Definition: $f \in BV(\mathbb{R}^2)$ means that the distributional gradient ∇f of $f(x)$ is a bounded Radon measure.

Then

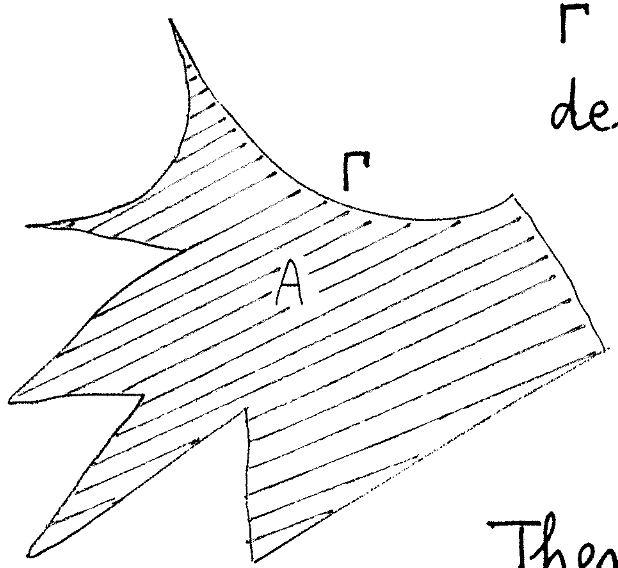
$\|f\|_{BV}$ = total mass of $|\nabla f|$ = $\int_{\mathbb{R}^2} |\nabla f|\, dx$ (when $\nabla f \in L^1(\mathbb{R}^2)$).

Example: $f(x) = \mathbb{1}_E(x) \in BV$ iff the de Giorgi reduced boundary $\partial^* E$ has a finite length (one-dimensional Hausdorff measure).

$$BV(\mathbb{R}^2) \subset L^{2,1}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$$

Lorentz space

Example of a BV-function



Γ is a rectifiable Jordan curve delimiting a bounded domain A

$f(x) = \mathbb{1}_A(x)$ is the indicator function of A

Then

$$\boxed{\|f\|_{BV} = \text{length}(\Gamma)}$$

Cartoon images

A cartoon image $u(x)$ is smooth inside some domains $D_1, \dots, D_N$ with jump discontinuities across the boundaries $\partial D_1, \dots, \partial D_N$.

These domains $D_1, \dots, D_N$ are the objects to be detected and $\partial D_1, \dots, \partial D_N$ are the edges or contours of these objects.

$$\boxed{\|u\|_{BV} \doteq \sum_1^N \iint_{D_j} |\nabla u| \, dx + \sum_1^N \int_{\partial D_j} |\text{jump}(u)| \, ds.}$$

Motivation

The Osher-Rudin model in image analysis

(Stanley Osher & Leonid Rudin)

$f(x)$ = grey level of a planar image

We want to decompose $f(x)$ into a sum $f(x) = u(x) + v(x)$

where

$u(x)$ = objects or shapes contained in $f(x)$

$v(x)$ = less organized structures (texture + noise).

How does one find u and v ?

We need a tuning parameter $\lambda > 0$.

Small objects with size $< \frac{1}{\lambda}$ will be disregarded and treated as noise.

We then fix a large parameter $\lambda \gg 1$ and we want to find the <u>optimal</u> <u>decomposition</u> $f(x) = u(x) + v(x)$ for which $\|u\|_{BV} + \lambda \|v\|_2^2$ is <u>minimal</u>.

Then $u(x)$ denotes the <u>geometrical</u> <u>content</u> of the given image.