address: `jaffard@univ-paris12.fr`