MULTIFRACTAL SPECTRUM ESTIMATION FOR PROCESSES

Rudolf H. Riedi, Paulo Goncalves and Richard G. Baraniuk

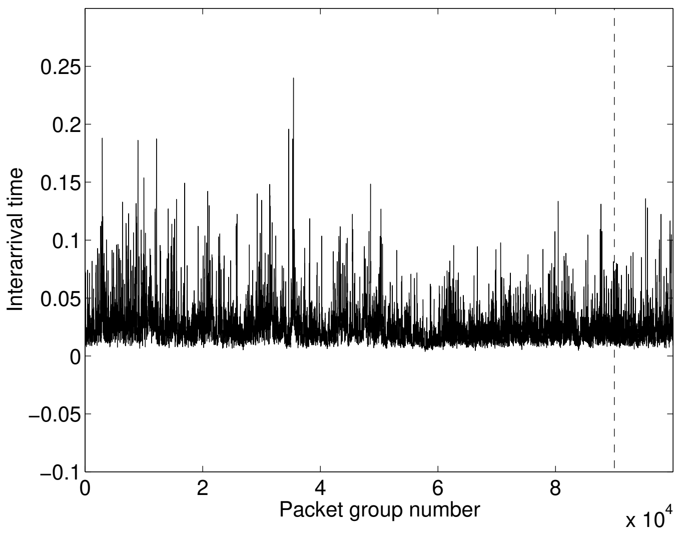
Rice University, Houston TX

INRIA Rocquencourt, France

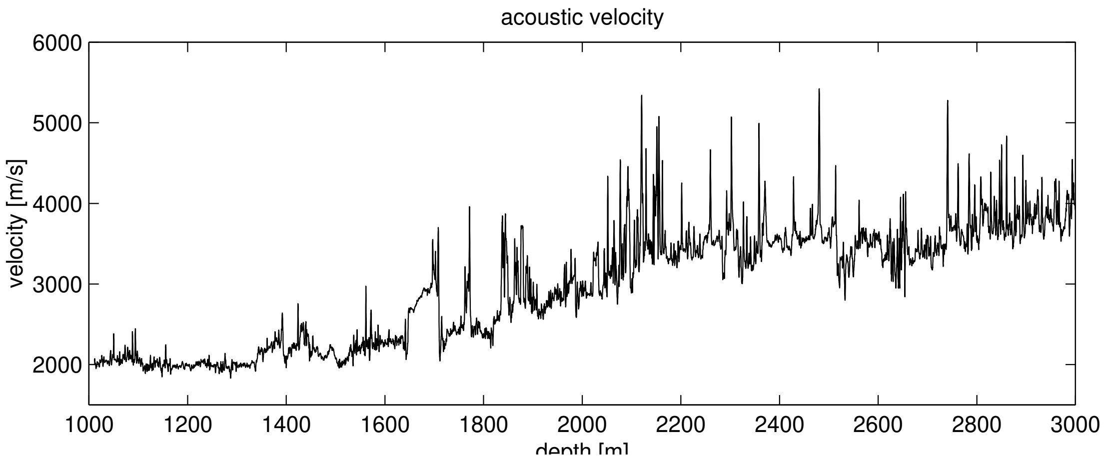
Supported by
National Science Foundation
Office of Naval Research
Rice Consortium for Computational Seismic Interpretation

A Gaussian World?

Network traffic



Well data



Traffic at large scales: Client behavior

Renewal Reward Process $X_t := W_k$ for $U_{k-1} < t < U_k$.

with workload W_k: i.i.d. with values 0 and 1 $\Rightarrow$ **ON/OFF**

inter-arrival times U_k: i.i.d. $P[U \geq u] \simeq u^{-\gamma}$ $(1 < \gamma < 2)$.

Self-similarity through aggregation:

$$Y_T(M) := \sum_{t=1}^{T} \sum_{i=1}^{M} X_t^{(i)}$$

Mandelbrot (1969), and Taqqu & Levy (1986):

$M >> T \to \infty \Rightarrow Y_T(M) \simeq$ **fBm** with $H = (3-\gamma)/2$

$T >> M \to \infty \Rightarrow Y_T(M) \simeq \gamma$-**stable** Lévy motion ($1/\gamma$-sssi)

Small time scale: Network flow control

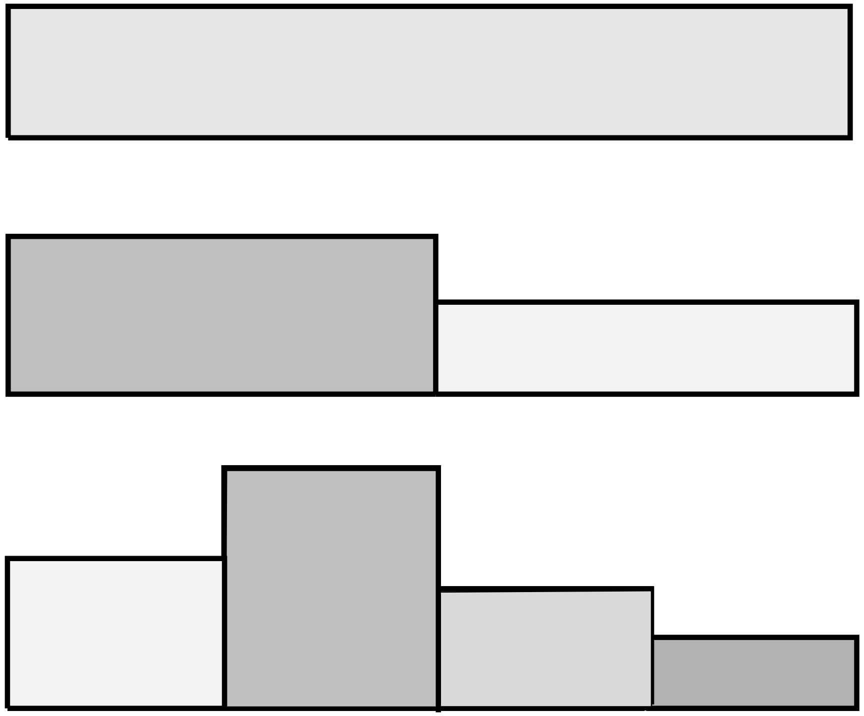
ON/OFF model $\rightarrow$ Large time scale

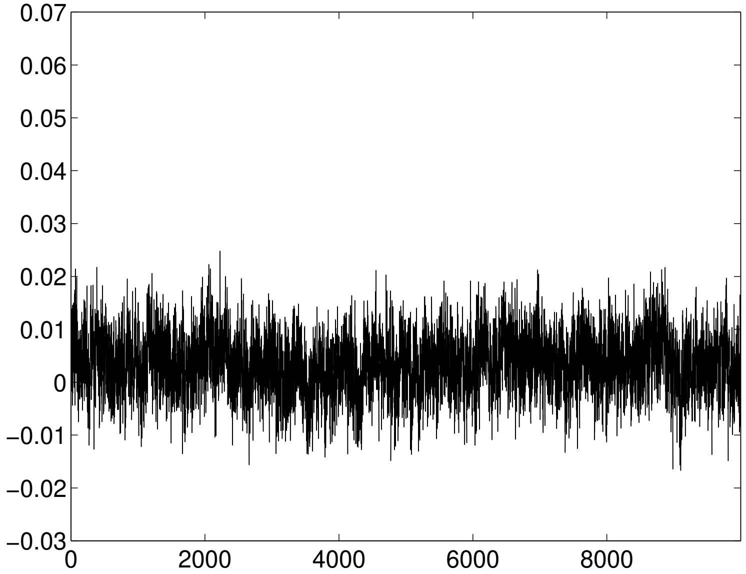
- reflects client behavior
- characterized by one parameter H
- is 'self-similar'

Small scales:

- Network control mechanism
- Still additive?

Additive Iteration



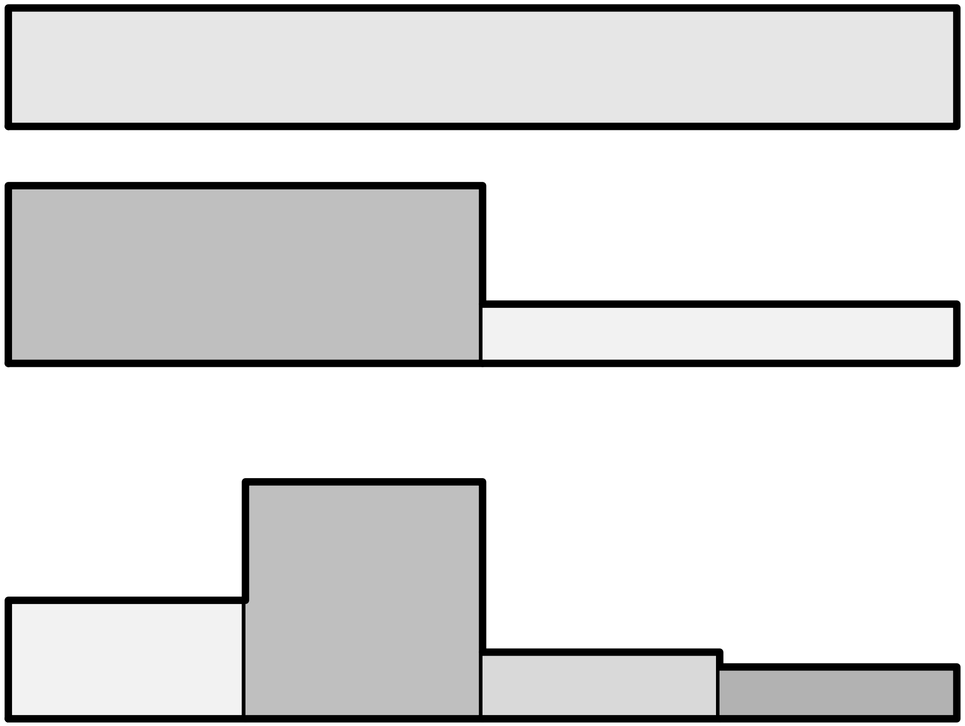


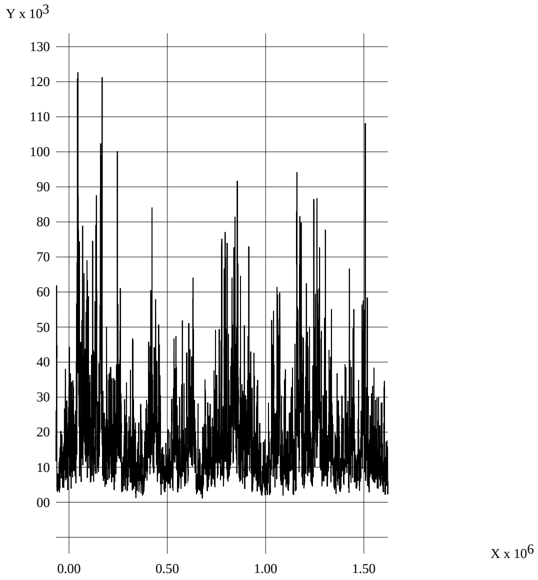
Scaling $\rightarrow$ Strong dependence

Additive $\rightarrow$ Gaussian

Tree $\rightarrow$ wavelets

Multiplicative Iteration





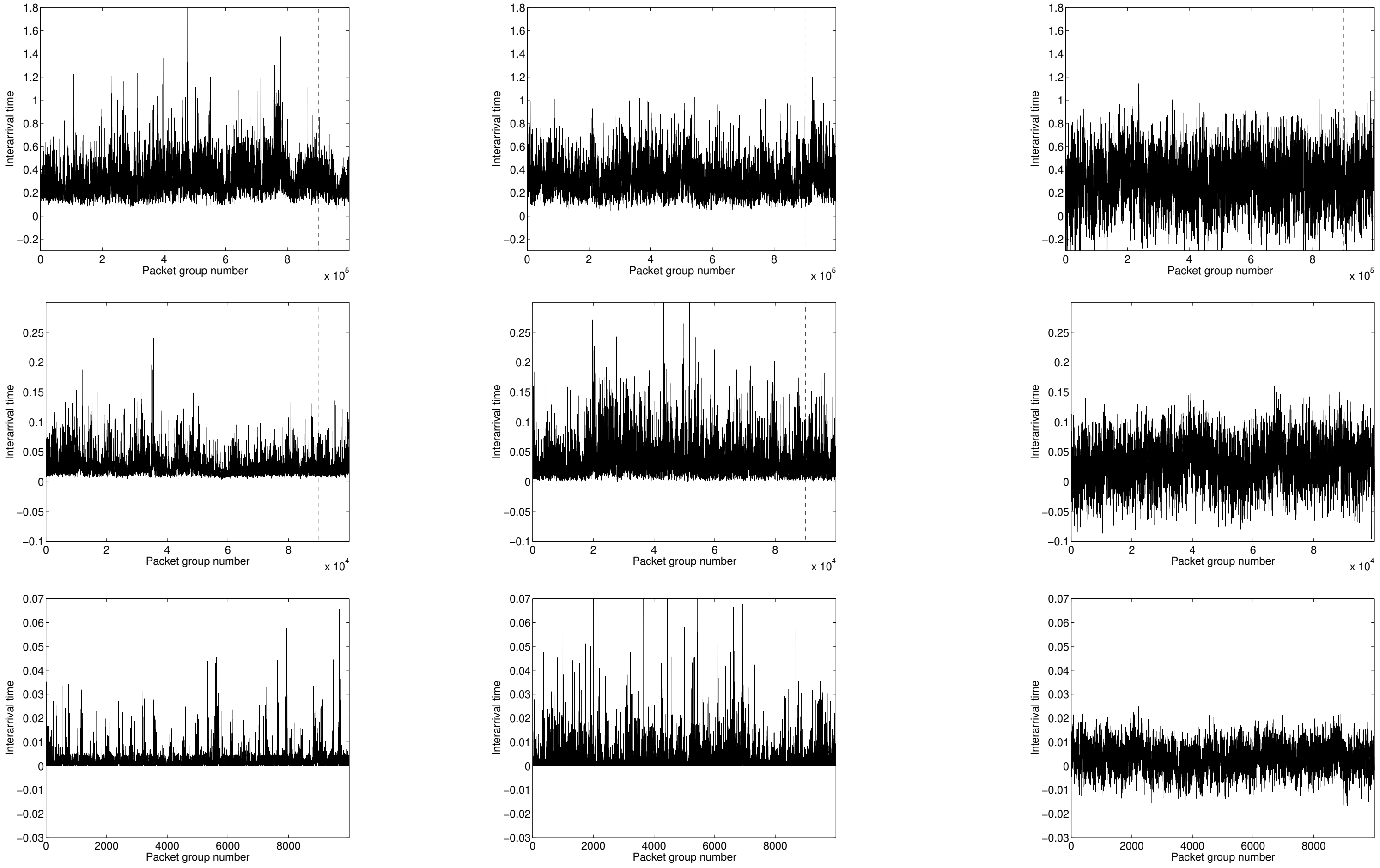
Scaling $\rightarrow$ Strong dependence

Multiplication $\rightarrow$ Log-Normal

Tree $\rightarrow$ fast algorithms

Reality check

(a) Network traffic (b) Multiplicative Noise (c) fractional Gaussian



Grip on Multiplicative Non-Gaussianity

Scaling of Moments

$$X^{(n)}[k_n] = M^n_{k_n} \cdot M^{n-1}_{k_{n-1}} \cdots M^1_{k_1} \cdot M^0_0 \quad \text{is stationary}$$

$$\mathbb{E}\left[(X^{(n)})^q\right] = (\mathbb{E}[M^q])^n$$

Exponential rate:

$$T(q) = -\frac{1}{n}\log_2 \mathbb{E}\left[(X^{(n)})^q\right] = -\log_2\left(\mathbb{E}[M^q]\right)$$

is strictly concave.

Additive Schemes $\rightarrow$ Gaussian

$$X^{(n)}[k_n] = W^n_{k_n} + W^{n-1}_{k_{n-1}} + \ldots + W^1_{k_1} + U^0_0 \quad \text{is stationary}$$

$$\text{CLT} \;\Rightarrow\; X^{(n)} \overset{d}{\sim} \mathcal{N}(0, \sigma_n^2) = \sigma_n \cdot \mathcal{N}(0,1)$$

$$\Rightarrow\; \mathbb{E}\left|X^{(n)}\right|^q = \sigma_n^q \cdot \mathbb{E}\left|\mathcal{N}(0,1)\right|^q$$

- Exponential rate linear: degenerate

$$T(q) = -\frac{1}{n}\log_2 \mathbb{E}\left[(X^{(n)})^q\right] = qH \qquad H = \lim_n -\frac{1}{n}\log_2 \sigma_n$$

- Example: fractional Gaussian noise $X^{(n)} \overset{d}{=} 2^{-nH}\mathcal{N}(0, \sigma_1)$

Multifractal formalism

- Strength of local singularity

$$\alpha_k^n := -\frac{1}{n}\log_2 X_k^{(n)}$$

- LDP:

$$\#\{\alpha_k^n \simeq \alpha\} \sim 2^{nf(\alpha)}$$

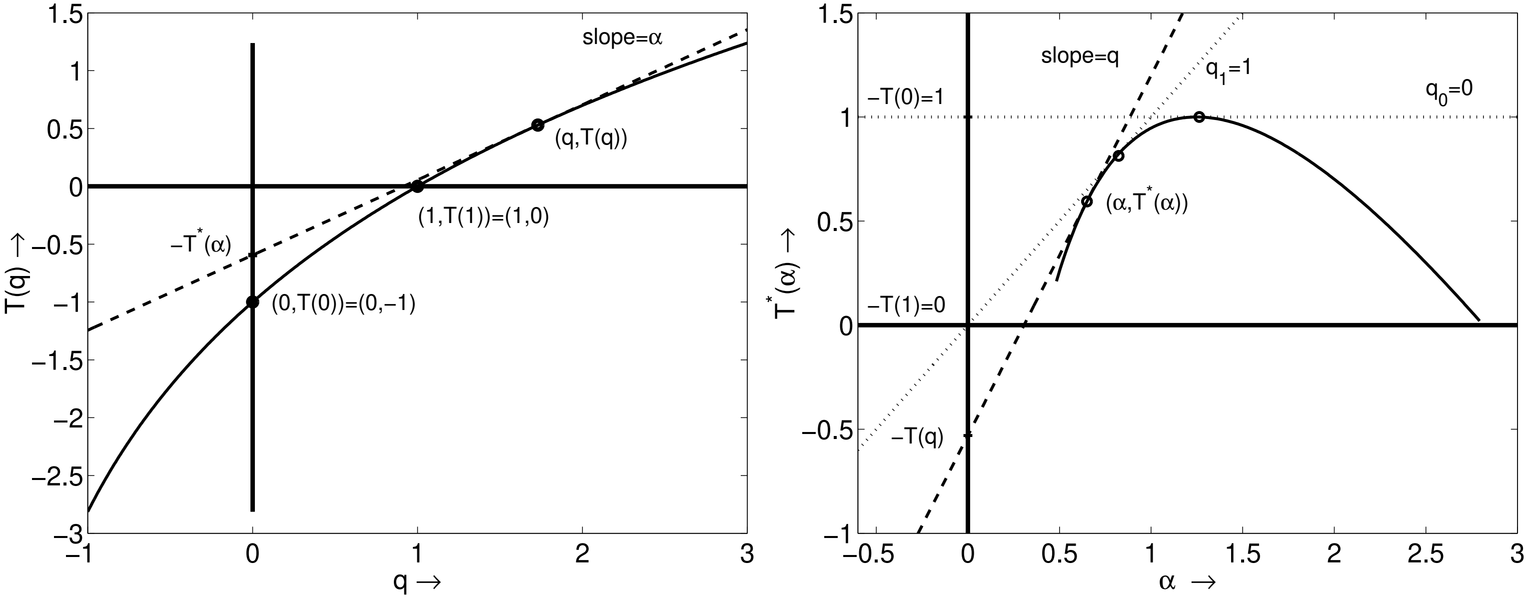
- Multifractal formalism:

$$T(q) - 1 = \inf_{\alpha}\,(q\alpha - f(\alpha))$$

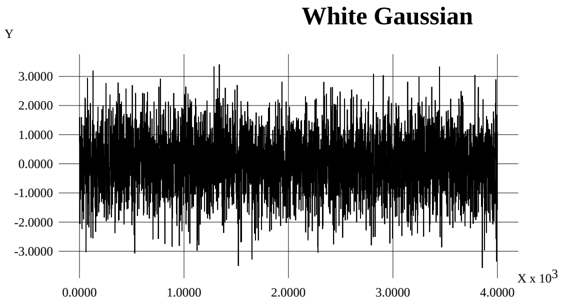
$$2^{-n(T(q)-1)} \sim \sum_{k=1}^{2^n}(X_k^{(n)})^q = \sum_\alpha \sum_{\alpha_k^n\simeq\alpha} 2^{-nq\alpha} \sim \sum_\alpha 2^{nf(\alpha)}2^{-nq\alpha} \sim 2^{-n\,\inf_\alpha(q\alpha - f(\alpha))}$$

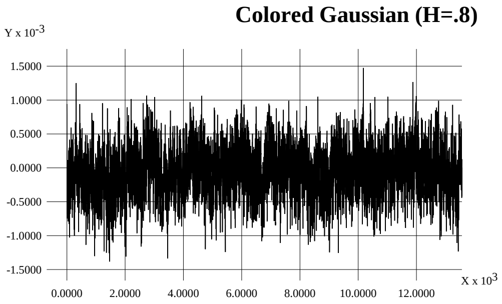
Multifractal formalism

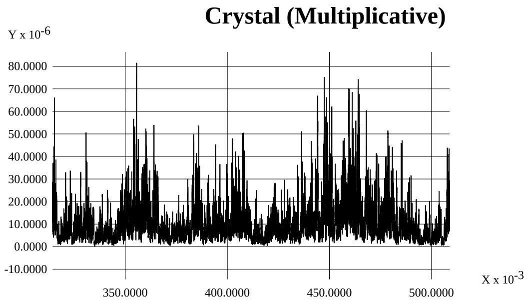
Moments give local behavior



White, Colored, Crystal Music


White Gaussian
Y
3.0000
2.0000
1.0000
0.0000
-1.0000
-2.0000
-3.0000
X x 10^3
0.0000
1.0000
2.0000
3.0000
4.0000


Colored Gaussian (H=.8)
Y x 10^-3
1.5000
1.0000
0.5000
0.0000
-0.5000
-1.0000
-1.5000
X x 10^3
0.0000
2.0000
4.0000
6.0000
8.0000
10.0000
12.0000


Crystal (Multiplicative)
Y x 10^-6
80.0000
70.0000
60.0000
50.0000
40.0000
30.0000
20.0000
10.0000
0.0000
-10.0000
X x 10^-3
350.0000
400.0000
450.0000
500.0000

Estimators

Wavelets more appropriate than marginals for processes with oscillations or trends

$$\hat{\mathbb{E}}|d_{n,k}|^q := \frac{1}{N2^n} \sum_{k=0}^{N2^n-1} |d_{n,k}|^q$$

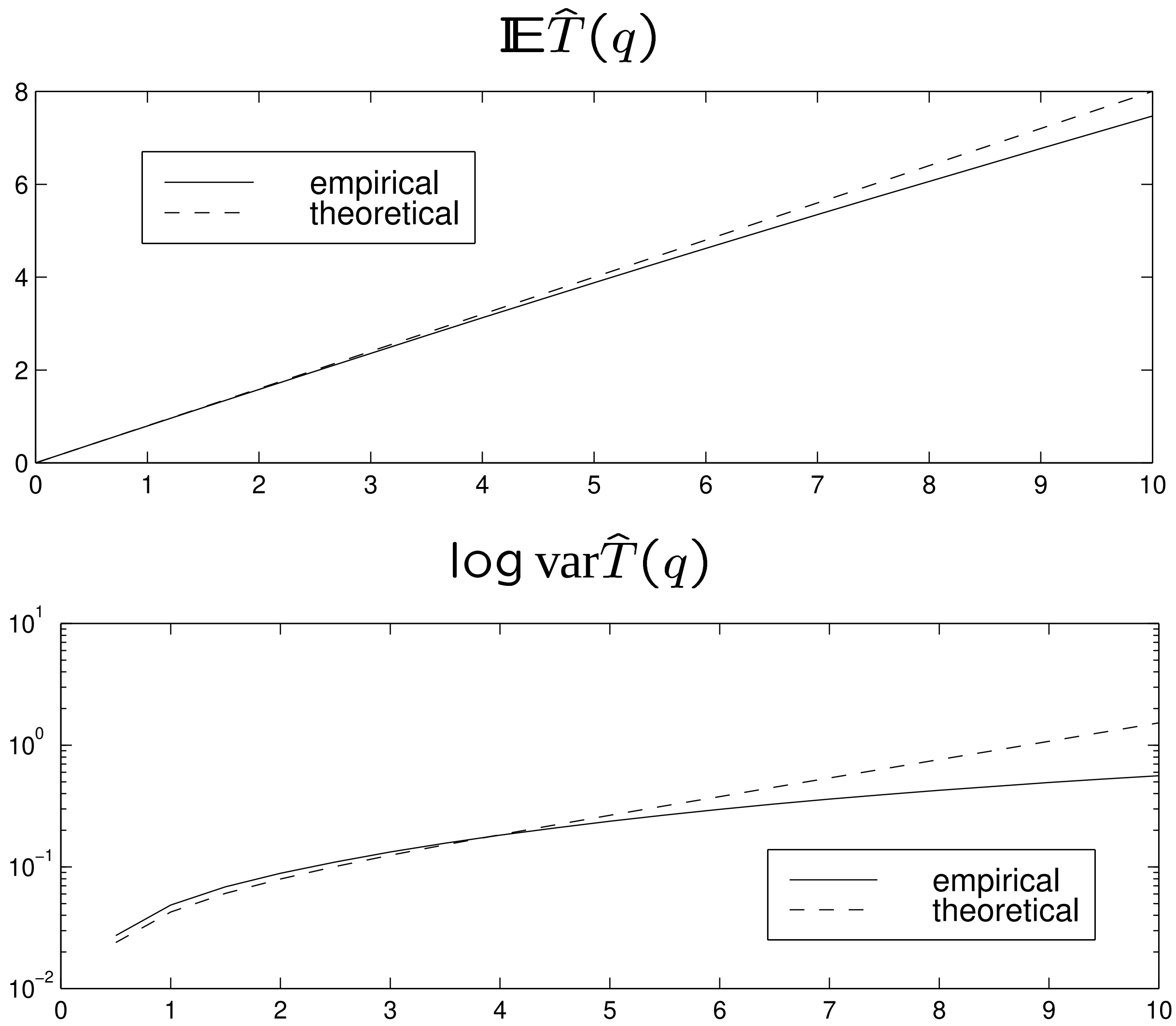
Estimators for the multifractal spectrum:

$$\hat{T}(q) := \sum_{n=j_1}^{j_2} a_n \log_2 \hat{\mathbb{E}}|d_{n,k}|^q$$

Asymptotically *unbiased* if wavelet coefficients uncorrelated within scale and if $\mathbb{E}|d_{n,k}|^q = 2^{-nT(q)}$

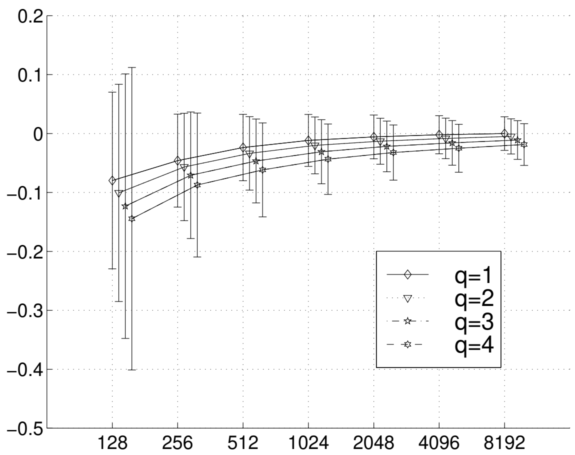
strong LLN, $\sum_n n \cdot a_n = 1$

Multifractal spectrum estimation for fBm

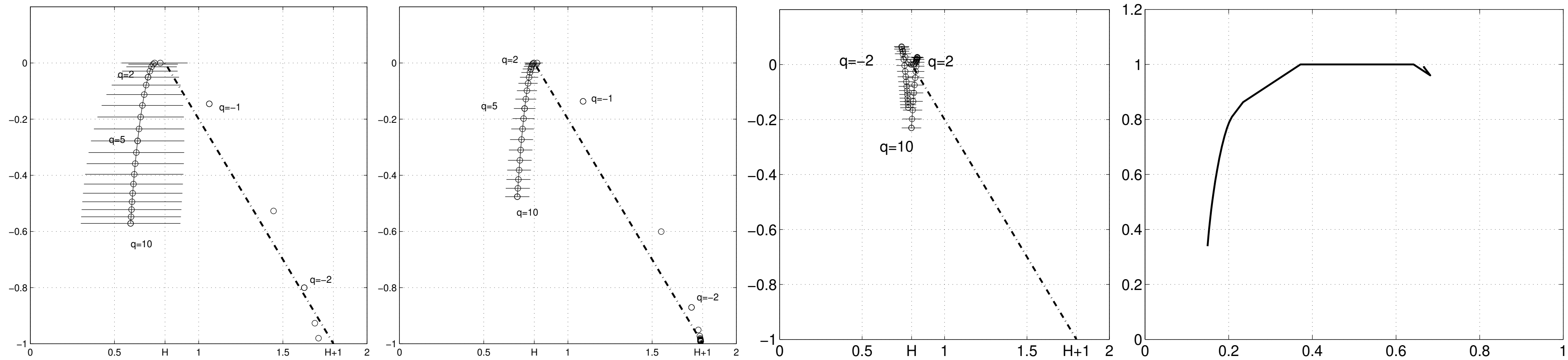


Is fBm Multifractal or Self-similar?

$$H \ - \ \mathbb{E}\widehat{\alpha}(q)$$



$\widehat{f}(\widehat{\alpha}(q))$: expectation and variance



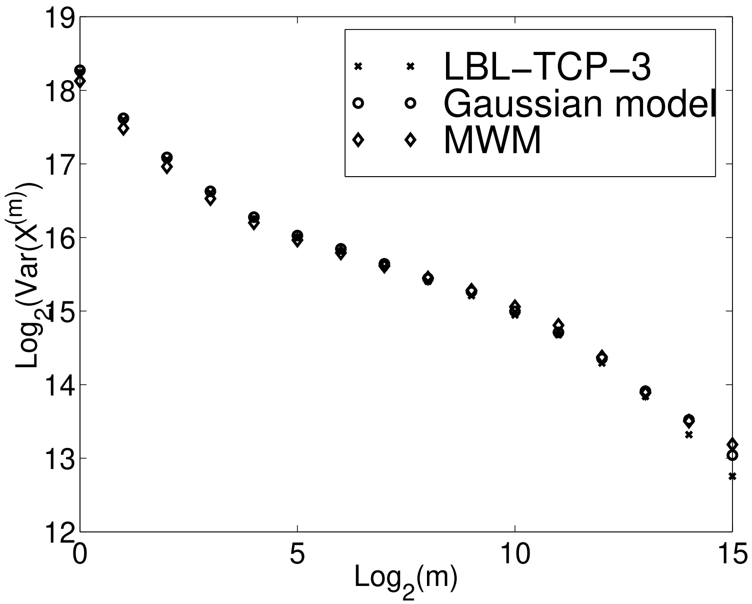
Diverging moments

For $q < -1$, $|d_{j,k}|^q$ is asymptotically $\gamma-$stable distributed with $\gamma = -1/q$.

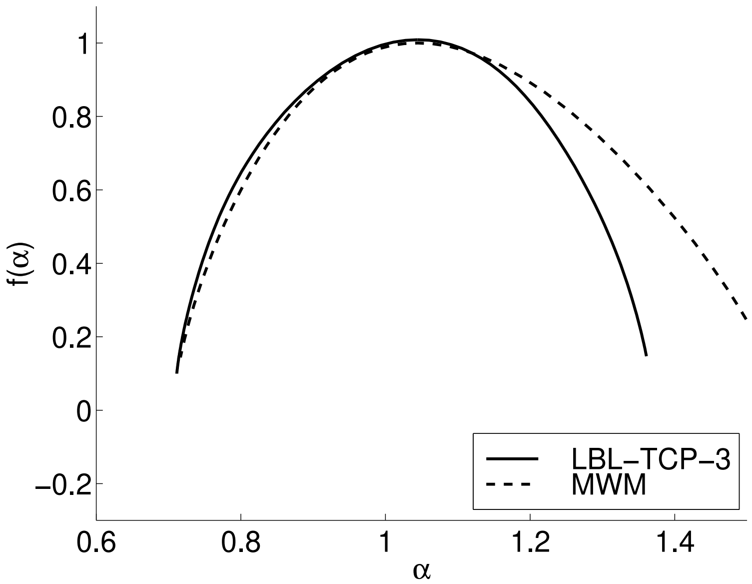
Koutrouvelis stable test on $|d_{j,k}|^q$.

q	-10	-5	-3	-1	1
$-1/q$	0.1	0.2	0.3	1	2
γ-test	0.125	0.215	0.345	0.75	1.81

Dependence Structure



Variance-time plot

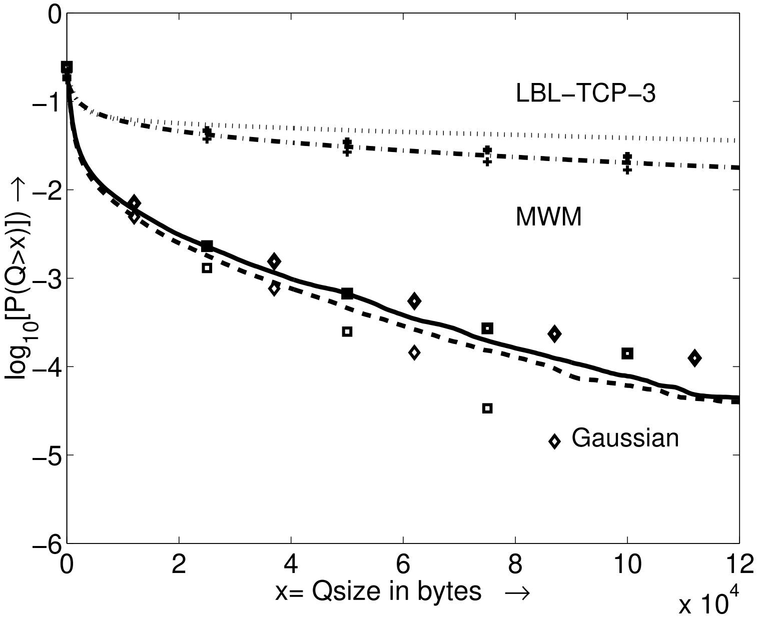


Multifractal spectrum

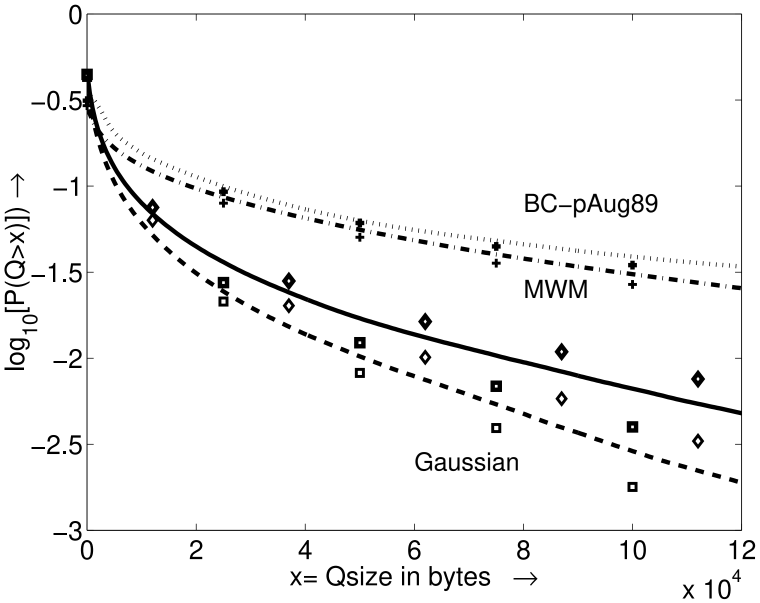
- correlation structure

- higer order dependence
- burstiness
- marginals

Queuing Analysis



Berkeley '94: WAN (TCP)



Bellcore '89: Ethernet

Marginals matter

Multiplicative modeling accurate

Conclusions

- Multiplicative model is novel, fast O(N), inherently accurate.
- Relevant for real networking applications
  - simulation / synthesis
  - prediction