Variants on the Osher-Rudin algorithm.

The Besov space $\dot{B}_1^{1,\infty}(\mathbb{R}^n)$.

Definition. A function f belongs to $\dot{B}_1^{1,\infty}$ iff there exists a constant C such that

$$\boxed{\int_{\mathbb{R}^n} |f(x+y)+f(x-y)-2f(x)|\,dx \le C|y| \quad (*)}$$

for every $y \in \mathbb{R}^n$.

Then $\dot{B}_1^{1,\infty} \subset L^{n/n-1,\infty}$ if $n \ge 2$,

and $BV \subset \dot{B}_1^{1,\infty}$.

Moreover $\dot{B}_1^{1,\infty}$ is characterized by

$$\boxed{\|\Delta_j(f)\|_1 \le C' 2^{-j}, \quad j \in \mathbb{Z},}$$

where $\Delta_j = S_{j+1} - S_j$ are the dyadic blocks in the Littlewood-Paley decomposition.

The norm of f in $\dot{B}_1^{1,\infty}$ is the lower bound of the constants C in the RHS of $(*)$.

Example: $\quad |x|^{-(n-1)} \in \dot{B}_1^{1,\infty}$.

A wavelet characterization of $\dot{B}_1^{1,\infty}(\mathbb{R}^2)$.

Let $2^j \psi(2^j x - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$, $\psi \in \{\psi_1, \psi_2, \psi_3\} \subset \mathcal{S}(\mathbb{R}^3)$ be an orthonormal basis of $L^2(\mathbb{R}^3)$.

We then have

Theorem. A function f belongs to $\dot{B}_1^{1,\infty}$ iff its wavelet coefficients

$$c(j,k) = 2^j \int_{\mathbb{R}^2} f(x)\, \psi(2^j x - k)\, dx$$

satisfy

$$\boxed{\sup_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |c(j,k)| < \infty.}$$

In other words $\dot{B}_1^{1,\infty} = \ell^\infty(\ell^1)$.

A simple function f belongs to S_N, $N \geq 1$, if $f(x) = c_1 \chi_{E_1} + \cdots + c_N \chi_{E_N}$ where χ_E is the characteristic (or indicator function) of the Borel set E.

Theorem (Gérard Bourdaud, Y.M.).

There exists a constant $C = C(n)$ such that for every $N \geq 1$ and every simple function $f \in S_N$, the following estimate holds

$$\boxed{\| f \|_{BV} \leq C(n)\, N \, \| f \|_{\dot{B}^{1,\infty}_1}}.$$

Observe that

$$\| f \|_{\dot{B}^{1,\infty}_1} \leq 2 \| f \|_{BV}$$

which implies $\| f \|_{BV} \cong \| f \|_{\dot{B}^{1,\infty}_1}$ on S_N.

[S_N is not a vector space and we do not know if $C(n)\, N$ is optimal in the RHS]

New algorithms:

The optimal decomposition $f = u + v$ minimizes $\|u\|_{\dot{B}^{1,\infty}_{1}} + \lambda \|v\|_2^2$.

Wavelet formulation:

$u(j,k)$ = wavelet coefficients of u ...

The optimal decomposition

$$f(j,k) = u(j,k) + v(j,k)$$

minimizes

$$\sup_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |u(j,k)| + \lambda \sum_{j,k} |v(j,k)|^2.$$

Simple and efficient algorithms provide us with this optimal decomposition.

Donoho's wavelet shrinkage

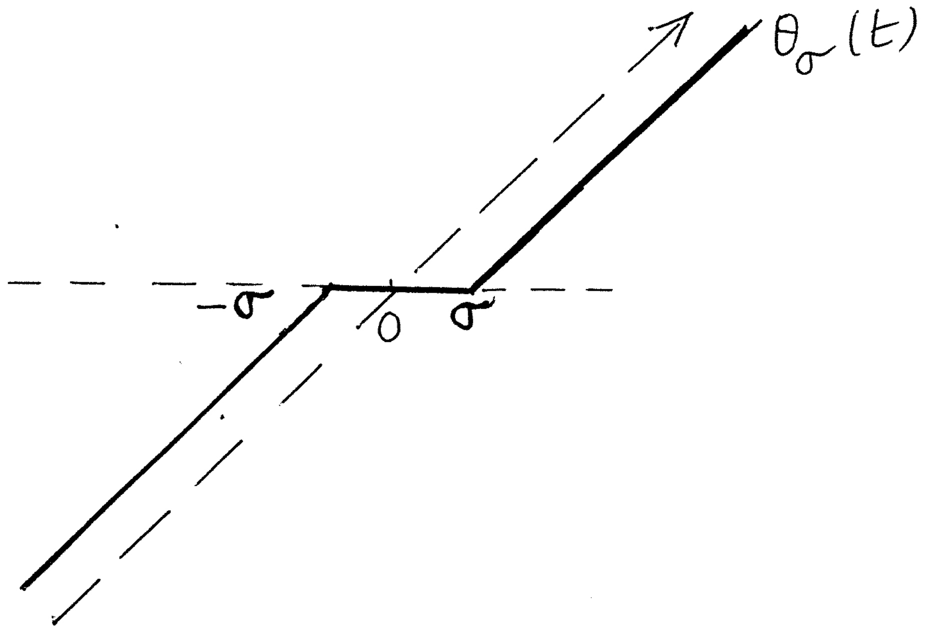
f = given image

$$f = u + v$$

u = objects contained in this image

v = texture + noise

$\sigma > 0$ is a tuning parameter



$$f(x) = \sum_{j,k} \alpha(j,k)\, \psi_{j,k}(x)$$

$$u = \sum_{j,k} \theta_\sigma(\alpha(j,k))\, \psi_{j,k}(x)$$